Single Identities for Lattice Theory and for Weakly Associative Lattices

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March 1, 1995

Abstract. We present a single identity for the variety of all lattices that is much simpler than those previously known. We also show that the variety of weakly associative lattices is one-based, and we present a generalized one-based theorem for subvarieties of weakly associative lattices that can be defined with absorption laws. The automated theorem-proving program OTTER was used in a substantial way to obtain the results.

1 Introduction

Equational identities are, perhaps, the simplest form of sentences expressing many basic properties of algebras. Several familiar classes of algebras like semigroups, groups, rings, lattices and Boolean algebras are defined by equational identities. Such a class of algebras is known as an equational

[*]Supported by the Office of Scientific Computing, U.S. Department of Energy, under Contract W-31-109-Eng-38.
[†]Supported by an operating grant from NSERC of Canada (#A8215).

class of algebras or a variety of algebras, (for mathematical properties of such classes, see [3], [10], and [11]).

An equational theory T is said to be finitely based if it can be defined by a finite set of identities. If an equational theory T is finitely based, we may wish to determine the least number of equations needed to define the theory T, in particular to determine whether T is one-based. It is known that every finitely based variety of lattices is 2-based and it is one-based iff it is either the variety of all lattices or else the trivial variety of one-element lattices defined by $x = y$ (Cf. [7] and [8]).

This paper has two parts. First, we improve several results on the variety of all lattices, in particular, simplifying McKenzie's absorption basis for lattice theory, presenting a reduction schema containing two unary functions for single identities, and finding a single identity for lattice theory much shorter than those previously reported. Second, and more significantly, we show that the theory of weakly associative lattices is one-based and that every subvariety of weakly associative lattices that is definable with absorption laws is one-based.

The automated theorem-proving program OTTER [6] was used in a substantial way to obtain the results in this paper, and we present some of the proofs found by OTTER. Because the practical use of automated theorem proving is new and not widely known, we also try give some indications of how it was used. In many cases, our goal was to *find* identities that have certain properties as well as to prove properties of identities; in some of those cases, we used OTTER to generate a large set of candidates and ran OTTER searches with members or subsets of the candidates. We stress that OTTER was not used as a proof checker; rather, it searched for proofs, and we think of those proofs as OTTER's.

2 Single Identities for Lattice Theory

An absorption equality has a variable as one size of the equality symbol and a term with at least one other variable as the other side. Most "nice" varieties satisfy absorption laws, for example,

$$\begin{aligned} &x - (x - y) = y, && \text{in abelian groups and rings} \\ &x \wedge (x \vee y) = x, && \text{in lattices.} \end{aligned}$$

By way of contrast, the variety of all semigroups does not satisfy any absorption law.

If a lattice variety is one based, it must be definable by a single absorption law, and as a prelude, we must have a basis consisting entirely of absorption identities. The simplest previously known absorptive basis for lattice theory (in terms of $\wedge$ and $\vee$) was Ralph McKenzie's [7], consisting of the four identities

$$y \vee (x \wedge (y \wedge z)) = y, \tag{1}$$

$$y \wedge (x \vee (y \vee z)) = y, \tag{2}$$

$$((x \wedge y) \vee (y \wedge z)) \vee y = y, \tag{3}$$

$$((x \vee y) \wedge (y \vee z)) \wedge y = y. \tag{4}$$

McKenzie first constructed a single identity, containing 34 variables, for lattice theory [7]; then G. Grätzer asked for a short identity defining lattices [2]. Padmanabhan's reduction schema [9], along with McKenzie's absorption basis above yields a single identity with just seven variables; however, written without abbreviation, it has length 355 (where *length* is the count of symbols, including variables, "$\wedge$", "$\vee$", and "$=$", but not parentheses).

2.1 A Simpler Absorption Basis for Lattice Theory

The program MACE [5], which searches for models and counterexamples, was used to show that McKenzie's absorption basis (1)–(4) is independent. The automated theorem-proving program OTTER [6] was then used to examine variants of (1)–(4) in which terms are commuted. After many OTTER searches, it was discovered that the pair (1) and (4) can be replaced with the following variant of (4):

$$(((y \vee x) \wedge (y \vee z)) \wedge y) = y. \tag{5}$$

Theorem 1 *The set of identities* {(2), (3), (5)} *is a basis for the variety of all lattices.*

We list here an equational proof found by OTTER, starting with {(2), (3), (5)} and deriving {(1), (4)}. The justification "$m \rightarrow n$" indicates paramodulation from m into n, that is, equality substitution, using (an instance of) m, into a subterm of (an instance of) n; ":i, j, …" indicates simplification with $i, j, \ldots$; and "flip" indicates that the equation is reversed so that the more complex side is on the left.

Proof (found by Otter 3.0.3d on gyro at 3.33 seconds).

3	$x \wedge (y \vee (x \vee z)) = x$	[(2)]
5	$((x \wedge y) \vee (y \wedge z)) \vee y = y$	[(3)]
8,7	$((x \vee y) \wedge (x \vee z)) \wedge x = x$	[(5)]
9	$((x \wedge y) \vee y) \vee y = y$	[$3 \rightarrow 5$]
11	$x \wedge (x \vee y) = x$	[$5 \rightarrow 3$]
13	$((x \wedge y) \vee (y \wedge z)) \wedge y = (x \wedge y) \vee (y \wedge z)$	[$5 \rightarrow 11$]
24,23	$(x \vee y) \wedge x = x$	[$3 \rightarrow 7$]
27	$(x \vee (x \wedge y)) \vee x = x$	[$7 \rightarrow 5$]
30,29	$x \vee (x \wedge y) = x$	[$7 \rightarrow 13$:24,8, flip]
31	$x \vee x = x$	[27 :30]
38,37	$x \wedge (y \vee x) = x$	[$31 \rightarrow 3$]
40,39	$(x \wedge y) \vee y = y$	[$9 \rightarrow 23$:38, flip]
45	$(x \vee y) \vee x = x \vee y$	[$23 \rightarrow 29$]
55	$x \vee (y \vee x) = y \vee x$	[$37 \rightarrow 39$]
67	$(x \wedge y) \wedge (z \vee y) = x \wedge y$	[$39 \rightarrow 3$]
69	$x \vee (y \wedge x) = x$	[$39 \rightarrow 45$:40]
106,105	$((x \vee y) \wedge (y \vee z)) \wedge y = y$	[$55 \rightarrow 7$]
224	$(x \vee y) \vee (z \wedge y) = x \vee y$	[$67 \rightarrow 69$]
677	$x \vee (y \wedge (x \wedge z)) = x$	[$29 \rightarrow 224$:30]

Line 106 is (4), and line 677 is (1).

2.2 A New Reduction Schema

Padmanabhan's reduction schema in [9] is

$$p(p(x, y, y), u, p(p(x, y, y), f(y), z)) = y. \tag{6}$$

where p is a majority polynomial. This is equivalent to the set

$$\{p(y, y, x) = y,\ p(y, x, y) = y,\ p(x, y, y) = y,\ f(y) = y\}. \tag{7}$$

(To build a single identity for a theory satisfying a majority polynomial and axiomatized with an absorption basis, say of size n, one can apply the lemma in [9] $n - 1$ times, then plug the result into the reduction schema.)

Lemma 1 *The identity*

$$p(p(x,y,y),p(x,p(y,z,f(y)),g(y)),u) = y \tag{8}$$

is equivalent to the set

$$\{p(y,y,x) = y,\ p(y,x,y) = y,\ p(x,y,y) = y,\ f(y) = y,\ g(y) = y\}. \tag{9}$$

Proof (found by Otter 3.0.3d on gyro at 0.28 seconds).

4,3	$p(p(x,y,y),p(x,p(y,z,f(y)),g(y)),u) = y$	[(8)]
5	$p(p(x,p(y,z,z),p(y,z,z)),p(x,z,g(p(y,z,z))),u) = p(y,z,z)$	[3 → 3]
7	$p(p(x,y,y),p(x,p(z,p(y,u,f(y)),g(y)),g(y)),v) = y$	[3 → 5 :4,4,4]
10,9	$p(p(p(x,y,y),y,y),y,z) = y$	[3 → 7]
16,15	$p(x,x,y) = x$	[9 → 9]
19,18	$p(p(x,y,y),p(x,y,g(y)),z) = y$	[9 → 5 :10,10,10]
22	$p(p(x,y,y),p(x,p(z,y,g(y)),g(y)),u) = y$	[15 → 7]
33,32	$p(x,y,g(y)) = y$	[18 → 18 :19,16, flip]
40	$p(x,p(y,z,f(y)),g(y)) = y$	[3 → 18 :4,16, flip]
42	$p(p(x,y,y),y,z) = y$	[22 :33,33]
44	$p(x,y,f(x)) = x$	[42 → 40]
47,46	$p(x,y,y) = y$	[42 → 44, flip]
50,49	$f(x) = x$	[44 → 46, flip]
53,52	$p(x,y,x) = x$	[44 :50]
55	$g(x) = x$	[32 → 52, flip]

Lines 16, 47, 50, 53, and 55 establish one direction of the equivalence; the other is established by inspection.

The reduction schema (8) was found by automatically generating candidate schemas with Otter, then with each, searching with Otter for a proof of the preceding type.

With the new reduction schema, because we have $f(y) = y$ and $g(y) = y$, we need to apply the lemma of [9] $n-2$ times instead of $n-1$ times, and with the new lattice theory absorption basis of size $n = 3$, instead of $n = 4$, we can build a single identity with just one application of the lemma. Using a variant of the majority polynomial used to build the length 355 identity, we can build a single identity of length 139, again with 7 variables, for lattice theory.

2.3 A Simpler Single Identity for Lattice Theory

The reduction schema (8) and the majority polynomial

$$p(x, y, z) = (x \wedge z) \vee (y \wedge (x \vee z)) \tag{10}$$

satisfy the three identities

$$(x \wedge y) \vee (x \wedge (x \vee y)) = x, \tag{11}$$

$$(x \wedge x) \vee (y \wedge (x \vee x)) = x, \tag{12}$$

$$(x \wedge y) \vee (y \wedge (x \vee y)) = y. \tag{13}$$

We can use the fact that these identities are "built-in" to the majority polynomial to simplify construction of a single identity (producing a shorter one). In particular, if an absorption basis for lattice theory contains the above three identities, those three identities need not be used in the construction of a single identity.

Theorem 2 *The absorption identities*

$$((x \wedge y) \vee (y \wedge z)) \vee y = y, \tag{14}$$

$$((x \vee (y \vee z)) \wedge (u \vee y)) \wedge y = y, \tag{15}$$

union the three built-in identities $\{(11), (12), (13)\}$ *are a basis for the variety of all lattices.*

Note that (14) is the same as (3).

Proof (found by Otter 3.0.3d on gyro at 0.82 seconds).

3	$(x \wedge y) \vee (x \wedge (x \vee y)) = x$	[(11)]
5	$(x \wedge x) \vee (y \wedge (x \vee x)) = x$	[(12)]
7	$(x \wedge y) \vee (y \wedge (x \vee y)) = y$	[(13)]
9	$((x \wedge y) \vee (y \wedge z)) \vee y = y$	[(14)]
11	$((x \vee (y \vee z)) \wedge (u \vee y)) \wedge y = y$	[(15)]
14,13	$x \vee x = x$	[$7 \rightarrow 9$]
15	$(x \wedge x) \vee (y \wedge x) = x$	[5 :14]
19	$((x \wedge y) \vee (y \wedge z)) \wedge y = (x \wedge y) \vee (y \wedge z)$	[$9 \rightarrow 3$:14]
26,25	$x \wedge x = x$	[$13 \rightarrow 7$:14]

27	$x \vee (y \wedge x) = x$	[15 :26]
31	$(x \vee (x \wedge y)) \vee x = x$	[25 $\rightarrow$ 9]
39	$((x \vee y) \wedge (z \vee x)) \wedge x = x$	[13 $\rightarrow$ 11]
41	$((x \vee (y \vee z)) \wedge y) \wedge y = y$	[13 $\rightarrow$ 11]
47	$(x \wedge y) \vee y = y$	[27 $\rightarrow$ 9]
51	$(x \wedge (y \wedge x)) \vee x = x$	[27 $\rightarrow$ 3 :26]
54,53	$(x \wedge y) \wedge y = x \wedge y$	[47 $\rightarrow$ 3 :14]
56,55	$(x \vee (y \vee z)) \wedge y = y$	[41 :54]
57	$(x \vee (x \wedge y)) \wedge x = x \vee (x \wedge y)$	[31 $\rightarrow$ 3 :14]
78,77	$(x \vee y) \wedge x = x$	[51 $\rightarrow$ 55]
79	$x \vee (x \wedge y) = x$	[57 :78, flip]
94,93	$x \wedge (x \wedge y) = x \wedge y$	[3 $\rightarrow$ 77]
99	$(x \vee y) \vee x = x \vee y$	[77 $\rightarrow$ 79]
112,111	$(x \wedge y) \vee x = x$	[79 $\rightarrow$ 3 :94,26]
115	$(x \wedge y) \wedge x = x \wedge y$	[111 $\rightarrow$ 19 :112]
160,159	$x \wedge (y \vee (x \vee z)) = x$	[55 $\rightarrow$ 115 :56]
223,222	$((x \vee y) \wedge (x \vee z)) \wedge x = x$	[99 $\rightarrow$ 39]

Lines 9, 160, and 223 are equations (3), (2), and (5), respectively.

The pair (14) and (15) was found by automatically searching with many candidate sets for proofs of the preceding type.

With (14) and (15), we need not apply the lemma in [9] to build a single identity. With the reduction schema (8) we can simply plug in (14) and (15) for $f(y)$ and $g(y)$. If the single identity is written in terms of $\wedge$ and $\vee$, it has length 79, again with 7 variables. It is

$$\begin{aligned}&(((x \wedge y) \vee (y \wedge (x \vee y))) \wedge z) \vee (((x \wedge (((x_1 \wedge y) \vee (y \wedge x_2)) \vee \\ &\quad y)) \vee (((y \wedge (((x_1 \vee (y \vee x_2)) \wedge (x_3 \vee y)) \wedge y)) \vee (u \wedge (y \vee \\ &\quad (((x_1 \vee (y \vee x_2)) \wedge (x_3 \vee y)) \wedge y)))) \wedge (x \vee (((x_1 \wedge y) \vee (y \wedge \\ &\quad x_2)) \vee y)))) \wedge (((x \wedge y) \vee (y \wedge (x \vee y))) \vee z)) = y.\end{aligned} \tag{16}$$

3 Weakly Associative Lattices

Here we prove that the variety of all weakly associative lattices (WAL) is one-based. We then show that every finitely based subvariety of the WAL which are definable by absorption laws is also one-based.

3.1 WAL is One-based

We take as our starting point the following (self-dual) basis for WAL [1]. [[RP: is that the correct reference for these axioms?]]

$$
\begin{array}{lll}
x \wedge x = x, & x \vee x = x, & \text{(W1,W1}') \\
x \wedge y = y \wedge x, & x \vee y = y \vee x, & \text{(W2,W2}') \\
((x \vee z) \wedge (y \vee z)) \wedge z = z, & ((x \wedge z) \vee (y \wedge z)) \vee z = z, & \text{(W3,W3}') \\
(x \vee z) \wedge z = z, & (x \wedge z) \vee z = z. & \text{(W4,W4}')
\end{array}
$$

Clearly, (W4) follows from (W1) and (W3), and dually, (W4′) follows from (W1′) and (W3′); henceforth we omit (W4,W4′).

The ternary polynomial (10) is a majority polynomial for WAL as well as for lattice theory. Therefore, if we can find an absorption basis for WAL, we can construct a single identity for WAL. Since the "built-in" identities (11)–(13) hold for WAL, we can attempt the same sort of construction as for lattice theory in Sec. 2.3.

With a sequence of OTTER searches, we tested many pairs of absorption identities, adding each pair to (11)–(13) and attempting to derive the set (W1, W1′, W2, W2′, W3, W3′). OTTER succeeded with pair

$$((x \vee y) \wedge (z \vee x)) \wedge x = x, \tag{17}$$

$$((x \wedge y) \vee (z \wedge x)) \vee x = x. \tag{18}$$

Lemma 2 *The set* {(11), (12), (13), (17), (18)} *is a basis for the variety of all weakly associative lattices.*

Because (17) and (18) are simply commuted variants of (W3) and (W3′), it is sufficient to derive idempotence and commutativity of the two operations.

Proof (found by Otter 3.0.3d on gyro at 2.51 seconds).

4	$(x \wedge y) \vee (x \wedge (x \vee y)) = x$	[(11)]
6	$(x \wedge x) \vee (y \wedge (x \vee x)) = x$	[(12)]
8	$(x \wedge y) \vee (y \wedge (x \vee y)) = y$	[(13)]
10	$((x \vee y) \wedge (z \vee x)) \wedge x = x$	[(17)]
12	$((x \wedge y) \vee (z \wedge x)) \vee x = x$	[(18)]

14	$(x \wedge (y \vee (z \wedge x))) \wedge (z \wedge x) = z \wedge x$	[8 → 10]
16	$(x \wedge (y \vee (x \wedge z))) \wedge (x \wedge z) = x \wedge z$	[4 → 10]
21,20	$x \vee (x \wedge (((x \vee y) \wedge (z \vee x)) \vee x)) = x$	[10 → 8]
24	$((x \wedge y) \vee x) \vee x = x$	[10 → 12]
27,26	$((x \vee y) \wedge x) \wedge x = x$	[12 → 10]
30	$(((x \wedge y) \vee x) \wedge x) \vee (x \wedge x) = x$	[24 → 8]
34	$(x \wedge (y \wedge x)) \wedge (y \wedge x) = y \wedge x$	[8 → 26]
45,44	$(x \wedge x) \wedge x = x$	[26 → 34 :27,27]
47,46	$(x \wedge (y \vee x)) \wedge x = x$	[26 → 14 :27,27]
48	$(x \wedge x) \wedge (x \wedge x) = x \wedge x$	[30 → 14]
63,62	$x \wedge (x \wedge (y \vee x)) = x \wedge (y \vee x)$	[8 → 46 :47]
64	$x \vee (x \wedge ((x \wedge (y \vee x)) \vee x)) = x$	[46 → 8]
70	$((x \wedge x) \vee (x \wedge x)) \vee (x \wedge x) = x \wedge x$	[48 → 24]
74	$((x \wedge x) \wedge (y \vee x)) \wedge x = x$	[44 → 16 :45,45]
89,88	$x \wedge x = x$	[24 → 74 :45]
95,94	$(x \vee x) \vee x = x$	[70 :89,89,89,89]
106	$x \vee (y \wedge (x \vee x)) = x$	[6 :89]
114,113	$x \vee x = x$	[88 → 20 :95,89]
116	$x \vee (y \wedge x) = x$	[106 :114]
119,118	$(x \wedge (((x \vee y) \wedge (z \vee x)) \vee x)) \vee x = x$	[20 → 4 :63,89]
123,122	$x \wedge (y \wedge x) = y \wedge x$	[116 → 14 :89]
128	$(x \wedge y) \vee y = y$	[116 → 4 :123,89]
131,130	$(x \wedge y) \wedge y = x \wedge y$	[128 → 4 :114]
134	$(x \vee y) \wedge x = x$	[26 :131]
136	$x \wedge (((x \vee y) \wedge (z \vee x)) \vee x) = x$	[20 → 64 :47,21,47,119, flip]
141,140	$x \wedge ((x \wedge y) \vee x) = (x \wedge y) \vee x$	[24 → 134]
142	$x \vee ((x \wedge (y \vee x)) \vee x) = x$	[64 :141]
144	$(x \vee (x \vee y)) \vee (x \vee y) = x \vee y$	[134 → 24]
152	$((x \wedge y) \vee (z \wedge y)) \vee y = y$	[122 → 12]
172	$(x \wedge (y \vee x)) \vee x = x$	[142 → 136 :141]
179,178	$x \wedge (y \vee x) = x$	[172 → 4 :47,47,114, flip]
180	$(x \wedge y) \vee x = x$	[140 :179, flip]
189,188	$(x \vee y) \vee y = x \vee y$	[178 → 116]
194	$(x \vee (y \wedge (z \vee x))) \vee (z \vee x) = z \vee x$	[178 → 152]
198	$x \vee (x \vee y) = x \vee y$	[144 :189]
200	$(x \wedge y) \wedge x = x \wedge y$	[180 → 4 :114]
212	$x \wedge (x \vee y) = x$	[198 → 4 :114]
216	$(x \vee y) \vee x = x \vee y$	[212 → 116]

236	$x \vee (x \wedge y) = x$	[200 → 116]
243,242	$(x \wedge y) \wedge (y \wedge x) = y \wedge x$	[236 → 14]
286	$x \wedge y = y \wedge x$	[242 → 200 :243,243]
288,287	$(x \vee y) \vee (y \vee x) = y \vee x$	[212 → 194]
349	$x \vee y = y \vee x$	[287 → 216 :288,288]

Lines 89, 114, 286, and 349 establish the result.

Theorem 3 *The variety all weakly associative lattices is one-based.*

Proof Using the reduction schema (8), we can construct a single identity for WAL by plugging in (17) and (18) for $f(y)$ and $g(y)$. The identity, written in terms of $\wedge$ and $\vee$ has length 75 and 6 variables:

$$\begin{aligned}&(((x \wedge y) \vee (y \wedge (x \vee y))) \wedge z) \vee (((x \wedge (((y \wedge x_1) \vee (x_2 \wedge y))\vee \\ &\quad y)) \vee (((y \wedge (((y \vee x_1) \wedge (x_2 \vee y)) \wedge y)) \vee (u \wedge (y \vee (((y \vee x_1)\wedge \\ &\quad (x_2 \vee y)) \wedge y)))) \wedge (x \vee (((y \wedge x_1) \vee (x_2 \wedge y)) \vee y)))) \wedge (((x\wedge \\ &\quad y) \vee (y \wedge (x \vee y))) \vee z)) = y.\end{aligned} \tag{19}$$

3.2 Subvarieties of WAL

Let gWAL stand for the equational theory of all WAL's satisfying a further absorption law, say $g(x_1, x_2, \cdots, x_n) = x_1$. To find a single identity for gWAL, we have to somehow embed the additional absorption law into a single identity for WAL. The idea of embedding a unary function inside an identity goes back to G. Higman and B.H. Neumann (see [4] where they prove that every finitely based variety of group theory is one-based). Such a technique was first applied to lattices by Padmanabhan in [8].

Theorem 4 *The variety of all* gWAL *is one-based.*

Proof We use the reduction schema (8), which has two unary functions:

$$p(p(x, y, y), p(x, p(y, z, f(y)), g(y)), u) = y,$$

and let us write (17) and (18) as $W_1(y) = y$ and $W_2(y) = y$. Using the lemma in [9], we can bundle the identities (17) and (18) as $p(W_1(y), W_2(y), x)$, and substitute this for $f(y)$; we keep $g(y)$ as a place-holder for the further absorption law $g(x_1, x_2, \cdots, x_n) = x_1$. The resulting single identity, written in terms of $\wedge$ and $\vee$ (using the same majority polynomial as above) is

$$\begin{aligned}&(((x \wedge y) \vee (y \wedge (x \vee y))) \wedge z) \vee (((x \wedge g(y)) \vee (((y \wedge (((((y \vee x_1) \wedge \\ &\quad (x_2 \vee y)) \wedge y) \wedge u) \vee ((((y \wedge x_1) \vee (x_2 \wedge y)) \vee y) \wedge ((((y \vee \\ &\quad x_1) \wedge (x_2 \vee y)) \wedge y) \vee u)))) \vee (v \wedge (y \vee (((((y \vee x_1) \wedge (x_2 \vee \\ &\quad y)) \wedge y) \wedge u) \vee ((((y \wedge x_1) \vee (x_2 \wedge y)) \vee y) \wedge ((((y \vee x_1) \wedge \\ &\quad (x_2 \vee y)) \wedge y) \vee u)))))) \wedge (x \vee g(y)))) \wedge (((x \wedge y) \vee (y \wedge (x \vee \\ &\quad y))) \vee z)) = y.\end{aligned} \tag{20}$$

The preceding single identity for gWAL has length 109 and 7 variables. Note the two occurrences of $g(y)$, which represent the further absorption law. If a further *set* of absorption laws, instead of a single law, is added, the set can be bundled, as above, with applications of the lemma in [9].

The theory of all lattices may be viewed as a weakly associative lattice satisfying the additional absorption law

$$x \wedge (y \vee (x \vee z)) = x \tag{21}$$

Lemma 3 {WAL, (21)} $\Rightarrow$ {*the two associative laws for* $\wedge$ *and* $\vee$}.

Proved by OTTER, but the proof is omitted because of length.

Therefore, as a corollary to the Theorem 4, we have that lattice theory is one-based (also, [7], [9], and Sec. 2).

4 Open Questions

1. Are there a single identities for lattice theory, WAL, or gWAL simpler than the ones presented here (16, 19, 20)?

2. Is the variety of near lattices (NL) one-based? A basis for NL is the following set.

$$\begin{array}{ll} x \wedge x = x, & x \vee x = x, \\ x \wedge y = y \wedge x, & x \vee y = y \vee x, \\ x \wedge (x \wedge y) = x \wedge y, & x \vee (x \vee y) = x \vee y, \\ x \wedge (x \vee y) = x, & x \vee (x \wedge y) = x. \end{array}$$

3. Is the variety of transitive near lattices (TNL) one-based? We have a basis TNL by adjoining the following pair to NL:

$$x \wedge (y \vee (x \vee z)) = x, \qquad x \vee (y \wedge (x \wedge z)) = x.$$

Appendix

The programs OTTER and MACE are in the public domain and are available through internet by FTP. See the file

`ftp://info.mcs.anl.gov/pub/Otter/README`

for more information.

Several OTTER input files, which produce proofs related to this paper, are available by FTP. These include input files for the OTTER proofs that appear here, input files for the omitted proof of Lemma 3, and input files that were used to double check, directly, that (16), (19), and (20) are single identities for lattice theory, WAL, and gWAL, respectively. See

`ftp://info.mcs.anl.gov/pub/Otter/LT-WAL/README`

for specific information on the input files.

References

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