An Anthropomorphized Version of
McCune's Machine Proof that Robbins Algebras are Boolean Algebras

Abbreviations Used

In the following, the effect of $\widehat{}$ on a term $t(x, y, z, \ldots)$ is to substitute x for all variables in the term, i.e., one obtains $t(x, x, x, \ldots)$.

$$
\begin{array}{rcl|rcl}
A & := & x+y & \widehat{A} & := & x+x \\
B & := & \mathbf{N}(x+y) & & & \\
C & := & \mathbf{N}x+y & & & \\
D & := & \mathbf{N}(\mathbf{N}x+y) & \widehat{D} & := & \mathbf{N}(\mathbf{N}x+x) \\
E & := & D+A & \widehat{E} & := & \widehat{D}+\widehat{A} \\
F & := & E+y & & & \\
G & := & F+D & & & \\
H & := & \mathbf{N}F+D & & & \\
I & := & H+z & & & \\
J & := & \mathbf{N}(y+z) & & & \\
K & := & \mathbf{N}(I+J) & & & \\
L & := & y+z & & & \\
M & := & x+L & & & \\
P & := & \mathbf{N}(\widehat{E}+x)+x & & & \\
Q & := & \widehat{E}+x & & & \\
R & := & Q+x & & & \\
S & := & \mathbf{N}(R+\widehat{D})+x & & & \\
T & := & D+E+y+y & \widehat{T} & := & \widehat{D}+\widehat{E}+x+x \\
U & := & N\widehat{T}+x & & &
\end{array}
$$

AXIOMS

$$x+y = y+x \qquad \text{(C)}$$

$$x+(y+z) = (x+y)+z \qquad \text{(A)}$$

$$\mathbf{N}[\mathbf{N}(\mathbf{N}x+y)+\mathbf{N}(x+y)] = y \qquad \text{(RX).}$$

PROOF

$$\mathbf{N}[\mathbf{N}(x+y)+\mathbf{N}(\mathbf{N}x+y)] = y \qquad \text{apply (C) to (RX).} \tag{1}$$

$$\mathbf{N}[\mathbf{N}C+B] = \mathbf{N}[\mathbf{N}(\mathbf{N}x+y)+\mathbf{N}(x+y)] \qquad \text{by def. of } C, B \tag{2}$$

$$= y \qquad \text{by (RX)} \tag{3}$$

$$\mathbf{N}[\mathbf{N}C+B] = y \tag{4}$$

$$B = \mathbf{N}[\mathbf{N}(\mathbf{N}C+B)+\mathbf{N}(C+B)] \qquad \text{by (RX)} \tag{5}$$

$$= \mathbf{N}[y+\mathbf{N}(C+B)] \qquad \text{apply (4)} \tag{6}$$

$$\mathbf{N}[y+\mathbf{N}(C+B)] = B \tag{7}$$

$$\mathbf{N}[y+\mathbf{N}(B+C)] = B \qquad \text{apply (C)} \tag{8}$$

$$\mathbf{N}[y+\mathbf{N}(\mathbf{N}(x+y)+\mathbf{N}x+y)] = \mathbf{N}(x+y) \qquad \text{by def. of } C, B \tag{9}$$

$$y = \mathbf{N}[\mathbf{N}(x+y)+\mathbf{N}(\mathbf{N}x+y)] \qquad \text{by (1)} \tag{10}$$

$$= \mathbf{N}[D+\mathbf{N}A] \qquad \text{by def. of } A, D \tag{11}$$

$$\mathbf{N}[D+\mathbf{N}A] = y \tag{12}$$

$$D = \mathbf{N}[\mathbf{N}(A+D)+\mathbf{N}(\mathbf{N}A+D)] \qquad \text{by (1)} \tag{13}$$

$$= \mathbf{N}[\mathbf{N}E+\mathbf{N}(\mathbf{N}A+D)] \qquad \text{by def. of } E \tag{14}$$

$$= \mathbf{N}[\mathbf{N}E+y] \qquad \text{by (12)} \tag{15}$$

$$\mathbf{N}[\mathbf{N}E+y] = D \tag{16}$$

$$y = \mathbf{N}[\mathbf{N}(E+y)+\mathbf{N}(\mathbf{N}E+y)] \qquad \text{by (1)} \tag{17}$$

$$= \mathbf{N}[\mathbf{N}(E+y)+D] \qquad \text{by (16)} \tag{18}$$

$$\mathbf{N}[\mathbf{N}(E+y)+D] = y \tag{19}$$

$$\mathbf{N}[\mathbf{N}F+D] = y \qquad \text{by def. of } F \tag{20}$$

$$\mathbf{N}H = y \qquad \text{by def. of } H \tag{21}$$

$$z = \mathbf{N}[\mathbf{N}(H+z)+\mathbf{N}(\mathbf{N}H+z)] \qquad \text{by (1)} \tag{22}$$

$$= \mathbf{N}[\mathbf{N}(H+z)+\mathbf{N}(y+z)] \qquad \text{by (21)} \tag{23}$$

$$\mathbf{N}[\mathbf{N}(H+z)+\mathbf{N}(y+z)] = z \tag{24}$$

$$\mathbf{N}[\mathbf{N}I+J] = z \qquad \text{by def. of } I, J \tag{25}$$

$$D = \mathbf{N}[\mathbf{N}(F+D)+\mathbf{N}(\mathbf{N}F+D)] \qquad \text{by (1)} \tag{26}$$

$$= \mathbf{N}[\mathbf{N}(F+D)+y] \qquad \text{by (20)} \tag{27}$$

$$= \mathbf{N}[\mathbf{N}G+y] \qquad \text{by def. of } G \tag{28}$$

$$\mathbf{N}[\mathbf{N}G+y] = D \tag{29}$$

$$y = \mathbf{N}[\mathbf{N}(G+y)+\mathbf{N}(\mathbf{N}G+y)] \qquad \text{by (1)} \tag{30}$$

$$= \mathbf{N}[\mathbf{N}(G+y)+D] \qquad \text{by (29)} \tag{31}$$

$$\mathbf{N}[\mathbf{N}(G+y)+D] = y \tag{32}$$

$$J = \mathbf{N}[\mathbf{N}(I+J)+\mathbf{N}(\mathbf{N}I+J)] \qquad \text{by (1)} \tag{33}$$

$$= \mathbf{N}[\mathbf{N}(I+J)+z] \qquad \text{by (25)} \tag{34}$$

$$
\begin{aligned}
\mathbf{N}[\mathbf{N}(I+J)+z] &= J && (35)\\
\mathbf{N}[K+z] &= J \quad \text{by def. of } K && (36)\\[1ex]
u &= \mathbf{N}[\mathbf{N}((K+z)+u)+\mathbf{N}(\mathbf{N}(K+z)+u)] \quad \text{by (1)} && (37)\\
&= \mathbf{N}[\mathbf{N}(K+z+u)+\mathbf{N}(J+u)] \quad \text{by (36)} && (38)\\
\mathbf{N}[\mathbf{N}(K+z+u)+\mathbf{N}(J+u)] &= u && (39)\\[1ex]
\mathbf{N}[L+\mathbf{N}(\mathbf{N}(x+L)+\mathbf{N}x+L)] &= \mathbf{N}[x+L] \quad \text{subs } y \leftarrow L \text{ in (9)} && (40)\\
\mathbf{N}[L+\mathbf{N}(\mathbf{N}M+\mathbf{N}x+L)] &= \mathbf{N}M \quad \text{by def. of } M && (41)\\
\mathbf{N}[A+\mathbf{N}(\mathbf{N}M+\mathbf{N}z+A)] &= \mathbf{N}M \quad \text{subs: } x \leftarrow z, y \leftarrow x, z \leftarrow y && (42)\\
\mathbf{N}[x+y+\mathbf{N}(\mathbf{N}(x+y+z)+\mathbf{N}z+x+y)] &= \mathbf{N}[x+y+z] \quad \text{by def. of } A, M && (43)\\[1ex]
\widehat{J} &= \mathbf{N}\widehat{A} \quad \text{as both sides} = \mathbf{N}[x+x] && (44)\\
\mathbf{N}[\mathbf{N}\widehat{A}+\widehat{D}] &= \mathbf{N}[\mathbf{N}(x+x)+\mathbf{N}(\mathbf{N}x+x)] \quad \text{by def. } \widehat{A}, \widehat{D} && (45)\\
&= x \quad \text{by (1)} && (46)\\
\widehat{K} &= \mathbf{N}[\widehat{I}+\widehat{J}] \quad \text{by def. of } \widehat{K} && (47)\\
&= \mathbf{N}[\widehat{H}+x+\mathbf{N}\widehat{A}] \quad \text{by def. of } \widehat{I}\text{, and (44)} && (48)\\
&= \mathbf{N}[\mathbf{N}\widehat{F}+\widehat{D}+x+\mathbf{N}\widehat{A}] \quad \text{by def. of } \widehat{H} && (49)\\
&= \mathbf{N}[\mathbf{N}(\widehat{E}+x)+\widehat{D}+x+\mathbf{N}\widehat{A}] \quad \text{by def. of } \widehat{F} && (50)\\
&= \mathbf{N}[\mathbf{N}(\widehat{D}+\widehat{A}+x)+\widehat{D}+x+\mathbf{N}\widehat{A}] \quad \text{by def. of } \widehat{E} && (51)\\
\mathbf{N}[\widehat{D}+x+\mathbf{N}(\mathbf{N}(\widehat{D}+x+\widehat{A})+\mathbf{N}\widehat{A}+\widehat{D}+x)] &= \mathbf{N}[\widehat{D}+x+\widehat{A}] \quad \text{by (43)} && (52)\\
& \quad \text{using } x \leftarrow \widehat{D};\ y \leftarrow x;\ z \leftarrow \widehat{A} &&\\
\mathbf{N}[\widehat{D}+x+\widehat{K}] &= \mathbf{N}[\widehat{D}+x+\widehat{A}] \quad \text{by (51)} && (53)\\
\mathbf{N}[\mathbf{N}(K+z+u)+\mathbf{N}(J+u)] &= u \quad \text{by (39)} && (54)\\
\mathbf{N}[\mathbf{N}(\widehat{K}+x+u)+\mathbf{N}(\widehat{J}+u)] &= u \quad \text{subs } y, z \leftarrow x && (55)\\
\mathbf{N}[\mathbf{N}(\widehat{K}+x+u)+\mathbf{N}(\mathbf{N}\widehat{A}+u)] &= u \quad \text{by (44)} && (56)\\
\mathbf{N}[\mathbf{N}(\widehat{K}+x+\widehat{D})+\mathbf{N}(\mathbf{N}\widehat{A}+\widehat{D})] &= \widehat{D} \quad \text{subs } u \leftarrow \widehat{D} && (57)\\
\mathbf{N}[\mathbf{N}(\widehat{D}+x+\widehat{A})+\mathbf{N}(\mathbf{N}\widehat{A}+\widehat{D})] &= \widehat{D} \quad \text{by (53)} && (58)\\
\mathbf{N}[\mathbf{N}(\widehat{D}+x+\widehat{A})+x] &= \widehat{D} \quad \text{by (46)} && (59)\\
\mathbf{N}[\mathbf{N}(\widehat{E}+x)+x] &= \widehat{D} \quad \text{by def. of } \widehat{E} && (60)\\
\mathbf{N}P &= \widehat{D} \quad \text{by def. of } P && (61)\\[1ex]
y &= \mathbf{N}[\mathbf{N}(P+y)+\mathbf{N}(\mathbf{N}P+y)] \quad \text{by (1)} && (62)\\
&= \mathbf{N}[\mathbf{N}(P+y)+\mathbf{N}(\widehat{D}+y)] \quad \text{by (61)} && (63)\\
\mathbf{N}[\mathbf{N}(P+y)+\mathbf{N}(\widehat{D}+y)] &= y && (64)\\[1ex]
\mathbf{N}[\mathbf{N}Q+x] &= \widehat{D} \quad \text{by (60)} && (65)\\
x &= \mathbf{N}[\mathbf{N}(Q+x)+\mathbf{N}(\mathbf{N}Q+x)] \quad \text{by (1)} && (66)\\
&= \mathbf{N}[\mathbf{N}(Q+x)+\widehat{D}] \quad \text{by (65)} && (67)\\
\mathbf{N}[\mathbf{N}(Q+x)+\widehat{D}] &= x && (68)\\
\mathbf{N}[\mathbf{N}R+\widehat{D}] &= x \quad \text{by def. of } R && (69)\\[1ex]
\widehat{D} &= \mathbf{N}[\mathbf{N}(R+\widehat{D})+\mathbf{N}(\mathbf{N}R+\widehat{D})] \quad \text{by (1)} && (70)\\
&= \mathbf{N}[\mathbf{N}(R+\widehat{D})+x] \quad \text{by (69)} && (71)\\
\mathbf{N}[\mathbf{N}(R+\widehat{D})+x] &= \widehat{D} && (72)\\
\mathbf{N}S &= \widehat{D} \quad \text{by def. of } S && (73)
\end{aligned}
$$

$$
\begin{aligned}
y &= \mathbf{N}[\mathbf{N}(S+y)+\mathbf{N}(\mathbf{N}S+y)] \qquad \text{by (1)} && (74)\\
&= \mathbf{N}[\mathbf{N}(S+y)+\mathbf{N}(\widehat{D}+y)] \qquad \text{by (73)} && (75)\\
\mathbf{N}[\mathbf{N}(S+y)+\mathbf{N}(\widehat{D}+y)] &= y && (76)
\end{aligned}
$$

$$
\begin{aligned}
T &= D+F+y \qquad \text{by def. of } T, F && (77)\\
&= G+y \qquad \text{by def. of } G && (78)\\
\mathbf{N}[\mathbf{N}T+D] &= y \qquad \text{by (32)} && (79)\\
\widehat{T} &= \widehat{D}+Q+x \qquad \text{by def. of } Q, \widehat{T} && (80)\\
\mathbf{N}[\mathbf{N}\widehat{T}+\widehat{D}] &= x \qquad \text{subs } y \leftarrow x && (81)\\
y &= \mathbf{N}[\mathbf{N}(P+y)+\mathbf{N}(\widehat{D}+y)] \qquad (64) && (82)\\
&= \mathbf{N}[\mathbf{N}(\mathbf{N}(\widehat{E}+x)+x+y)+\mathbf{N}(\widehat{D}+y)] \qquad \text{by def. of } P && (83)\\
&= \mathbf{N}[\mathbf{N}(y+\mathbf{N}(\widehat{E}+x)+x)+\mathbf{N}(\widehat{D}+y)] \qquad \text{by AC} && (84)\\
\mathbf{N}\widehat{T} &= \mathbf{N}[\mathbf{N}(\mathbf{N}\widehat{T}+\mathbf{N}(\widehat{E}+x)+x]+\underline{\mathbf{N}(\widehat{D}+\mathbf{N}\widehat{T})}) \qquad y \leftarrow \mathbf{N}\widehat{T} \text{ in (84)} && (85)\\
&= \mathbf{N}[\mathbf{N}(\mathbf{N}\widehat{T}+\mathbf{N}(\widehat{E}+x)+x)+x] \qquad \text{by (81)} && (86)\\
&= \mathbf{N}[\mathbf{N}(\mathbf{N}\widehat{T}+\mathbf{N}Q+x)+x] \qquad \text{by def. of } Q && (87)\\
\mathbf{N}[\mathbf{N}(\mathbf{N}\widehat{T}+\mathbf{N}Q+x)+x] &= \mathbf{N}\widehat{T} && (88)\\
\mathbf{N}[\mathbf{N}(U+\mathbf{N}Q)+x] &= \mathbf{N}\widehat{T} \qquad \text{by def. of } U && (89)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{N}[\mathbf{N}(E+y)+D] &= y \qquad \text{by (19)} && (90)\\
\mathbf{N}[\mathbf{N}(\widehat{E}+x)+\widehat{D}] &= x \qquad \text{subs } y \leftarrow x && (91)\\
\mathbf{N}[\mathbf{N}Q+\widehat{D}] &= x \qquad \text{by def. of } Q && (92)\\
\mathbf{N}[\widehat{D}+\mathbf{N}Q] &= x \qquad \text{apply (C)} && (93)\\
\widehat{T} &= \widehat{D}+Q+x \qquad \text{by (80)} && (94)\\
&= \widehat{D}+R \qquad \text{by def. of } R && (95)\\
&= R+\widehat{D} \qquad \text{apply (C)} && (96)\\
\mathbf{N}[\mathbf{N}\widehat{T}+x] &= \widehat{D} \qquad \text{apply (96) to (72)} && (97)\\
\mathbf{N}U &= \widehat{D} \qquad \text{by def. of } U && (98)\\
\mathbf{N}Q &= \mathbf{N}[\mathbf{N}(\mathbf{N}U+\mathbf{N}Q)+\mathbf{N}(U+\mathbf{N}Q)) \qquad \text{by (RX)} && (99)\\
&= \mathbf{N}[\mathbf{N}(\widehat{D}+\mathbf{N}Q)+\mathbf{N}(U+\mathbf{N}Q)] \qquad \text{apply (97)} && (100)\\
&= \mathbf{N}[x+\mathbf{N}(U+\mathbf{N}Q)] \qquad \text{apply (92)} && (101)\\
&= \mathbf{N}[\mathbf{N}(U+\mathbf{N}Q)+x] \qquad \text{by (C)} && (102)\\
&= \mathbf{N}\widehat{T} \qquad \text{apply (89)} && (103)\\
&= \mathbf{N}[\widehat{D}+Q+x] \qquad \text{by (80)} && (104)\\
\mathbf{N}[Q+\widehat{D}+x] &= \mathbf{N}Q && (105)
\end{aligned}
$$

This shows that every Robbins algebra has a pair of elements a, b such that $\mathbf{N}(a+b) = \mathbf{N}a$, thus proving (by earlier results of Winker) that the Robbins algebras are Boolean algebras.