# FAST AND SLOW DYNAMICS FOR THE COMPUTATIONAL SINGULAR PERTURBATION METHOD 

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#### Abstract

The Computational Singular Perturbation (CSP) method of Lam and Goussis is an iterative method to reduce the dimensionality of systems of ordinary differential equations with multiple time scales. In [J. Nonlin. Sci., to appear], the authors showed that each iteration of the CSP algorithm improves the approximation of the slow manifold by one order. In this paper, it is shown that the CSP method simultaneously approximates the tangent spaces to the fast fibers along which solutions relax to the slow manifold. Again, each iteration adds one order of accuracy. In some studies, the output of the CSP algorithm is postprocessed by linearly projecting initial data onto the slow manifold along these approximate tangent spaces. These projections, in turn, also become successively more accurate.


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## 1 Introduction

The Computational Singular Perturbation (CSP) method is one of several so-called reduction methods developed in chemistry to systematically decrease the size and complexity of systems of chemical kinetics equations. The method was first proposed by Lam and Goussis $[3,6,7,8,9]$ and is widely used, for example, in combustion modeling $[4,10,11,12,18,19]$.

The CSP method is generally applicable to systems of nonlinear ordinary differential equations (ODEs) with simultaneous fast and slow dynamics where the long-term dynamics evolve on a low-dimensional slow manifold in the phase space. The method is essentially an algorithm to find successive
approximations to the slow manifold and match the initial conditions to the dynamics on the slow manifold.

In a previous paper [20], we focused on the slow manifold and the accuracy of the CSP approximation for fast-slow systems of ODEs. In such systems, the ratio of the characteristic fast and slow times is made explicit by a small parameter $\varepsilon$, and the quality of the approximation can be measured in terms of $\varepsilon$. By comparing the CSP manifold with the slow manifold found in Fenichel's geometric singular perturbation theory (GSPT, [2, 5]), we showed that each application of the CSP algorithm improves the asymptotic accuracy of the CSP manifold by one order of $\varepsilon$.

In this paper, we complete the analysis of the CSP method by focusing on the fast dynamics. According to Fenichel's theory, the fast-slow systems we consider have, besides a slow manifold, a family of fast stable fibers along which initial conditions tend toward the slow manifold. The base points of these fibers lie on the slow manifold, and the dynamics near the slow manifold can be decomposed into a fast contracting component along the fast fibers and a slow component governed by the motion of the base points on the slow manifold. By comparing the CSP fibers with the tangent spaces of the fast fibers at their base points, we show that each application of the CSP algorithm also improves the asymptotic accuracy of the CSP fibers by one order of $\varepsilon$.

Summarizing the results of [20] and the present investigation, we conclude that the CSP method provides for the simultaneous approximation of the slow manifold and the tangents to the fast fibers at their base points. If one is interested only in the slow manifold, then it suffices to implement a reduced (one-step) version of the algorithm. On the other hand, if one is interested in both the slow and fast dynamics, then it is necessary to use the full (two-step) CSP algorithm. Moreover, only the full CSP algorithm allows for a linear matching of any initial data with the dynamics on the slow manifold.

This paper is organized as follows. In Section 2, we recall the relevant results from Fenichel's theory and set the framework for the CSP method. In Section 3, we outline the CSP algorithm and state the main results: Theorem 3.1 concerning the approximation of the slow manifold, which is a verbatim restatement of [20, Theorem 3.1]; and Theorem 3.2 concerning the approximation of the tangent spaces of the fast fibers. The proof of Theorem 3.2 is given in Section 4. In Section 5, we revisit the Michaelis-Menten-Henri mechanism of enzyme kinetics to illustrate the CSP method and the results of this article. Section 6 is devoted to a discussion of methods for linearly projecting initial conditions on the slow manifold.

## 2 Slow Manifolds and Fast Fibers

Consider a general system of ODEs,

$$
\begin{equation*}
\frac{d x}{d t}=g(x), \tag{2.1}
\end{equation*}
$$

for a vector-valued function $x \equiv x(t) \in \mathbf{R}^{m+n}$ in a smooth vector field $g$. For the present analysis, we assume that $n$ components of $x$ evolve on a time scale characterized by the "fast" time $t$, while the remaining $m$ components evolve on a time scale characterized by the "slow" time $\tau=\varepsilon t$, where $\varepsilon$ is a small parameter. (The explicit identification of a small parameter $\varepsilon$ is not necessary for the applicability of the CSP method; a separation of time scales is sufficient.) We collect the slow variables in $y \in \mathbf{R}^{m}$ and the fast variables in $z \in \mathbf{R}^{n}$. Thus, the system (2.1) is equivalent to either the "fast system"

$$
\begin{align*}
y^{\prime} & =\varepsilon g_{1}(y, z, \varepsilon),  \tag{2.2}\\
z^{\prime} & =g_{2}(y, z, \varepsilon) \tag{2.3}
\end{align*}
$$

or the "slow system"

$$
\begin{align*}
\dot{y} & =g_{1}(y, z, \varepsilon),  \tag{2.4}\\
\varepsilon \dot{z} & =g_{2}(y, z, \varepsilon) . \tag{2.5}
\end{align*}
$$

(A prime' denotes differentiation with respect to $t$, a dot ${ }^{\cdot}$ differentiation with respect to $\tau$.) The fast system is more appropriate for the short-term dynamics, the slow system for the long-term dynamics of the system (2.1).

In the limit as $\varepsilon$ tends to 0 , the fast system reduces formally to a single equation for the fast variable $z$,

$$
\begin{equation*}
z^{\prime}=g_{2}(y, z, 0), \tag{2.6}
\end{equation*}
$$

where $y$ is a parameter, while the slow system reduces to a differential equation for the slow variable $y$,

$$
\begin{equation*}
\dot{y}=g_{1}(y, z, 0) \tag{2.7}
\end{equation*}
$$

with the algebraic constraint $g_{2}(y, z, 0)=0$.
We assume that there exist a compact domain $K$ and a smooth function $h_{0}$ defined on $K$ such that

$$
\begin{equation*}
g_{2}\left(y, h_{0}(y), 0\right)=0, \quad y \in K . \tag{2.8}
\end{equation*}
$$

The graph of $h_{0}$ defines a critical manifold $\mathcal{M}_{0}$,

$$
\begin{equation*}
\mathcal{M}_{0}=\left\{(y, z) \in \mathbf{R}^{m+n}: z=h_{0}(y), y \in K\right\} \tag{2.9}
\end{equation*}
$$

and with each point $p=\left(y, h_{0}(y)\right) \in \mathcal{M}_{0}$ is associated a fast fiber $\mathcal{F}_{0}^{p}$,

$$
\begin{equation*}
\mathcal{F}_{0}^{p}=\left\{(y, z) \in \mathbf{R}^{m+n}: z \in \mathbf{R}^{n}\right\}, \quad p \in \mathcal{M}_{0} . \tag{2.10}
\end{equation*}
$$

The points of $\mathcal{M}_{0}$ are fixed points of Eq. (2.6). If the real parts of the eigenvalues of $D_{z} g_{2}\left(y, h_{0}(y), 0\right)$ are all negative, as we assume, then $\mathcal{M}_{0}$ is asymptotically stable, and all solutions on $\mathcal{F}_{0}^{p}$ contract exponentially toward $p$.

If $\varepsilon$ is positive but arbitrarily small, Fenichel's theory [2,5] guarantees that there exists a function $h_{\varepsilon}$ whose graph is a slow manifold $\mathcal{M}_{\varepsilon}$,

$$
\begin{equation*}
\mathcal{M}_{\varepsilon}=\left\{(y, z) \in \mathbf{R}^{m+n}: z=h_{\varepsilon}(y), y \in K\right\} \tag{2.11}
\end{equation*}
$$

This manifold is locally invariant under the system dynamics, and the dynamics on $\mathcal{M}_{\varepsilon}$ are governed by the equation

$$
\begin{equation*}
\dot{y}=g_{1}\left(y, h_{\varepsilon}(y), \varepsilon\right), \tag{2.12}
\end{equation*}
$$

as long as $y \in K$. Fenichel's theory also guarantees that there exists an invariant family $\mathcal{F}_{\varepsilon}$,

$$
\begin{equation*}
\mathcal{F}_{\varepsilon}=\bigcup_{p \in \mathcal{M}_{\varepsilon}} \mathcal{F}_{\varepsilon}^{p} \tag{2.13}
\end{equation*}
$$

of fast stable fibers $\mathcal{F}_{\varepsilon}^{p}$ along which solutions relax to $\mathcal{M}_{\varepsilon}$. The family is invariant in the sense that, if $\phi_{t}$ denotes the time- $t$ map associated with Eq. (2.1), then

$$
\begin{equation*}
\phi_{t}\left(\mathcal{F}_{\varepsilon}^{p}\right) \subset \mathcal{F}_{\varepsilon}^{\phi_{t}(p)}, \quad p \in \mathcal{M}_{\varepsilon} . \tag{2.14}
\end{equation*}
$$

The collection of fast fibers $\mathcal{F}_{\varepsilon}^{p}$ foliates a neighborhood of $\mathcal{M}_{\varepsilon}$. Hence, the motion of any point on $\mathcal{F}_{\varepsilon}^{p}$ decomposes into a fast contracting component along the fiber and a slow component governed by the motion of the base point of the fiber. Also, $\mathcal{M}_{\varepsilon}$ is $\mathcal{O}(\varepsilon)$-close to $\mathcal{M}_{0}$, with

$$
\begin{equation*}
h_{\varepsilon}(y)=h_{0}(y)+\varepsilon h_{1}(y)+\varepsilon^{2} h_{2}(y)+\cdots, \quad \varepsilon \downarrow 0, \tag{2.15}
\end{equation*}
$$

and $\mathcal{F}_{\varepsilon}^{p}$ is $\mathcal{O}(\varepsilon)$-close to $\mathcal{F}_{0}^{p}$ in any compact neighborhood of $\mathcal{M}_{\varepsilon}$.
Remark 2.1. Typically, the manifold $\mathcal{M}_{\varepsilon}$ is not unique; there is a family of slow manifolds, all having the same asymptotic expansion (2.15) to all orders in $\varepsilon$ but differing by exponentially small amounts $\left(\mathcal{O}\left(e^{-c / \varepsilon}\right), c>0\right)$.

## 3 The CSP Method

The CSP method focuses on the dynamics of the vector field $g(x)$, rather than on the dynamics of the vector $x$ itself.

Writing a single differential equation like (2.1) as a system of equations amounts to choosing a basis in the vector space. For example, in Eqs. (2.2)(2.3), the basis consists of the ordered set of unit vectors in $\mathbf{R}^{m+n}$. The coordinates of $g$ relative to this basis are $\varepsilon g_{1}$ and $g_{2}$. If we collect the basis vectors in a matrix in the usual way, then we can express the relation between $g$ and its coordinates in the form

$$
g=\left(\begin{array}{cc}
I_{m} & 0  \tag{3.1}\\
0 & I_{n}
\end{array}\right)\binom{\varepsilon g_{1}}{g_{2}}
$$

Note that the basis chosen for this representation is the same at every point of the phase space. The CSP method is based on a generalization of this idea, where the basis is allowed to vary from point to point, so it can be tailored to the local dynamics near $\mathcal{M}_{\varepsilon}$.

Suppose that we choose, instead of a fixed basis, a (point-dependent) basis $A$ for $\mathbf{R}^{m+n}$. The relation between the vector field $g$ and the vector $f$ of its coordinates relative to this basis is

$$
\begin{equation*}
g=A f \tag{3.2}
\end{equation*}
$$

Conversely,

$$
\begin{equation*}
f=B g, \tag{3.3}
\end{equation*}
$$

where $B$ is the left inverse of $A, B A=I$ on $\mathbf{R}^{m+n}$. In the convention of the CSP method, $A$ is a matrix of column vectors (vectors in $\mathbf{R}^{m+n}$ ) and $B$ a matrix of row vectors (functionals on $\mathbf{R}^{m+n}$ ).

The CSP method focuses on the dynamics of the vector $f$. Along a trajectory of the system (2.1), $f$ satisfies the ODE

$$
\begin{equation*}
\frac{d f}{d t}=\Lambda f \tag{3.4}
\end{equation*}
$$

where $\Lambda$ is a linear operator $[13,20]$,

$$
\begin{equation*}
\Lambda=B(D g) A+\frac{d B}{d t} A=B(D g) A-B \frac{d A}{d t}=B[A, g] \tag{3.5}
\end{equation*}
$$

Here, $D g$ is the Jacobian of $g, d B / d t=(D B) g, d A / d t=(D A) g$, and $[A, g]$ is the Lie bracket of $A$ (taken column by column) and $g$. The Lie bracket of any two vectors $a$ and $g$ is $[a, g]=(D g) a-(D a) g$; see [14].

It is clear from Eq. (3.4) that the dynamics of $f$ are governed by $\Lambda$, so the CSP method focuses on the structure of $\Lambda$.

Remark 3.1. It is useful to see how $\Lambda$ transforms under a change of basis. If $C$ is an invertible square matrix representing a coordinate transformation in $\mathbf{R}^{m+n}$, and $\hat{A}=A C$ and $\hat{B}=C^{-1} B$, then

$$
\begin{align*}
\hat{\Lambda} & =\hat{B}(D g) \hat{A}-\hat{B} \frac{d \hat{A}}{d t}=C^{-1} B(D g) A C-C^{-1} B \frac{d(A C)}{d t} \\
& =C^{-1} B(D g) A C-C^{-1} B\left(\frac{d A}{d t} C+A \frac{d C}{d t}\right) \\
& =C^{-1} \Lambda C-C^{-1} \frac{d C}{d t} . \tag{3.6}
\end{align*}
$$

Hence, $\Lambda$ does not transform as a matrix, unless $C$ is constant.

### 3.1 Decompositions

Our goal is to decompose the vector $f$ into its fast and slow components. Suppose, therefore, that we have a decomposition of this type, $f=\binom{f^{1}}{f^{2}}$, where $f^{1}$ and $f^{2}$ are of length $n$ and $m$, respectively, but not necessarily fast and slow everywhere. The decomposition suggests corresponding decompositions of the matrices $A$ and $B$, namely $A=\left(A_{1}, A_{2}\right)$ and $B=\binom{B^{1}}{B^{2}}$, where $A_{1}$ is an $(m+n) \times n$ matrix, $A_{2}$ an $(m+n) \times m$ matrix, $B^{1}$ an $n \times(m+n)$ matrix, and $B^{2}$ an $m \times(m+n)$ matrix. Then, $f^{1}=B^{1} g$ and $f^{2}=B^{2} g$.

The decompositions of $A$ and $B$ lead, in turn, to a decomposition of $\Lambda$,

$$
\Lambda=\left(\begin{array}{ll}
\Lambda^{11} & \Lambda^{12}  \tag{3.7}\\
\Lambda^{21} & \Lambda^{22}
\end{array}\right)=\left(\begin{array}{ll}
B^{1}\left[A_{1}, g\right] & B^{1}\left[A_{2}, g\right] \\
B^{2}\left[A_{1}, g\right] & B^{2}\left[A_{2}, g\right]
\end{array}\right) .
$$

The off-diagonal blocks $\Lambda^{12}$ and $\Lambda^{21}$ are, in general, not zero, so the equations governing the evolution of the coordinates $f^{1}$ and $f^{2}$ are coupled. Consequently, $f^{1}$ and $f^{2}$ cannot be identified with the fast and slow coordinates of $g$ globally along trajectories. The objective of the CSP method is to construct local coordinate systems (that is, matrices $A$ and $B$ ) that lead to a blockdiagonal structure of $\Lambda$. We will see, in the next section, that such a structure is associated with a decomposition in terms of the slow manifold and the fast fibers.

Remark 3.2. Note that the identity $B A=I$ on $\mathbf{R}^{m+n}$ implies four identities, which are summarized in the matrix identity

$$
\left(\begin{array}{cc}
B^{1} A_{1} & B^{1} A_{2}  \tag{3.8}\\
B^{2} A_{1} & B^{2} A_{2}
\end{array}\right)=\left(\begin{array}{cc}
I_{n} & 0 \\
0 & I_{m}
\end{array}\right) .
$$

### 3.2 Block-Diagonalization of $\Lambda$

In this section we analyze the properties of $\Lambda$ relative to a fast-slow decomposition of the dynamics near $\mathcal{M}_{\varepsilon}$.

Let $\mathcal{T}_{p} \mathcal{F}_{\varepsilon}$ and $\mathcal{T}_{p} \mathcal{M}_{\varepsilon}$ denote the tangent spaces to the fast fiber and the slow manifold, respectively, at the base point $p$ of the fiber on $\mathcal{M}_{\varepsilon}$. (Note that $\operatorname{dim} \mathcal{T}_{p} \mathcal{F}_{\varepsilon}=n$ and $\operatorname{dim} \mathcal{T}_{p} \mathcal{M}_{\varepsilon}=m$.) These two linear spaces intersect transversally, because $\mathcal{M}_{\varepsilon}$ is normally hyperbolic and compact, so

$$
\begin{equation*}
\mathbf{R}^{m+n}=\mathcal{T}_{p} \mathcal{F}_{\varepsilon} \oplus \mathcal{T}_{p} \mathcal{M}_{\varepsilon}, \quad p \in \mathcal{M}_{\varepsilon} \tag{3.9}
\end{equation*}
$$

Let $A_{f}$ be an $(m+n) \times n$ matrix whose columns form a basis for $\mathcal{T}_{p} \mathcal{F}_{\varepsilon}$ and $A_{s}$ an $(m+n) \times m$ matrix whose columns form a basis for $\mathcal{T}_{p} \mathcal{M}_{\varepsilon}$, and let $A=\left(A_{f}, A_{s}\right)$. (We omit the subscript p.) Then $A$ is a (point-dependent) basis for $\mathbf{R}^{m+n}$ that respects the decomposition (3.9). We recall that $\mathcal{T} \mathcal{M}_{\varepsilon} \equiv \bigcup_{p \in \mathcal{M}_{\varepsilon}}\left(p, \mathcal{T}_{p} \mathcal{M}_{\varepsilon}\right)$ and $\mathcal{T} \mathcal{F}_{\varepsilon} \equiv \bigcup_{p \in \mathcal{M}_{\varepsilon}}\left(p, \mathcal{T}_{p} \mathcal{F}_{\varepsilon}\right)$ are the tangent bundles of the slow manifold and the family of the fast fibers, respectively. (A general treatment of tangent bundles of manifolds is given in [1, Section 1.7].)

The decomposition (3.9) induces a dual decomposition,

$$
\begin{equation*}
\mathbf{R}^{m+n}=\mathcal{N}_{p} \mathcal{M}_{\varepsilon} \oplus \mathcal{N}_{p} \mathcal{F}_{\varepsilon}, \quad p \in \mathcal{M}_{\varepsilon} \tag{3.10}
\end{equation*}
$$

where $\mathcal{N}_{p} \mathcal{M}_{\varepsilon}$ and $\mathcal{N}_{p} \mathcal{F}_{\varepsilon}$ are the duals of $\mathcal{T}_{p} \mathcal{M}_{\varepsilon}$ and $\mathcal{T}_{p} \mathcal{F}_{\varepsilon}$, respectively, in $\mathbf{R}^{m+n}$. (Note that $\operatorname{dim} \mathcal{N}_{p} \mathcal{M}_{\varepsilon}=n$ and $\operatorname{dim} \mathcal{N}_{p} \mathcal{F}_{\varepsilon}=m$.) The corresponding decomposition of $B$ is $B=\binom{B^{s \perp}}{B^{f \perp}}$, where the rows of $B^{s \perp}$ form a basis for $\mathcal{N}_{p} \mathcal{M}_{\varepsilon}$ and the rows of $B^{f \perp}$ a basis for $\mathcal{N}_{p} \mathcal{F}_{\varepsilon}$. Furthermore,

$$
\left(\begin{array}{cc}
B^{s \perp} A_{f} & B^{s \perp} A_{s}  \tag{3.11}\\
B^{f \perp} A_{f} & B^{f \perp} A_{s}
\end{array}\right)=\left(\begin{array}{cc}
I_{n} & 0 \\
0 & I_{m}
\end{array}\right)
$$

The decompositions of $A$ and $B$ lead, in turn, to a decomposition of $\Lambda$,

$$
\Lambda=\left(\begin{array}{cc}
B^{s \perp}\left[A_{f}, g\right] & B^{s \perp}\left[A_{s}, g\right]  \tag{3.12}\\
B^{f \perp}\left[A_{f}, g\right] & B^{f \perp}\left[A_{s}, g\right]
\end{array}\right)
$$

This decomposition is similar to, but different from, the decomposition (3.7). The following lemma shows that its off-diagonal blocks are zero.

Lemma 3.1 The off-diagonal blocks in the representation (3.12) of $\Lambda$ are zero at each point $p \in \mathcal{M}_{\varepsilon}$.

Proof. Since $B^{s \perp} A_{s}=0$ on $\mathcal{M}_{\varepsilon}$ and $\mathcal{M}_{\varepsilon}$ is invariant, we have

$$
\begin{equation*}
\frac{d}{d t}\left(B^{s \perp} A_{s}\right)=D\left(B^{s \perp} A_{s}\right) g=\left(D B^{s \perp}\right)\left(g, A_{s}\right)+B^{s \perp}\left(\left(D A_{s}\right) g\right)=0 . \tag{3.13}
\end{equation*}
$$

( $D B^{s \perp}$ is a symmetric bilinear form; its action on a matrix must be understood as column-wise action.)

Also, $g \in \mathcal{T} \mathcal{M}_{\varepsilon}$, so $B^{s \perp} g=0$ on $\mathcal{M}_{\varepsilon}$. Hence, the directional derivative along $A_{s}$ (taken column by column) at points on $\mathcal{M}_{\varepsilon}$ also vanishes,

$$
\begin{equation*}
D\left(B^{s \perp} g\right) A_{s}=\left(D B^{s \perp}\right)\left(A_{s}, g\right)+B^{s \perp}(D g) A_{s}=0 . \tag{3.14}
\end{equation*}
$$

Subtracting Eq. (3.13) from Eq. (3.14), we obtain the identity

$$
\begin{equation*}
B^{s \perp}\left[A_{s}, g\right]=B^{s \perp}\left((D g) A_{s}-\left(D A_{s}\right) g\right)=0 . \tag{3.15}
\end{equation*}
$$

The proof for the lower left block is more involved, since the fast fibers are invariant as a family. Assume that the fiber $\mathcal{F}_{\varepsilon}^{p}$ at $p \in \mathcal{M}_{\varepsilon}$ is given implicitly by the equation $F(q ; p)=0, q \in \mathcal{F}_{\varepsilon}^{p}$. Then the rows of $\left(D_{q} F\right)(q ; p)$ form a basis for $\mathcal{N}_{q} \mathcal{F}_{\varepsilon}$, so there exists an invertible matrix $C$ such that $B^{f \perp}=C\left(D_{q} F\right)$.

Since the rows of $\left(D_{q} F\right)(q ; p)$ span $\mathcal{N}_{q} \mathcal{F}_{\varepsilon}$, we have $\left(D_{q} F\right)(q ; p) A_{f}(q)=0$. This identity holds, in particular, along solutions of (2.1), so

$$
\begin{align*}
\frac{d}{d t}\left(\left(D_{q} F\right)(q ; p) A_{f}(q)\right)= & \left(\left(D_{q}^{2} F\right)(q ; p)\right)\left(g(q), A_{f}(q)\right) \\
& +\left(\left(D_{p q} F\right)(q ; p)\right)\left(g(p), A_{f}(q)\right) \\
& +\left(\left(D_{q} F\right)(q ; p)\right)\left(D A_{f}(q)\right) g(q) \\
= & 0 . \tag{3.16}
\end{align*}
$$

The family of the fast fibers is invariant under the flow associated with (2.1), so if $F(q ; p)=0$, then also $F(q(t) ; p(t))=0$ and, hence,

$$
\begin{equation*}
\frac{d F(q ; p)}{d t}=\left(\left(D_{q} F\right)(q ; p)\right) g(q)+\left(\left(D_{p} F\right)(q ; p)\right) g(p)=0 . \tag{3.17}
\end{equation*}
$$

Next, we take the directional derivative of both members of this equation along $A_{f}$, keeping in mind that $(D g)(p) A_{f}(q)=0$ because the base point $p$ does not vary along $A_{f}$. (Recall that the columns of $A_{f}(q) \operatorname{span} \mathcal{T}_{q} \mathcal{F}_{\varepsilon}$.) We find

$$
\begin{align*}
& \left(\left(D_{q}^{2} F\right)(q ; p)\right)\left(A_{f}(q), g(q)\right)+\left(\left(D_{q} F\right)(q ; p)\right)(D g(q)) A_{f}(q) \\
& +\left(\left(D_{p q} F\right)(q ; p)\right)\left(A_{f}(q), g(p)\right)=0 . \tag{3.18}
\end{align*}
$$

But the bilinear forms $D_{q}^{2} F$ and $D_{p q} F$ are symmetric, so subtracting Eq. (3.16) from Eq. (3.18) and letting $q=p$, we obtain the identity

$$
\begin{equation*}
\left(D_{q} F\right)(p ; p)\left((D g) A_{f}-\left(D A_{f}\right) g\right)(p)=0 . \tag{3.19}
\end{equation*}
$$

Hence, $B^{f \perp}\left[A_{f}, g\right](p)=C\left(D_{q} F\right)(p ; p)\left[A_{f}, g\right](p)=0$, and the proof of the lemma is complete.

The lemma implies that the representation (3.12) is block-diagonal,

$$
\Lambda=\left(\begin{array}{cc}
B^{s \perp}\left[A_{f}, g\right] & 0  \tag{3.20}\\
0 & B^{f \perp}\left[A_{s}, g\right]
\end{array}\right)
$$

Consequently, the decomposition (3.9) reduces $\Lambda$. In summary, if we can construct bases $A_{f}$ and $A_{s}$, then we will have achieved a representation of $\Lambda$ where the fast and slow components remain separated at all times and the designation of fast and slow takes on a global meaning.

### 3.3 The CSP Algorithm

The CSP method is a constructive algorithm to approximate $A_{f}$ and $A_{s}$. One typically initializes the algorithm with a constant matrix $A^{(0)}$,

$$
A^{(0)}=\left(A_{1}^{(0)}, A_{2}^{(0)}\right)=\left(\begin{array}{ll}
A_{11}^{(0)} & A_{12}^{(0)}  \tag{3.21}\\
A_{21}^{(0)} & A_{22}^{(0)}
\end{array}\right)
$$

Here, $A_{11}^{(0)}$ is an $m \times n$ matrix, $A_{22}^{(0)}$ an $n \times m$ matrix, and the off-diagonal blocks $A_{12}^{(0)}$ and $A_{21}^{(0)}$ are full-rank square matrices of order $m$ and $n$, respectively. A common choice is $A_{11}^{(0)}=0$. We follow this convention and assume, henceforth, that $A_{11}^{(0)}=0$,

$$
A^{(0)}=\left(A_{1}^{(0)}, A_{2}^{(0)}\right)=\left(\begin{array}{cc}
0 & A_{12}^{(0)}  \tag{3.22}\\
A_{21}^{(0)} & A_{22}^{(0)}
\end{array}\right) .
$$

(Other choices are discussed in [20].) The left inverse of $A^{(0)}$ is

$$
\begin{align*}
B_{(0)} & =\binom{B_{(0)}^{1}}{B_{(0)}^{2}}=\left(\begin{array}{cc}
B_{(0)}^{11} & B_{(0)}^{12} \\
B_{(0)}^{21} & 0
\end{array}\right) \\
& =\left(\begin{array}{cc}
-\left(A_{21}^{(0)}\right)^{-1} A_{22}^{(0)}\left(A_{12}^{(0)}\right)^{-1} & \left(A_{21}^{(0)}\right)^{-1} \\
\left(A_{12}^{(0)}\right)^{-1} & 0
\end{array}\right) \tag{3.23}
\end{align*}
$$

The algorithm proceeds iteratively. For $q=0,1, \ldots$, one first defines the operator $\Lambda_{(q)}$ in accordance with Eq. (3.5),

$$
\Lambda_{(q)}=B_{(q)}(D g) A^{(q)}-B_{(q)} \frac{d A^{(q)}}{d t}=\left(\begin{array}{cc}
\Lambda_{(q)}^{11} & \Lambda_{(q)}^{12}  \tag{3.24}\\
\Lambda_{(q)}^{21} & \Lambda_{(q)}^{22}
\end{array}\right)
$$

and matrices $U_{(q)}$ and $L_{(q)}$,

$$
U_{(q)}=\left(\begin{array}{cc}
0 & \left(\Lambda_{(q)}^{11}\right)^{-1} \Lambda_{(q)}^{12}  \tag{3.25}\\
0 & 0
\end{array}\right), \quad L_{(q)}=\left(\begin{array}{cc}
0 & 0 \\
\Lambda_{(q)}^{21}\left(\Lambda_{(q)}^{11}\right)^{-1} & 0
\end{array}\right) .
$$

Then one updates $A^{(q)}$ and $B_{(q)}$ according to the formulas

$$
\begin{align*}
& A^{(q+1)}=A^{(q)}\left(I-U_{(q)}\right)\left(I+L_{(q)}\right),  \tag{3.26}\\
& B_{(q+1)}=\left(I-L_{(q)}\right)\left(I+U_{(q)}\right) B_{(q)}, \tag{3.27}
\end{align*}
$$

and returns to Eq. (3.24) for the next iteration.
Remark 3.3. Lam and Goussis [6] perform the update (3.26)-(3.27) in two steps. The first step corresponds to the postmultiplication of $A^{(q)}$ with $I-U_{(q)}$ and premultiplication of $B_{(q)}$ with $I+U_{(q)}$, the second step to the subsequent postmultiplication of $A^{(q)}\left(I-U_{(q)}\right)$ with $I+L_{(q)}$ and premultiplication of ( $I+$ $\left.U_{(q)}\right) B_{(q)}$ with $I-L_{(q)}$.

### 3.4 Approximation of the Slow Manifold

After $q$ iterations, the CSP condition

$$
\begin{equation*}
B_{(q)}^{1} g=0, \quad q=0,1, \ldots, \tag{3.28}
\end{equation*}
$$

identifies those points where the fast amplitudes vanish with respect to the then current basis. These points define a manifold that is an approximation for the slow manifold $\mathcal{M}_{\varepsilon}$.

For $q=0, B_{(0)}^{1}$ is constant and given by Eq. (3.23). Hence, the CSP condition (3.28) reduces to the constraint $g_{2}(y, z, \varepsilon)=0$. In general, this constraint is satisfied by a function $z=\psi_{(0)}(y, \varepsilon)$. The graph of this function defines $\mathcal{K}_{\varepsilon}^{(0)}$, the CSP manifold (CSPM) of order zero. Since the constraint reduces at leading order to the equation $g_{2}(y, z, 0)=0$, which is satisfied by the function $z=h_{0}(y), \mathcal{K}_{\varepsilon}^{(0)}$ may be chosen to coincide with $\mathcal{M}_{0}$ to leading order; see Eq. (2.9).

For $q=1,2, \ldots$, the CSP condition takes the form

$$
\begin{equation*}
B_{(q)}^{1}\left(y, \psi_{(q-1)}(y, \varepsilon), \varepsilon\right) g(y, z, \varepsilon)=0, \quad q=1,2, \ldots \tag{3.29}
\end{equation*}
$$

The condition is satisfied by a function $z=\psi_{(q)}(y, \varepsilon)$, and the manifold

$$
\begin{equation*}
\mathcal{K}_{\varepsilon}^{(q)}=\left\{(y, z): z=\psi_{(q)}(y, \varepsilon), y \in K\right\}, \quad q=0,1, \ldots \tag{3.30}
\end{equation*}
$$

defines the CSP manifold (CSPM) of order $q$, which is an approximation of $\mathcal{M}_{\varepsilon}$. The following theorem regarding the quality of the approximation was proven in [20].

Theorem 3.1 [20, Theorem 3.1] The asymptotic expansions of the CSP manifold $\mathcal{K}_{\varepsilon}^{(q)}$ and the slow manifold $\mathcal{M}_{\varepsilon}$ agree up to and including terms of $\mathcal{O}\left(\varepsilon^{q}\right)$,

$$
\begin{equation*}
\psi_{(q)}(\cdot, \varepsilon)=\sum_{j=0}^{q} \varepsilon^{j} h_{j}+\mathcal{O}\left(\varepsilon^{q+1}\right), \quad \varepsilon \downarrow 0, \quad q=0,1, \ldots \tag{3.31}
\end{equation*}
$$

### 3.5 Approximation of the Fast Fibers

We now turn our attention to the fast fibers. The columns of $A_{f}\left(y, h_{\varepsilon}(y)\right)$ span the tangent space to the fast fiber with base point $p=\left(y, h_{\varepsilon}(y)\right)$, so we expect that $A_{1}^{(q)}$ defines an approximation for the same space after $q$ applications of the CSP algorithm. We denote this approximation by $\mathcal{L}_{\varepsilon}^{(q)}(y)$ and refer to it as the CSP fiber (CSPF) of order $q$ at $p$,

$$
\begin{equation*}
\mathcal{L}_{\varepsilon}^{(q)}(y)=\operatorname{span}\left(\operatorname{cols}\left(A_{1}^{(q)}\left(y, \psi_{(q)}(y, \varepsilon), \varepsilon\right)\right)\right) \tag{3.32}
\end{equation*}
$$

We will shortly estimate the asymptotic accuracy of the approximation, but before doing so we need to make an important observation.

Each application of the CSP algorithm involves two steps, see Remark 3.3. The first step involves $U$ and serves to push the order of magnitude of the upper right block of $\Lambda$ up by one, the second step involves $L$ and serves the same purpose for the lower left block. The two steps are consecutive. At the first step of the $q$ th iteration, one evaluates $B_{(q)}^{1}$ on $\mathcal{K}_{\varepsilon}^{(q-1)}$ to find $\mathcal{K}_{\varepsilon}^{(q)}$ by solving the CSP condition (3.28) for the function $\psi_{(q)}$. One then uses this expression in the second step to update $A$ and $B$, thus effectively evaluating $A_{1}^{(q)}$ on $\mathcal{K}_{\varepsilon}^{(q)}$ rather than on $\mathcal{K}_{\varepsilon}^{(q-1)}$.

The following theorem contains our main result.

Theorem 3.2 The asymptotic expansions of $\mathcal{L}_{\varepsilon}^{(q)}(y)$ and $\mathcal{T}_{p} \mathcal{F}_{\varepsilon}$, where $p=$ $\left(y, h_{\varepsilon}(y)\right) \in \mathcal{M}_{\varepsilon}$, agree up to and including terms of $\mathcal{O}\left(\varepsilon^{q}\right)$, for all $y \in K$ and for $q=0,1, \ldots$.

Theorem 3.2 implies that the family $\mathcal{L}_{\varepsilon}^{(q)} \equiv \bigcup_{p \in \mathcal{M}_{\varepsilon}}\left(p, \mathcal{L}_{\varepsilon}^{(q)}(y)\right)$ is an $\mathcal{O}\left(\varepsilon^{q}\right)$ approximation to the tangent bundle $\mathcal{T} \mathcal{F}_{\varepsilon}$.

The proof of Theorem 3.2 is given in Section 4. The essential idea is to show that, at each iteration, the asymptotic order of the off-diagonal blocks of $\Lambda_{(q)}$ increases by one and $A_{1}^{(q)}$ and $B_{(q)}^{2}$ become fast and fast ${ }^{\perp}$, respectively, to one higher order. As a consequence, in the limit as $q \rightarrow \infty, \Lambda_{(q)} \rightarrow \Lambda$, $A^{(q)} \rightarrow A$, and $B_{(q)} \rightarrow B$, where $\Lambda, A$, and $B$ are ideal in the sense described in Section 3.2.

Remark 3.4. If, in the second step of the CSP algorithm, $A_{1}^{(q)}$ were evaluated on $\mathcal{K}_{\varepsilon}^{(q-1)}$ instead of on $\mathcal{K}_{\varepsilon}^{(q)}$, the approximation of $\mathcal{T} \mathcal{F}_{\varepsilon}$ might be only $\mathcal{O}\left(\varepsilon^{q-1}\right)$ accurate. However, see Section 5 for an example where the approximation is still $\mathcal{O}\left(\varepsilon^{q}\right)$.

## 4 Proof of Theorem 3.2

The proof of Theorem 3.2 is by induction on $q$. Section 4.1 contains an auxiliary lemma that shows that each successive application of the CSP algorithm pushes $\Lambda$ closer to block-diagonal form. The induction hypothesis is formulated in Section 4.2, the hypothesis is shown to be true for $q=0$ in Section 4.3, and the induction step is taken in Section 4.4.

### 4.1 Asymptotic Estimates of $\Lambda$

As stated in Section 3, the goal of the CSP method is to reduce $\Lambda$ to blockdiagonal form. This goal is approached by the repeated application of a twostep algorithm. As shown in [20], the first step of the algorithm is engineered so that each application increases the asymptotic accuracy of the upper-right block $\Lambda_{(q)}^{12}$ by one order of $\varepsilon$; in particular, $\Lambda_{(q)}^{12}=\mathcal{O}\left(\varepsilon^{q}\right)$ on $\mathcal{K}_{\varepsilon}^{(q)}$ [20, Eq. (5.25)]. We now complete the picture and show that each application of the second step increases the asymptotic accuracy of the lower-left block $\Lambda_{(q)}^{21}$ by one order of $\varepsilon$, when the information obtained in the first step of the same iteration is used. In particular, $\Lambda_{(q)}^{21}=\mathcal{O}\left(\varepsilon^{q+1}\right)$ on $\mathcal{K}_{\varepsilon}^{(q+1)}$, where $\mathcal{K}_{\varepsilon}^{(q+1)}$ has been obtained in the first step of the $(q+1)$ th refinement.

Lemma 4.1 For $q=0,1, \ldots$,

$$
\Lambda_{(q)}=\left(\begin{array}{cc}
\Lambda_{(0,0)}^{11}+\mathcal{O}(\varepsilon) & \varepsilon^{q} \Lambda_{(q, q)}^{12}  \tag{4.1}\\
\varepsilon^{q+1} \Lambda_{(q, q+1)}^{21} & \varepsilon \Lambda_{(1,1)}^{22}+\mathcal{O}\left(\varepsilon^{2}\right)
\end{array}\right),
$$

when $\Lambda_{(q)}$ is evaluated on $\mathcal{K}_{\varepsilon}^{(q+1)}$.

Proof. The proof is by induction. The desired estimates of $\Lambda_{(q)}^{11}, \Lambda_{(q)}^{12}$, and $\Lambda_{(q)}^{22}$ on $\mathcal{K}_{\varepsilon}^{(q)}$ were established in [20, Eqs. (5.24), (5.25), (5.27)]. Since the asymptotic expansions of $\mathcal{K}_{\varepsilon}^{(q+1)}$ and $\mathcal{K}_{\varepsilon}^{(q)}$ differ only at terms of $\mathcal{O}\left(\varepsilon^{q+1}\right)$ or higher $\left(\left[20\right.\right.$, Theorem 3.1]), these estimates of $\Lambda_{(q)}^{11}, \Lambda_{(q)}^{12}$, and $\Lambda_{(q)}^{22}$ are true also on $\mathcal{K}_{\varepsilon}^{(q+1)}$. It only remains to estimate $\Lambda_{(q)}^{21}$.

Consider the case $q=0$. Let $\Lambda_{(0, j)}^{21}$ be the coefficient of $\varepsilon^{j}$ in the asymptotic expansion of $\Lambda_{(0)}^{21}\left(y, \psi_{(1)}(y), \varepsilon\right)$. The estimate $\Lambda_{(0)}^{21}=\mathcal{O}(\varepsilon)$ on $\mathcal{K}_{\varepsilon}^{(1)}$ follows if we can show that $\Lambda_{(0,0)}^{21}=0$. It is already stated in [20, Eq. (4.30)] that $\Lambda_{(0,0)}^{21}=0$ on $\mathcal{K}_{\varepsilon}^{(0)}$. Furthermore, [20, Theorem 3.1] implies that the asymptotic expansions of $\psi_{(1)}$ and $\psi_{(0)}$ agree to leading order. Thus, the asymptotic expansions of $\Lambda_{(0)}^{21}\left(y, \psi_{(0)}(y), \varepsilon\right)$ and $\Lambda_{(0)}^{21}\left(y, \psi_{(1)}(y), \varepsilon\right)$ also agree to leading order, and the result follows.

Now, assume that the asymptotic estimate holds for $0,1, \ldots, q$. From Eq. (3.6) we obtain

$$
\begin{align*}
\Lambda_{(q+1)}^{21}= & \Lambda_{(q)}^{21}-L_{(q)} \Lambda_{(q)}^{11}+\Lambda_{(q)}^{22} L_{(q)}-L_{(q)} \Lambda_{(q)}^{12} L_{(q)}-\Lambda_{(q)}^{21} U_{(q)} L_{(q)} \\
& -L_{(q)} U_{(q)} \Lambda_{(q)}^{21}+L_{(q)} \Lambda_{(q)}^{11} U_{(q)} L_{(q)}-L_{(q)} U_{(q)} \Lambda_{(q)}^{22} L_{(q)} \\
& +L_{(q)} U_{(q)} \Lambda_{(q)}^{21} U_{(q)} L_{(q)}+\left(D L_{(q)}\right) g+L_{(q)}\left(\left(D U_{(q)}\right) g\right) L_{(q)} . \tag{4.2}
\end{align*}
$$

The first two terms in the right member sum to zero, by virtue of the definition (3.25) of $L_{(q)}$. The next seven terms are all $\mathcal{O}\left(\varepsilon^{q+2}\right)$ or higher, by virtue of the induction hypothesis. Finally, the last two terms are also $\mathcal{O}\left(\varepsilon^{q+2}\right)$ or higher, by the induction hypothesis and [20, Lemma A.2].

### 4.2 The Induction Hypothesis

The CSPF of order $q, \mathcal{L}_{\varepsilon}^{(q)}(y)$, is defined in Eq. (3.32) to be the linear space spanned by the columns of the fast component, $A_{1}^{(q)}\left(y, \psi_{(q)}, \varepsilon\right)$, of the basis $A^{(q)}$. Thus, to prove Theorem 3.2, it suffices to show that the asymptotic expansions of $A_{1}^{(q)}\left(y, \psi_{(q)}, \varepsilon\right)$ and the space tangent to the fast fiber, $\mathcal{T}_{p} \mathcal{F}_{\varepsilon}$, agree up to and including terms of $\mathcal{O}\left(\varepsilon^{q}\right)$, for $p=\left(y, h_{\varepsilon}(y)\right)$ and for $q=0,1, \ldots$. The central idea of the proof is to show that each successive application of the CSP method pushes the projection of $A_{1}^{(q)}$ on $\mathcal{T} \mathcal{M}_{\varepsilon}$ along $\mathcal{T} \mathcal{F}_{\varepsilon}$ to one higher order in $\varepsilon$.

We express $A^{(q)}$, generated after $q$ applications of the CSP algorithm, in terms of the basis $A$,

$$
\begin{equation*}
A^{(q)}(y, z, \varepsilon)=A\left(y, h_{\varepsilon}, \varepsilon\right) Q^{(q)}(y, z, \varepsilon), \quad q=0,1, \ldots \tag{4.3}
\end{equation*}
$$

Since $B_{(q)}$ and $B$ are the left inverses of $A^{(q)}$ and $A$, respectively, we also have

$$
\begin{equation*}
B_{(q)}(y, z, \varepsilon)=R_{(q)}(y, z, \varepsilon) B\left(y, h_{\varepsilon}, \varepsilon\right), \quad q=0,1, \ldots, \tag{4.4}
\end{equation*}
$$

where $R_{(q)} \equiv\left(Q^{(q)}\right)^{-1}$. Introducing the block structure of $Q^{(q)}$ and $R_{(q)}$,

$$
Q^{(q)}=\left(\begin{array}{cc}
Q_{1 f}^{(q)} & Q_{2 f}^{(q)}  \tag{4.5}\\
Q_{1 s}^{(q)} & Q_{2 s}^{(q)}
\end{array}\right), \quad R_{(q)}=\left(\begin{array}{cc}
R_{(q)}^{1 s \perp} & R_{(q)}^{1 f \perp} \\
R_{(q)}^{2 s \perp} & R_{(q)}^{2 f \perp}
\end{array}\right),
$$

we rewrite Eqs. (4.3) and (4.4) as

$$
\begin{equation*}
A_{1}^{(q)}=A_{f} Q_{1 f}^{(q)}+A_{s} Q_{1 s}^{(q)}, \quad A_{2}^{(q)}=A_{f} Q_{2 f}^{(q)}+A_{s} Q_{2 s}^{(q)} \tag{4.6}
\end{equation*}
$$

and

$$
\begin{equation*}
B_{(q)}^{1}=R_{(q)}^{1 s \perp} B^{s \perp}+R_{(q)}^{1 f \perp} B^{f \perp}, \quad B_{(q)}^{2}=R_{(q)}^{2 s \perp} B^{s \perp}+R_{(q)}^{2 f \perp} B^{f \perp} \tag{4.7}
\end{equation*}
$$

for $q=0,1, \ldots$.
Equation (4.7) shows that $A_{s} Q_{1 s}^{(q)}$ is the projection of $A_{1}^{(q)}$ on $\mathcal{T} \mathcal{M}_{\varepsilon}$. Thus, to establish Theorem 3.2, we only need to prove the asymptotic estimate $Q_{1 s}^{(q)}=$ $\mathcal{O}\left(\varepsilon^{q+1}\right)$. The proof is by induction on $q$, where the induction hypothesis is

$$
\begin{align*}
Q^{(q)}\left(\cdot, \psi_{(q)}, \varepsilon\right) & =\left(\begin{array}{cc}
\mathcal{O}(1) & \mathcal{O}\left(\varepsilon^{q}\right) \\
\mathcal{O}\left(\varepsilon^{q+1}\right) & \mathcal{O}(1)
\end{array}\right),  \tag{4.8}\\
R_{(q)}\left(\cdot, \psi_{(q)}, \varepsilon\right) & =\left(\begin{array}{cc}
\mathcal{O}(1) & \mathcal{O}\left(\varepsilon^{q}\right) \\
\mathcal{O}\left(\varepsilon^{q+1}\right) & \mathcal{O}(1)
\end{array}\right), \quad q=0,1, \ldots \tag{4.9}
\end{align*}
$$

Remark 4.1. Although the estimate of $Q_{1 s}^{(q)}$ is sufficient to establish Theorem 3.2, we provide the estimates of all the blocks in Eqs. (4.8)-(4.9) because they will be required in the induction step.

The validity of Eqs. (4.8)-(4.9) for $q=0$ is shown in Section 4.3. The induction step is carried out in Section 4.4.

### 4.3 Proof of Theorem 3.2 for $q=0$

We fix $q=0$ and verify the induction hypothesis for $Q^{(0)}$ and $R_{(0)}$. By Eq. (4.3)

$$
\begin{equation*}
Q^{(0)}=B A^{(0)}, \tag{4.10}
\end{equation*}
$$

whence

$$
Q^{(0)}=\left(\begin{array}{cc}
B^{s \perp} A_{1}^{(0)} & B^{s \perp} A_{2}^{(0)}  \tag{4.11}\\
B^{f \perp} A_{1}^{(0)} & B^{f \perp} A_{2}^{(0)}
\end{array}\right)
$$

It suffices to show that the lower-left block is zero to leading order, since the other blocks are all $\mathcal{O}(1)$. We do this by showing that $Q_{1 s}^{(0,0)}=0$. By Eq. (4.11),

$$
\begin{equation*}
Q_{1 s}^{(0,0)}=B_{0}^{f \perp} A_{1}^{(0,0)} . \tag{4.12}
\end{equation*}
$$

$B_{0}^{f \perp}$ spans $\mathcal{N}_{p} \mathcal{F}_{0}$ for every $p \in \mathcal{K}_{\varepsilon}^{(0)}$. Also, $z$ is constant on $\mathcal{N}_{p} \mathcal{F}_{0}$, so $B_{0}^{f \perp}=$ $\left(B^{1 f \perp}, 0\right)$, where $B^{1 f \perp}$ is a full-rank matrix of size $m$. Last, $A_{1}^{(0,0)}=A_{1}^{(0)}=$ $\binom{0}{A_{21}^{(0)}}$, by Eq. (3.22). Substituting these expressions for $B_{0}^{f \perp}$ and $A_{1}^{(0,0)}$ into Eq. (4.12), we obtain that $Q_{1 s}^{(0,0)}=0$.

The induction hypothesis on $R_{(0)}$ can be verified either by a similar argument, or by recalling that $R_{(0)}=\left(Q^{(0)}\right)^{-1}$, where $Q^{(0)}$ was shown above to be block-triangular to leading order.

### 4.4 Proof of Theorem 3.2 for $q=1,2, \ldots$

We assume that the induction hypothesis (4.8)-(4.9) holds for $0,1, \ldots, q$ and show that it holds for $q+1$. The proof proceeds in four steps. In step 1 , we derive explicit expressions for $R_{(q+1)}$ and $Q^{(q+1)}$ in terms of $R_{(q)}$ and $Q^{(q)}$; these expressions also involve $U_{(q)}$ and $L_{(q)}$. In step 2, we derive the leading-order asymptotics of $U_{(q)}$, and in step 3 the leading-order asymptotics of $L_{(q)}$. Then, in step 4, we substitute these results into the expressions derived in step 1 to complete the induction.

Step 1. We derive the expressions for $Q^{(q+1)}$ and $R_{(q+1)}$. Equations (4.3) and (4.4), together with the update formulas (3.26) for $A^{(q)}$ and (3.27) for $B_{(q)}$, yield

$$
\begin{align*}
Q^{(q+1)} & =Q^{(q)}\left(I-U_{(q)}\right)\left(I+L_{(q)}\right)  \tag{4.13}\\
R_{(q+1)} & =\left(I-L_{(q)}\right)\left(I+U_{(q)}\right) R_{(q)} \tag{4.14}
\end{align*}
$$

In terms of the constituent blocks, we have

$$
\begin{align*}
Q_{1 f}^{(q+1)} & =Q_{1 f}^{(q)}+Q_{2 f}^{(q)} L_{(q)}-Q_{1 f}^{(q)} U_{(q)} L_{(q)},  \tag{4.15}\\
Q_{2 f}^{(q+1)} & =Q_{2 f}^{(q)}-Q_{1 f}^{(q)} U_{(q)},  \tag{4.16}\\
Q_{1 s}^{(q+1)} & =Q_{1 s}^{(q)}+Q_{2 s}^{(q)} L_{(q)}-Q_{1 s}^{(q)} U_{(q)} L_{(q)},  \tag{4.17}\\
Q_{2 s}^{(q+1)} & =Q_{2 s}^{(q)}-Q_{1 s}^{(q)} U_{(q)}, \tag{4.18}
\end{align*}
$$

and

$$
\begin{equation*}
R_{(q+1)}^{1 s \perp}=R_{(q)}^{1 s \perp}+U_{(q)} R_{(q)}^{2 s \perp} \tag{4.19}
\end{equation*}
$$

$$
\begin{align*}
R_{(q+1)}^{1 f \perp} & =R_{(q)}^{1 f \perp}+U_{(q)} R_{(q)}^{2 f \perp},  \tag{4.20}\\
R_{(q+1)}^{2 s \perp} & =R_{(q)}^{2 s \perp}-L_{(q)} R_{(q)}^{1 s \perp}-L_{(q)} U_{(q)} R_{(q)}^{2 s \perp},  \tag{4.21}\\
R_{(q+1)}^{2 f \perp} & =R_{(q)}^{2 f \perp}-L_{(q)} R_{(q)}^{1 f \perp}-L_{(q)} U_{(q)} R_{(q)}^{2 f \perp} . \tag{4.22}
\end{align*}
$$

Step 2. We derive the leading-order asymptotics of the matrix $U_{(q)}$.
Recall that $U_{(q)}=\left(\Lambda_{(q)}^{11}\right)^{-1} \Lambda_{(q)}^{12}$. Moreover, $\Lambda_{(q)}^{11}$ is strictly $\mathcal{O}(1)$ and $\Lambda_{(q)}^{12}$ is strictly $\mathcal{O}\left(\varepsilon^{q}\right)$ by Lemma 4.1. Hence, $U_{(q)}=U_{(q, q)} \varepsilon^{q}+\mathcal{O}\left(\varepsilon^{q+1}\right)$, with $U_{(q, q)}=$ $\left(\Lambda_{(q, 0)}^{11}\right)^{-1} \Lambda_{(q, q)}^{12}$. Therefore, it suffices to derive the leading order asymptotics of these blocks of $\Lambda$.

By definition, $\Lambda_{(q)}=B_{(q)}\left[A^{(q)}, g\right]$. Therefore,

$$
\Lambda_{(q)}=\left(\begin{array}{cc}
B_{(q)}^{1}\left[A_{1}^{(q)}, g\right] & B_{(q)}^{1}\left[A_{2}^{(q)}, g\right]  \tag{4.23}\\
B_{(q)}^{2}\left[A_{1}^{(q)}, g\right] & B_{(q)}^{2}\left[A_{2}^{(q)}, g\right]
\end{array}\right) .
$$

The individual blocks of $\Lambda_{(q)}$ are obtained by substituting Eqs. (4.6) and (4.7) into Eq. (4.23). We observe that one-half of all the terms would vanish, were they to be evaluated on $\mathcal{M}_{\varepsilon}$, by virtue of Lemma 3.1. Since they are evaluated on $\mathcal{K}_{\varepsilon}^{(q+1)}$, instead, which is $\mathcal{O}\left(\varepsilon^{q+1}\right)$-close to $\mathcal{M}_{\varepsilon}$, these terms are $\mathcal{O}\left(\varepsilon^{q+2}\right)$ and therefore of higher order for each of the blocks, recall Lemma 4.1. Thus,

$$
\begin{align*}
\Lambda_{(q)}^{11} & =R_{(q)}^{1 s \perp} B^{s \perp}\left[A_{f} Q_{1 f}^{(q)}, g\right]+R_{(q)}^{1 f \perp} B^{f \perp}\left[A_{s} Q_{1 s}^{(q)}, g\right],  \tag{4.24}\\
\Lambda_{(q)}^{12} & =R_{(q)}^{1 s \perp} B^{s \perp}\left[A_{f} Q_{2 f}^{(q)}, g\right]+R_{(q)}^{1 f \perp} B^{f \perp}\left[A_{s} Q_{2 s}^{(q)}, g\right],  \tag{4.25}\\
\Lambda_{(q)}^{21} & =R_{(q)}^{2 s \perp} B^{s \perp}\left[A_{f} Q_{1 f}^{(q)}, g\right]+R_{(q)}^{2 f \perp} B^{f \perp}\left[A_{s} Q_{1 s}^{(q)}, g\right], \tag{4.26}
\end{align*}
$$

where the remainders of $\mathcal{O}\left(\varepsilon^{q+2}\right)$ have been ommited for brevity. Recalling the definition of the Lie bracket, we rewrite Eq. (4.24) as

$$
\begin{align*}
\Lambda_{(q)}^{11}= & R_{(q)}^{1 s \perp} B^{s \perp}\left((D g) A_{f} Q_{1 f}^{(q)}-\frac{d}{d t}\left(A_{f} Q_{1 f}^{(q)}\right)\right) \\
& +R_{(q)}^{1 f \perp} B^{f \perp}\left((D g) A_{s} Q_{1 s}^{(q)}-\frac{d}{d t}\left(A_{s} Q_{1 s}^{(q)}\right)\right) \tag{4.27}
\end{align*}
$$

where we recall that all of the quantities are evaluated at $\left(y, \psi_{(q+1)}, \varepsilon\right)$. Next, $(D g) A_{s}$ and the two time derivatives in Eq. (4.27) are zero to leading order by Lemma A. 1 and [20, Lemma A.2], respectively. Therefore, to leading order Eq. (4.27) becomes

$$
\begin{equation*}
\Lambda_{(q, 0)}^{11}=R_{(q, 0)}^{1 s \perp} B_{0}^{s \perp}(D g)_{0} A_{f}^{0} Q_{1 f}^{(q, 0)} \tag{4.28}
\end{equation*}
$$

Here, $\Lambda_{(q, 0)}^{11}$ stands for the leading-order term in the asymptotic expansion of $\Lambda_{(q)}^{11}\left(y, \psi_{(q+1)}(y), \varepsilon\right)$, and the right member is the leading order term in the asymptotic expansion of $\left(R_{(q)}^{1 s \perp} B^{s \perp}(D g) A_{f} Q_{1 f}^{(q)}\right)\left(y, h_{\varepsilon}(y), \varepsilon\right)$.

We derive a similar formula for $\Lambda_{(q, q)}^{12}$. First, we rewrite Eq. (4.25) as

$$
\begin{align*}
\Lambda_{(q)}^{12}= & R_{(q)}^{1 s \perp} B^{s \perp}\left((D g) A_{f} Q_{2 f}^{(q)}-\frac{d}{d t}\left(A_{f} Q_{2 f}^{(q)}\right)\right) \\
& +R_{(q)}^{1 f \perp} B^{f \perp}\left((D g) A_{s} Q_{2 s}^{(q)}-\frac{d}{d t}\left(A_{s} Q_{2 s}^{(q)}\right)\right) . \tag{4.29}
\end{align*}
$$

Next, $Q_{2 f}^{(q)}=\mathcal{O}\left(\varepsilon^{q}\right), Q_{2 s}^{(q)}=\mathcal{O}(1), R_{(q)}^{1 s \perp}=\mathcal{O}(1)$, and $R_{(q)}^{1 f \perp}=\mathcal{O}\left(\varepsilon^{q}\right)$, by the induction hypothesis (4.8)-(4.9). Thus, [20, Lemma A.2] implies that the two terms in Eq. (4.29) involving time derivatives are $\mathcal{O}\left(\varepsilon^{q+1}\right)$ and therefore of higher order. Also, $(D g) A_{s}$ is zero to leading order by Lemma A.1, and thus

$$
\begin{equation*}
\Lambda_{(q, q)}^{12}=R_{(q, 0)}^{1 s \perp} B_{0}^{s \perp}(D g)_{0} A_{f}^{0} Q_{2 f}^{(q, q)} . \tag{4.30}
\end{equation*}
$$

We now substitute $\Lambda_{(q, 0)}^{11}$ and $\Lambda_{(q, q)}^{12}$ from Eqs. (4.28) and (4.30) in the expression $U_{(q, q)}=\left(\Lambda_{(q, 0)}^{11}\right)^{-1} \Lambda_{(q, q)}^{12}$ to find the desired expression for $U_{(q, q)}$ in terms of $Q^{(q)}$,

$$
\begin{equation*}
U_{(q, q)}=\left(Q_{1 f}^{(q, 0)}\right)^{-1} Q_{2 f}^{(q, q)} \tag{4.31}
\end{equation*}
$$

We also need an expression for $U_{(q, q)}$ in terms of blocks of $R_{(q)}$, which we will use in Eqs. (4.19)-(4.22). Since $R_{(q)}$ has the near block-diagonal structure given by the induction hypothesis (4.8)-(4.9) and $Q^{(q)}$ is its inverse, we find
$Q^{(q)}=\left(\begin{array}{cc}\left(R_{(q, 0)}^{1 s \perp}\right)^{-1} & -\varepsilon^{q}\left(R_{(q, 0)}^{1 s \perp}\right)^{-1} R_{(q, q)}^{1 f \perp}\left(R_{(q, 0)}^{2 f \perp}\right)^{-1} \\ -\varepsilon^{q+1}\left(R_{(q, 0)}^{2 f \perp}\right)^{-1} R_{(q, q+1)}^{2 s \perp}\left(R_{(q, 0)}^{1 s \perp}\right)^{-1} & \left(R_{(q, 0)}^{2 f \perp}\right)^{-1}\end{array}\right),($
to leading order for each of the blocks and for $q=1,2, \ldots$. Equations (4.31) and (4.32) lead to the desired expression for $U_{(q, q)}$ in terms of $R_{(q)}$,

$$
\begin{equation*}
U_{(q, q)}=-R_{(q, q)}^{1 f \perp}\left(R_{(q, 0)}^{2 f \perp}\right)^{-1} \tag{4.33}
\end{equation*}
$$

Step 3. We derive the leading-order asymptotics of the matrix $L_{(q)}$.
Recall that $L_{(q)}=\Lambda_{(q)}^{21}\left(\Lambda_{(q)}^{11}\right)^{-1}$. Moreover, by Lemma 4.1, $\Lambda_{(q)}^{11}$ is strictly $\mathcal{O}(1)$ and $\Lambda_{(q)}^{21}$ is strictly $\mathcal{O}\left(\varepsilon^{q+1}\right)$. Hence, $L_{(q)}=L_{(q, q+1)} \varepsilon^{q+1}+\mathcal{O}\left(\varepsilon^{q+2}\right)$, with
$\left.L_{(q, q+1)}=\Lambda_{(q, q+1)}^{21} \Lambda_{(q, 0)}^{11}\right)^{-1}$. An expression for $\Lambda_{(q, 0)}^{11}$ was derived in Eq. (4.28), so here we focus on $\Lambda_{(q, q+1)}^{21}$.

Equation (4.26) and the definition of the Lie bracket imply that

$$
\begin{align*}
\Lambda_{(q)}^{21}= & R_{(q)}^{2 s \perp} B^{s \perp}\left((D g) A_{f} Q_{1 f}^{(q)}-\frac{d}{d t}\left(A_{f} Q_{1 f}^{(q)}\right)\right) \\
& +R_{(q)}^{2 f \perp} B^{f \perp}\left((D g) A_{s} Q_{1 s}^{(q)}-\frac{d}{d t}\left(A_{s} Q_{1 s}^{(q)}\right)\right) . \tag{4.34}
\end{align*}
$$

Next, $Q_{1 f}^{(q)}=\mathcal{O}(1), Q_{1 s}^{(q)}=\mathcal{O}\left(\varepsilon^{q+1}\right), R_{(q)}^{2 s \perp}=\mathcal{O}\left(\varepsilon^{q+1}\right)$, and $R_{(q)}^{2 f \perp}=\mathcal{O}(1)$, by the induction hypothesis. Also, the time derivatives are $\mathcal{O}(\varepsilon)$ by [20, Lemma A.2], and thus the two terms in Eq. (4.34) that involve time derivatives are $\mathcal{O}\left(\varepsilon^{q+2}\right)$. Last, $(D g) A_{s}=\mathcal{O}(\varepsilon)$ by Lemma A.1. Thus, we find

$$
\begin{equation*}
\Lambda_{(q, q+1)}^{21}=R_{(q, q+1)}^{2 s \perp} B_{0}^{s \perp}(D g)_{0} A_{f}^{0} Q_{1 f}^{(q, 0)} \tag{4.35}
\end{equation*}
$$

Equations (4.28) and (4.35) yield the desired formula for $L_{(q, q+1)}$ in terms of the blocks of $R_{(q)}$,

$$
\begin{equation*}
L_{(q, q+1)}=\Lambda_{(q, q+1)}^{21}\left(\Lambda_{(q, 0)}^{11}\right)^{-1}=R_{(q, q+1)}^{2 s \perp}\left(R_{(q, 0)}^{1 s \perp}\right)^{-1} \tag{4.36}
\end{equation*}
$$

Next, we recast Eq. (4.36) in terms of blocks of $Q^{(q)}$, in order to use it in Eqs. (4.15)-(4.18). The matrix $R_{(q)}$ is the inverse of $Q^{(q)}$ and has the near block-diagonal form given in (4.9). Thus,

$$
R_{(q)}=\left(\begin{array}{cc}
\left(Q_{1 f}^{(q, 0)}\right)^{-1} & -\varepsilon^{q}\left(Q_{1 f}^{(q, 0)}\right)^{-1} Q_{2 f}^{(q, q)}\left(Q_{2 s}^{(q, 0)}\right)^{-1}  \tag{4.37}\\
-\varepsilon^{q+1}\left(Q_{2 s}^{(q, 0)}\right)^{-1} Q_{1 s}^{(q, q+1)}\left(Q_{1 f}^{(q, 0)}\right)^{-1} & \left(Q_{2 s}^{(q, 0)}\right)^{-1}
\end{array}\right),(
$$

to leading order for each block and for $q=1,2, \ldots$. Equations (4.36) and (4.37) lead to the desired expression for $L_{(q, q+1)}$ in terms of the blocks of $Q^{(q)}$,

$$
\begin{equation*}
L_{(q, q+1)}=-\left(Q_{2 s}^{(q, 0)}\right)^{-1} Q_{1 s}^{(q, q+1)} \tag{4.38}
\end{equation*}
$$

Step 4. We substitute the results obtained in Step 2 and Step 3 into the formulas (4.15)-(4.22) derived in Step 1.

Equations (4.15) and (4.18), together with the induction hypothesis and the estimates $U_{(q)}=\mathcal{O}\left(\varepsilon^{q}\right)$ and $L_{(q)}=\mathcal{O}\left(\varepsilon^{q+1}\right)$, imply that $Q_{1 f}^{(q+1)}$ and $Q_{2 s}^{(q+1)}$ remain $\mathcal{O}(1)$. This concludes the estimation of these blocks.

Next, we show that $Q_{2 f}^{(q+1)}=\mathcal{O}\left(\varepsilon^{q+1}\right)$. First, $Q_{2 f}^{(q+1)}$ and $Q_{2 f}^{(q)}$ are equal up to and including terms of $\mathcal{O}\left(\varepsilon^{q-1}\right)$, by Eq. (4.16) and the estimate on $U_{(q)}$.

Thus, $Q_{2 f}^{(q+1, i)}=0$ for $i=0,1, \ldots, q-1$, by the induction hypothesis on $Q_{2 f}^{(q)}$. It suffices to show that $Q_{2 f}^{(q+1, q)}=0$. Equation (4.16) implies that

$$
\begin{equation*}
Q_{2 f}^{(q+1, q)}=Q_{2 f}^{(q, q)}-Q_{1 f}^{(q, 0)} U_{(q, q)} . \tag{4.39}
\end{equation*}
$$

The right member of this equation is zero, by Eq. (4.31), and the estimation of $Q_{2 f}^{(q+1)}$ is complete.

Finally, we show that $Q_{1 s}^{(q+1)}=\mathcal{O}\left(\varepsilon^{q+2}\right)$ to complete the estimates on the blocks of $Q^{(q+1)}$. First, $Q_{1 s}^{(q+1)}$ and $Q_{1 s}^{(q)}$ are equal up to and including terms of $\mathcal{O}\left(\varepsilon^{q}\right)$, by Eq. (4.17) and the order estimates on $U_{(q)}$ and $L_{(q)}$. Thus, $Q_{1 s}^{(q+1, i)}=0$ for $i=0,1, \ldots, q$, by the induction hypothesis on $Q_{1 s}^{(q)}$. It suffices to show that $Q_{1 s}^{(q+1, q+1)}=0$. Equation (4.17) implies that

$$
\begin{equation*}
Q_{1 s}^{(q+1, q+1)}=Q_{1 s}^{(q, q+1)}+Q_{2 s}^{(q, 0)} L_{(q, q+1)}, \tag{4.40}
\end{equation*}
$$

where the right member of this equation is zero by Eq. (4.38). The estimation of $Q_{1 s}^{(q+1)}$ is complete.

The blocks of $R_{(q)}$ may be estimated in an entirely similar manner, using Eqs. (4.19)-(4.22), instead of Eqs. (4.15)-(4.18), and Eqs. (4.33) and (4.36), instead of Eqs. (4.31) and (4.38). The proof of Theorem 3.2 is complete.

## 5 The Michaelis-Menten-Henri Model

In this section, we illustrate Theorem 3.2 by applying the CSP method to the Michaelis-Menten-Henri (MMH) mechanism of enzyme kinetics [15, 16]. We consider the planar system of ODEs for a slow variable $s$ and a fast variable $c$,

$$
\begin{align*}
s^{\prime} & =\varepsilon(-s+(s+\kappa-\lambda) c),  \tag{5.1}\\
c^{\prime} & =s-(s+\kappa) c . \tag{5.2}
\end{align*}
$$

The parameters satisfy the inequalities $0<\varepsilon \ll 1$ and $\kappa>\lambda>0$. Only nonnegative values of $s$ and $c$ are relevant. The system of Eqs. (5.1)-(5.2) is of the form (2.2)-(2.3) with $m=1, n=1, y=s, z=c, g_{1}=-s+(s+\kappa-\lambda) c$, and $g_{2}=s-(s+\kappa) c$.

### 5.1 Slow Manifolds and Fast Fibers

In the limit as $\varepsilon \downarrow 0$, the dynamics of the MMH equations are confined to the reduced slow manifold

$$
\begin{equation*}
\mathcal{M}_{0}=\left\{(c, s): c=\frac{s}{s+\kappa}, s \geq 0\right\} \tag{5.3}
\end{equation*}
$$

The manifold $\mathcal{M}_{0}$ is asymptotically stable, so there exists a locally invariant slow manifold $\mathcal{M}_{\varepsilon}$ for all sufficiently small $\varepsilon$ that is $\mathcal{O}(\varepsilon)$ close to $\mathcal{M}_{0}$ on any compact set. Moreover, $\mathcal{M}_{\varepsilon}$ is the graph of a function $h_{\varepsilon}$,

$$
\begin{equation*}
\mathcal{M}_{\varepsilon}=\left\{(c, s): c=h_{\varepsilon}(s), s \geq 0\right\} \tag{5.4}
\end{equation*}
$$

and $h_{\varepsilon}$ admits an asymptotic expansion, $h_{\varepsilon}=h_{0}+\varepsilon h_{1}+\varepsilon^{2} h_{2}+\cdots$. The coefficients are found from the invariance equation,

$$
\begin{equation*}
s-(s+\kappa) h_{\varepsilon}(s)=\varepsilon h_{\varepsilon}^{\prime}(s)\left(-s+(s+\kappa-\lambda) h_{\varepsilon}(s)\right) . \tag{5.5}
\end{equation*}
$$

The first few coefficients are

$$
\begin{equation*}
h_{0}(s)=\frac{s}{s+\kappa}, \quad h_{1}(s)=\frac{\kappa \lambda s}{(s+\kappa)^{4}}, \quad h_{2}(s)=\frac{\kappa \lambda s\left(2 \kappa \lambda-3 \lambda s-\kappa s-\kappa^{2}\right)}{(s+\kappa)^{7}} . \tag{5.6}
\end{equation*}
$$

In the limit as $\varepsilon \downarrow 0$, each line of constant $s$ is trivially invariant under Eqs. (5.1)-(5.2). These are the (one-dimensional) fast fibers $\mathcal{F}_{0}^{p}$ with base point $p=\left(s, h_{0}(s)\right) \in \mathcal{M}_{0}$. All points on $\mathcal{F}_{0}^{p}$ contract exponentially fast to $p$ with rate constant $-(s+\kappa)$. The fast fiber $\mathcal{F}_{0}^{p}$ perturbs to a curve $\mathcal{F}_{\varepsilon}^{p}$ that is $\mathcal{O}(\varepsilon)$ close to $\mathcal{F}_{0}^{p}$ in any compact neighborhood of $\mathcal{M}_{\varepsilon}$. The fast fibers $\mathcal{F}_{\varepsilon}^{p}$, $p \in \mathcal{M}_{\varepsilon}$, form an invariant family.

### 5.2 Asymptotic Expansions of the Fast Fibers

To derive asymptotic information about the fast fibers, we look for general solutions of Eqs. (5.1)-(5.2) that are given by asymptotic expansions,

$$
\begin{equation*}
s(t ; \varepsilon)=\sum_{i=0} \varepsilon^{i} s_{i}(t), \quad c(t ; \varepsilon)=\sum_{i=0} \varepsilon^{i} c_{i}(t) \tag{5.7}
\end{equation*}
$$

where the coefficients $s_{i}$ and $c_{i}$ are determined order by order.
Consider the fast fiber $\mathcal{F}_{\varepsilon}^{p}$ with base point $p=\left(s, h_{\varepsilon}(s)\right)$, and let $\left(s^{A}, c^{A}\right)$ and $\left(s^{B}, c^{B}\right)$ be two points on it; let $\Delta s(t)=s^{B}(t)-s^{A}(t)$ and $\Delta c(t)=$
$c^{B}(t)-c^{A}(t)$. The distance between any two points on the same fast fiber will contract exponentially fast towards zero at the $\mathcal{O}(1)$ rate, as long as both points are chosen in a neighborhood of $\mathcal{M}_{\varepsilon}$. We may write

$$
\begin{equation*}
\Delta s(t ; \varepsilon)=\sum_{i=0} \varepsilon^{i} \Delta s_{i}(t), \quad \Delta c(t ; \varepsilon)=\sum_{i=0} \varepsilon^{i} \Delta c_{i}(t) \tag{5.8}
\end{equation*}
$$

where $\Delta s_{i}(t)=s_{i}^{B}(t)-s_{i}^{A}(t)$ and $\Delta c_{i}(t)=c_{i}^{B}(t)-c_{i}^{A}(t)$. The condition on fast exponential decay of $\Delta s(t)$ and $\Delta c(t)$ translates into

$$
\begin{equation*}
\Delta s_{i}(t)=\mathcal{O}\left(e^{-C_{s} t}\right), \quad \Delta c_{i}(t)=\mathcal{O}\left(e^{-C_{c} t}\right), \quad t \rightarrow \infty \tag{5.9}
\end{equation*}
$$

for some positive constants $C_{s}$ and $C_{c}$. We let $\left(s^{A}, c^{A}\right)$ and $\left(s^{B}, c^{B}\right)$ be infinitesimally close, since we are interested in vectors tangent to the fast fiber.

### 5.2.1 $\mathcal{O}(1)$ Fast Fibers

Substituting the expansions (5.7) into Eqs. (5.1)-(5.2) and equating $\mathcal{O}(1)$ terms, we find

$$
\begin{align*}
& s_{0}^{\prime}=0,  \tag{5.10}\\
& c_{0}^{\prime}=s_{0}-\left(s_{0}+\kappa\right) c_{0} . \tag{5.11}
\end{align*}
$$

The equations can be integrated,

$$
\begin{align*}
& s_{0}(t)=s_{0}(0)=s_{0}  \tag{5.12}\\
& c_{0}(t)=\frac{s_{0}}{s_{0}+\kappa}+\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right) e^{-\left(s_{0}+\kappa\right) t} \tag{5.13}
\end{align*}
$$

Hence,

$$
\begin{align*}
\Delta s_{0}(t) & =\Delta s_{0}(0)  \tag{5.14}\\
\Delta c_{0}(t) & =\Delta c_{0}(0) e^{-\left(s_{0}+\kappa\right) t}+\left(\partial_{s_{0}} c_{0}(t)\right) \Delta s_{0}(0)+\mathcal{O}\left(\left(\Delta s_{0}(0)\right)^{2}\right) \tag{5.15}
\end{align*}
$$

The points $A$ and $B$ lie on the same fiber if and only if

$$
\begin{equation*}
\Delta s_{0}(0)=0 \tag{5.16}
\end{equation*}
$$

Thus, Eq. (5.15) simplifies to

$$
\begin{equation*}
\Delta c_{0}(t)=\Delta c_{0}(0) e^{-\left(s_{0}+\kappa\right) t} \tag{5.17}
\end{equation*}
$$

and $\Delta c_{0}(t)$ decays exponentially towards zero, irrespective of the choice of $\Delta c_{0}(0)$. Hence, $\Delta c_{0}(0)$ is a free parameter.

We conclude that, to $\mathcal{O}(1)$, any vector $\binom{0}{\alpha}$ with $\alpha$ constant $(\alpha \neq 0)$ is tangent to every fast fiber at the base point.

### 5.2.2 $\mathcal{O}(\varepsilon)$ Fast Fibers

At $\mathcal{O}(\varepsilon)$, we obtain the equations

$$
\begin{align*}
s_{1}^{\prime} & =-s_{0}+\left(s_{0}+\kappa-\lambda\right) c_{0}  \tag{5.18}\\
c_{1}^{\prime} & =s_{1}-\left(s_{0}+\kappa\right) c_{1}-s_{1} c_{0} \tag{5.19}
\end{align*}
$$

Using Eqs. (5.12) and (5.13), we integrate Eq. (5.18) to obtain

$$
\begin{equation*}
s_{1}(t)=s_{1}(0)-\frac{\lambda s_{0}}{s_{0}+\kappa} t+\frac{s_{0}+\kappa-\lambda}{s_{0}+\kappa}\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right)\left(1-e^{-\left(s_{0}+\kappa\right) t}\right) . \tag{5.20}
\end{equation*}
$$

Therefore, at $\mathcal{O}(\varepsilon)$,

$$
\begin{equation*}
\Delta s_{1}(t)=\Delta s_{1}(0)+\frac{s_{0}+\kappa-\lambda}{s_{0}+\kappa} \Delta c_{0}(0)\left(1-e^{-\left(s_{0}+\kappa\right) t}\right) \tag{5.21}
\end{equation*}
$$

For the two points to have the same phase asymptotically, it is necessary that $\lim _{t \rightarrow \infty} \Delta s_{1}(t)=0$. This condition is satisfied if and only if

$$
\begin{equation*}
\Delta s_{1}(0)=-\frac{s_{0}+\kappa-\lambda}{s_{0}+\kappa} \Delta c_{0}(0) \tag{5.22}
\end{equation*}
$$

Next, $c_{1}(t)$ follows upon integration of Eq. (5.19),

$$
\begin{align*}
c_{1}(t)= & c_{1}(0) e^{-\left(s_{0}+\kappa\right) t} \\
& +\frac{\kappa}{\left(s_{0}+\kappa\right)^{2}}\left(s_{1}(0)+\frac{s_{0}+\kappa-\lambda}{s_{0}+\kappa}\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right)\right)\left(1-e^{-\left(s_{0}+\kappa\right) t}\right) \\
& -\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right)\left(s_{1}(0)+\frac{s_{0}+\kappa-\lambda}{s_{0}+\kappa}\left(c_{0}(0)+\frac{\kappa-s_{0}}{s_{0}+\kappa}\right)\right) t e^{-\left(s_{0}+\kappa\right) t} \\
& -\frac{s_{0}+\kappa-\lambda}{\left(s_{0}+\kappa\right)^{2}}\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right)^{2}\left(e^{-2\left(s_{0}+\kappa\right) t}-e^{-\left(s_{0}+\kappa\right) t}\right) \\
& +\frac{\lambda s_{0}}{2\left(s_{0}+\kappa\right)}\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right) t^{2} e^{-\left(s_{0}+\kappa\right) t} \\
& -\frac{\kappa \lambda s_{0}}{\left(s_{0}+\kappa\right)^{4}}\left(e^{-\left(s_{0}+\kappa\right) t}+\left(s_{0}+\kappa\right) t-1\right) . \tag{5.23}
\end{align*}
$$

We infer from this expression that $\lim _{t \rightarrow \infty} \Delta c_{1}(t)=0$, as long as Eqs. (5.22) and (5.16) hold. Hence, $\Delta c_{1}(0)$ is a free parameter, just like $\Delta c_{0}(0)$, and the only condition that arises at $\mathcal{O}(\varepsilon)$ is (5.22) on $\Delta s_{1}(0)$.

We conclude that any vector

$$
\begin{equation*}
\binom{0}{\alpha}+\varepsilon\binom{-\left(1-\frac{\lambda}{s_{0}+\kappa}\right) \alpha}{\beta} \tag{5.24}
\end{equation*}
$$

with $\alpha$ and $\beta$ constant $(\alpha \neq 0)$, is tangent to every fast fiber at the base point up to and including terms of $\mathcal{O}(\varepsilon)$. Any such vector may be written as the product of a free parameter and a constant vector (fixed by $s_{0}$ ),

$$
\begin{equation*}
(\alpha+\varepsilon \beta)\binom{-\varepsilon\left(1-\frac{\lambda}{s_{0}+\kappa}\right)}{1}+\mathcal{O}\left(\varepsilon^{2}\right) . \tag{5.25}
\end{equation*}
$$

### 5.2.3 $\mathcal{O}\left(\varepsilon^{2}\right)$ Fast Fibers

At $\mathcal{O}\left(\varepsilon^{2}\right)$, we obtain the equation

$$
\begin{equation*}
s_{2}^{\prime}=s_{1}\left(c_{0}-1\right)+\left(s_{0}+\kappa-\lambda\right) c_{1} . \tag{5.26}
\end{equation*}
$$

Direct integration yields

$$
\begin{align*}
s_{2}(t)= & s_{2}(0)+\left[\frac{\lambda}{\left(s_{0}+\kappa\right)^{2}}\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right)-\frac{\kappa\left(s_{0}+\kappa-\lambda\right)}{\left(s_{0}+\kappa\right)^{3}}\right] s_{1}(0) \\
& -\left[\frac{\kappa\left(s_{0}+\kappa-\lambda\right)\left(s_{0}+\kappa-2 \lambda\right)+\lambda^{2} s_{0}}{\left(s_{0}+\kappa\right)^{4}}\right]\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right) \\
& +\frac{\lambda\left(s_{0}+\kappa-\lambda\right)}{2\left(s_{0}+\kappa\right)^{3}}\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right)^{2} \\
& +\left(1-\frac{\lambda}{s_{0}+\kappa}\right)\left(c_{1}(0)-\frac{\kappa \lambda s_{0}}{\left(s_{0}+\kappa\right)^{4}}\right) \\
& -\frac{\kappa \lambda}{\left(s_{0}+\kappa\right)^{2}}\left[s_{1}(0)+\frac{s_{0}+\kappa-\lambda}{s_{0}+\kappa}\left(c_{0}(0)-\frac{2 s_{0}}{s_{0}+\kappa}\right)\right] t \\
& +\frac{\kappa \lambda^{2} s_{0}}{2\left(s_{0}+\kappa\right)^{3}} t^{2}+\mathcal{R}(t) \tag{5.27}
\end{align*}
$$

where the remainder $\mathcal{R}(t)$ involves the functions $e^{-\left(s_{0}+\kappa\right) t}, t e^{-\left(s_{0}+\kappa\right) t}, t^{2} e^{-\left(s_{0}+\kappa\right) t}$, and $e^{-2\left(s_{0}+\kappa\right) t}$. From this expression we find

$$
\begin{align*}
\Delta s_{2}(t)= & \Delta s_{2}(0)+\left(\partial_{s_{0}} s_{2}(t)\right) \Delta s_{0}(0)+\left(\partial_{c_{0}} s_{2}(t)\right) \Delta c_{0}(0) \\
& +\left(\partial_{s_{1}} s_{2}(t)\right) \Delta s_{1}(0)+\left(\partial_{c_{1}} s_{2}(t)\right) \Delta c_{1}(0)+\mathcal{O}(2)+\mathcal{O}\left(e^{-C t}\right), \tag{5.28}
\end{align*}
$$

for some $C>0$. Here, $\partial_{c_{0}}$ is an abbreviation for the partial derivative $\partial_{c_{0}(0)}$, and so on, and $\mathcal{O}(2)$ denotes quadratic terms in the multivariable Taylor expansion. First, we recall that $\Delta s_{0}(0)=0$ by Eq. (5.16). Next, we calculate the partial derivatives in each of the three remaining terms,

$$
\partial_{c_{0}} s_{2}(t)=\frac{\lambda s_{1}(0)}{\left(s_{0}+\kappa\right)^{2}}-\frac{\kappa\left(s_{0}+\kappa-\lambda\right)\left(s_{0}+\kappa-2 \lambda\right)+\lambda^{2} s_{0}}{\left(s_{0}+\kappa\right)^{4}}
$$

$$
\begin{align*}
& +\frac{\lambda\left(s_{0}+\kappa-\lambda\right)}{\left(s_{0}+\kappa\right)^{3}}\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right)-\frac{\kappa \lambda\left(s_{0}+\kappa-\lambda\right)}{\left(s_{0}+\kappa\right)^{3}} t  \tag{5.29}\\
\partial_{s_{1}} s_{2}(t)= & \frac{\lambda}{\left(s_{0}+\kappa\right)^{2}}\left(c_{0}(0)-\frac{s_{0}}{s_{0}+\kappa}\right)-\frac{\kappa\left(s_{0}+\kappa-\lambda\right)}{\left(s_{0}+\kappa\right)^{3}} \\
& -\frac{\kappa \lambda}{\left(s_{0}+\kappa\right)^{2}} t  \tag{5.30}\\
\partial_{c_{1}} s_{2}(t)= & 1-\frac{\lambda}{s_{0}+\kappa} . \tag{5.31}
\end{align*}
$$

We substitute these expressions into Eq. (5.28), recall Eq. (5.22), and carry out the algebra to obtain

$$
\begin{align*}
\Delta s_{2}(t)= & \Delta s_{2}(0)+\left(1-\frac{\lambda}{s_{0}+\kappa}\right) \Delta c_{1}(0) \\
& +\frac{\lambda}{\left(s_{0}+\kappa\right)^{2}}\left(s_{1}(0)+\frac{\kappa\left(s_{0}+\kappa-\lambda\right)-\lambda s_{0}}{\left(s_{0}+\kappa\right)^{2}}\right) \Delta c_{0}(0) \\
& +\mathcal{O}(2)+\mathcal{O}\left(e^{-C t}\right), \quad C>0 \tag{5.32}
\end{align*}
$$

In the limit $t \rightarrow \infty$, Eq. (5.32) yields the condition

$$
\begin{align*}
\Delta s_{2}(0)= & -\left(1-\frac{\lambda}{s_{0}+\kappa}\right) \Delta c_{1}(0) \\
& -\frac{\lambda}{\left(s_{0}+\kappa\right)^{2}}\left(s_{1}(0)+\frac{\kappa\left(s_{0}+\kappa-\lambda\right)-\lambda s_{0}}{\left(s_{0}+\kappa\right)^{2}}\right) \Delta c_{0}(0) \tag{5.33}
\end{align*}
$$

Finally, $\Delta c_{2}(t)$ vanishes exponentially, as follows directly from the conditions (5.22) and (5.33). Thus, no further conditions besides (5.33) arise at $\mathcal{O}\left(\varepsilon^{2}\right)$.

We conclude that any vector

$$
\begin{align*}
\binom{0}{\alpha} & +\varepsilon\binom{-\left(1-\frac{\lambda}{s_{0}+\kappa}\right) \alpha}{\beta} \\
& +\varepsilon^{2}\left(-\left(1-\frac{\lambda}{s_{0}+\kappa}\right) \beta-\frac{\lambda}{\left(s_{0}+\kappa\right)^{2}}\left(s_{1}(0)+\frac{\kappa\left(s_{0}+\kappa-\lambda\right)-\lambda s_{0}}{\left(s_{0}+\kappa\right)^{2}}\right) \alpha\right), \tag{5.34}
\end{align*}
$$

with $\alpha, \beta$, and $\gamma$ constant $(\alpha \neq 0)$, is tangent to every fiber at the base point, up to and including terms of $\mathcal{O}\left(\varepsilon^{2}\right)$.

### 5.3 CSP Approximations of the Fast Fibers

We choose the stoichiometric vectors as the basis vectors, so

$$
A^{(0)}=\left(A_{1}^{(0)}, A_{2}^{(0)}\right)=\left(\begin{array}{ll}
0 & 1  \tag{5.35}\\
1 & 0
\end{array}\right), \quad B_{(0)}=\binom{B_{(0)}^{1}}{B_{(0)}^{2}}=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right) .
$$

The CSP condition $B_{(0)}^{1} g=0$ is satisfied if $c=h_{0}(s)$, so the CSP manifold $\mathcal{K}_{\varepsilon}^{(0)}$ coincides with $\mathcal{M}_{0}$. With this choice of initial basis, we have

$$
\Lambda_{(0)}=B_{(0)}(D g) A^{(0)}=\left(\begin{array}{cc}
-(s+\kappa) & -(c-1)  \tag{5.36}\\
\varepsilon(s+\kappa-\lambda) & \varepsilon(c-1)
\end{array}\right) .
$$

### 5.3.1 First Iteration

At any point $(s, c)$, we have

$$
\begin{array}{ll}
A_{1}^{(1)}=\binom{0}{1}+\varepsilon \frac{s+\kappa-\lambda}{s+\kappa}\binom{-1}{\frac{c-1}{s+\kappa}}, & A_{2}^{(1)}=\binom{1}{-\frac{c-1}{s+\kappa}}, \\
B_{(1)}^{1}=\left(-A_{22}^{(1)}, A_{12}^{(1)}\right), & B_{(1)}^{2}=\left(A_{21}^{(1)},-A_{11}^{(1)}\right) . \tag{5.38}
\end{array}
$$

In the first step, we evaluate $A_{2}^{(1)}$ and $B_{(1)}^{1}$ on $\mathcal{K}_{\varepsilon}^{(0)}$ to obtain

$$
\begin{equation*}
A_{2}^{(1)}=\binom{1}{\frac{\kappa}{(s+\kappa)^{2}}}, \quad B_{(1)}^{1}=\left(-\frac{\kappa}{(s+\kappa)^{2}}, 1\right) . \tag{5.39}
\end{equation*}
$$

Hence, the CSP condition,

$$
\begin{equation*}
B_{(1)}^{1} g=s-(s+\kappa) c-\varepsilon \frac{\kappa(-s+(s+\kappa-\lambda) c)}{(s+\kappa)^{2}}=0 \tag{5.40}
\end{equation*}
$$

is satisfied if

$$
\begin{equation*}
c=\frac{s}{s+\kappa}+\varepsilon \frac{\kappa \lambda s}{(s+\kappa)^{4}}-\varepsilon^{2} \frac{\kappa^{2} \lambda s(s+\kappa-\lambda)}{(s+\kappa)^{7}}+\mathcal{O}\left(\varepsilon^{3}\right) \tag{5.41}
\end{equation*}
$$

Equation (5.41) defines $\mathcal{K}_{\varepsilon}^{(1)}$, the CSPM of order one, which agrees with $\mathcal{M}_{\varepsilon}$ up to and including terms of $\mathcal{O}(\varepsilon)$; recall Eq. (5.6).

Then, in the second step, the new fast basis vector, $A_{1}^{(1)}$, and its complement, $B_{(1)}^{2}$, in the dual basis are evaluated on $\mathcal{K}_{\varepsilon}^{(1)}$,

$$
\begin{align*}
A_{1}^{(1)} & =\binom{0}{1}-\varepsilon\binom{1}{\frac{\kappa(s+\kappa-\lambda)}{(s+\kappa)^{3}}}+\varepsilon^{2}\binom{0}{\frac{\kappa \lambda s(s+\kappa-\lambda)}{(s+\kappa)^{6}}}+\mathcal{O}\left(\varepsilon^{3}\right)  \tag{5.42}\\
B_{(1)}^{2} & =\left(A_{21}^{(1)},-A_{11}^{(1)}\right) \tag{5.43}
\end{align*}
$$

Thus, we see that $A_{1}^{(1)}$ is tangent to the fast fibers at their base points up to and including terms of $\mathcal{O}(\varepsilon)$ as Eq. (5.24) (with $\alpha=1, \beta=-\frac{\kappa(s+\kappa-\lambda)}{(s+\kappa)^{3}}$ ) implies. As a result, $\mathcal{L}_{\varepsilon}^{(1)}$ approximates $\mathcal{T} \mathcal{F}_{\varepsilon}$ also up to and including terms of $\mathcal{O}(\varepsilon)$.

Remark 5.1. If, in this particular example, one evaluates $A_{1}^{(1)}$ on $\mathcal{K}_{\varepsilon}^{(0)}$ as opposed to $\mathcal{K}_{\varepsilon}^{(1)}$ as we did above, then the approximation of $\mathcal{T} \mathcal{F}_{\varepsilon}$ is also accurate up to and including terms of $\mathcal{O}(\varepsilon)$.

### 5.3.2 Second Iteration

The blocks of $\Lambda_{(1)}$ are

$$
\begin{align*}
\Lambda_{(1)}^{11}= & -(s+\kappa)+\varepsilon \frac{(s+\kappa-\lambda)}{s+\kappa}\left[(c-1)+\left(c-\frac{s}{s+\kappa}\right)\right] \\
& +\varepsilon^{2} \frac{(c-1)(s+\kappa-\lambda)}{(s+\kappa)^{3}}[-\lambda(c-1)+[(s+\kappa-\lambda) c-s]],  \tag{5.44}\\
\Lambda_{(1)}^{12}= & \frac{s}{s+\kappa}-c+\varepsilon \frac{c-1}{(s+\kappa)^{2}}[\lambda(c-1)-[(s+\kappa-\lambda) c-s]]  \tag{5.45}\\
\Lambda_{(1)}^{21}= & \frac{\varepsilon^{2}}{(s+\kappa)^{2}}[(c-1)(s+\kappa-\lambda)(s+\kappa-2 \lambda) \\
& \left.+\lambda[(s+\kappa-\lambda) c-s]+(s+\kappa-\lambda)^{2}\left(c-\frac{s}{s+\kappa}\right)\right]  \tag{5.46}\\
\Lambda_{(1)}^{22}= & \frac{\varepsilon}{s+\kappa}\left[\lambda(c-1)+(s+\kappa-\lambda)\left(\frac{s}{s+\kappa}-c\right)\right] \\
& +\varepsilon^{2} \frac{(c-1)(s+\kappa-\lambda)}{(s+\kappa)^{3}}[\lambda(c-1)-[(s+\kappa-\lambda) c-s]] \tag{5.47}
\end{align*}
$$

with remainders of $\mathcal{O}\left(\varepsilon^{3}\right)$.
In the first step, we update $A_{2}^{(1)}$ and $B_{(1)}^{1}$ and evaluate the updated quantities on $\mathcal{K}_{\varepsilon}^{(1)}$, to obtain

$$
\begin{align*}
A_{12}^{(2)}= & 1+\varepsilon^{2} \frac{\kappa \lambda(2 s-\kappa)(s+\kappa-\lambda)}{(s+\kappa)^{6}},  \tag{5.48}\\
A_{22}^{(2)}= & \frac{\kappa}{(s+\kappa)^{2}}+\varepsilon \frac{\kappa \lambda(\kappa-3 s)}{(s+\kappa)^{5}} \\
& +\varepsilon^{2} \frac{\kappa^{2} \lambda(7 s-2 \kappa)(s+\kappa-\lambda)+\kappa \lambda^{2} s(s-2 \kappa)}{(s+\kappa)^{8}},  \tag{5.49}\\
B_{(2)}^{1}= & \left(-A_{22}^{(2)}, A_{12}^{(2)}\right), \tag{5.50}
\end{align*}
$$

up to and including terms of $\mathcal{O}\left(\varepsilon^{2}\right)$.
The CSP condition

$$
\begin{align*}
B_{(2)}^{1} g= & s-(s+\kappa) c-\varepsilon \frac{\kappa(-s+(s+\kappa-\lambda) c)}{(s+\kappa)^{2}} \\
& +\varepsilon^{2} \kappa \lambda\left(\frac{(3 s-\kappa)(-s+(s+\kappa-\lambda) c)}{(s+\kappa)^{5}}\right. \\
& \left.+\frac{(2 s-\kappa)(s+\kappa-\lambda)(s-(s+\kappa) c)}{(s+\kappa)^{6}}\right)+\mathcal{O}\left(\varepsilon^{3}\right) \\
= & 0 \tag{5.51}
\end{align*}
$$

is satisfied if

$$
\begin{equation*}
c=\frac{s}{s+\kappa}+\varepsilon \frac{\kappa \lambda s}{(s+\kappa)^{4}}+\varepsilon^{2} \frac{\kappa \lambda s\left(2 \kappa \lambda-3 \lambda s-\kappa s-\kappa^{2}\right)}{(s+\kappa)^{7}}+\mathcal{O}\left(\varepsilon^{3}\right) \tag{5.52}
\end{equation*}
$$

Equation (5.52) defines $\mathcal{K}_{\varepsilon}^{(2)}$, the CSPM of order two, which agrees with $\mathcal{M}_{\varepsilon}$ up to and including terms of $\mathcal{O}\left(\varepsilon^{2}\right)$; recall Eq. (5.6).

Then, in the second step, we update $A_{1}^{(1)}$ and $B_{(1)}^{2}$ to obtain

$$
\begin{align*}
A_{11}^{(2)}= & -\varepsilon \frac{s+\kappa-\lambda}{s+\kappa}-\varepsilon^{2} \frac{1}{(s+\kappa)^{3}}[(s+\kappa-\lambda)(s+\kappa-2 \lambda)(c-1) \\
& \left.+(s+\kappa-\lambda)^{2}\left(c-\frac{s}{s+\kappa}\right)+\lambda[(s+\kappa-\lambda) c-s]\right]  \tag{5.53}\\
A_{21}^{(2)}= & 1+\varepsilon \frac{(s+\kappa-\lambda)(c-1)}{(s+\kappa)^{2}}+\varepsilon^{2} \frac{1}{(s+\kappa)^{4}}[(s+\kappa-\lambda)[(s+\kappa-2 \lambda)(c-1) \\
& \left.\left.+(s+\kappa-\lambda)\left(c-\frac{s}{s+\kappa}\right)+\lambda c\right]-\lambda s\right]\left(2 c-\frac{2 s+\kappa}{s+\kappa}\right)  \tag{5.54}\\
B_{(2)}^{2}= & \left(A_{21}^{(2)},-A_{11}^{(2)}\right) \tag{5.55}
\end{align*}
$$

with remainders of $\mathcal{O}\left(\varepsilon^{3}\right)$. Evaluating these expressions on $\mathcal{K}_{\varepsilon}^{(2)}$, we obtain

$$
\begin{align*}
A_{11}^{(2)}= & -\varepsilon \frac{s+\kappa-\lambda}{s+\kappa}+\varepsilon^{2} \frac{\kappa(s+\kappa-2 \lambda)(s+\kappa-\lambda)+\lambda^{2} s}{(s+\kappa)^{4}},  \tag{5.56}\\
A_{21}^{(2)}= & 1-\varepsilon \frac{\kappa(s+\kappa-\lambda)}{(s+\kappa)^{3}} \\
& +\varepsilon^{2} \frac{(s+\kappa-\lambda)\left(\kappa^{2}(s+\kappa-2 \lambda)+\kappa \lambda s\right)+\kappa \lambda^{2} s}{(s+\kappa)^{6}},  \tag{5.57}\\
B_{(2)}^{2}= & \left(A_{21}^{(2)},-A_{11}^{(2)}\right) \tag{5.58}
\end{align*}
$$

with remainders of $\mathcal{O}\left(\varepsilon^{3}\right)$. Therefore, $A_{1}^{(2)}$ is tangent to the fast fibers at their base points up to and including terms of $\mathcal{O}\left(\varepsilon^{2}\right)$, according to Eq. (5.34) (with $\left.\alpha=1, \beta=-\frac{\kappa(s+\kappa-\lambda)}{(s+\kappa)^{3}}, \gamma=\frac{(s+\kappa-\lambda)\left(\kappa^{2}(s+\kappa-2 \lambda)+\kappa \lambda s\right)+\kappa \lambda^{2} s}{(s+\kappa)^{6}}\right)$, and $\mathcal{L}_{\varepsilon}^{(2)}$ is an $\mathcal{O}\left(\varepsilon^{2}\right)$-accurate approximation to $\mathcal{T} \mathcal{F}_{\varepsilon}$.

Remark 5.2. If one evaluates, in this particular example, $A_{1}^{(2)}$ on $\mathcal{K}_{\varepsilon}^{(1)}$ instead of on $\mathcal{K}_{\varepsilon}^{(2)}$ as we did above, then the approximation of $\mathcal{T} \mathcal{F}_{\varepsilon}$ is also accurate up to and including terms of $\mathcal{O}\left(\varepsilon^{2}\right)$.

## 6 Linear Projection of Initial Conditions

The main result of this article, Theorem 3.2, states that after $q$ iterations the CSP method successfully identifies $\mathcal{T} \mathcal{F}_{\varepsilon}$ up to and including terms of $\mathcal{O}\left(\varepsilon^{q+1}\right)$, where this approximation is given explicitly by $A_{1}^{(q)}$. This information is postprocessed to project the initial conditions on the CSPM of order $q$. In this section, we discuss the accuracy and limitations of this linear projection.

Geometrically, one knows from Fenichel's theory that any given initial condition $x_{0}$ sufficiently close to $\mathcal{M}_{\varepsilon}$ lies on a (generally nonlinear) fiber $\mathcal{F}_{\varepsilon}^{p}$ with base point $p$ on $\mathcal{M}_{\varepsilon}$. Hence, the ideal projection would be $\pi_{F}\left(x_{0}\right)=p$ (the subscript $F$ stands for fiber or Fenichel) and this is, in general, a nonlinear projection.

Within the framework of an algorithm that yields only linearized information about the fast fibers, one must ask how best to approximate this ideal. A consistent approach is to identify a point on the slow manifold such that the approximate linearized fiber through it also goes through the given initial condition. This approach was used, for example, by Roberts [17] for systems with asymptotically stable center manifolds, where we note that a different method is first used to approximate the center manifold. Also, this approach is exact in the special case that the perturbed fast fibers are hyperplanes which need not be vertical. In general, if $x_{0}$ lies on the linearized fiber $\mathcal{L}_{\varepsilon}^{p_{1}}$ and if $\pi_{F}\left(x_{0}\right)=p_{2}$, then the error $\left\|p_{1}-p_{2}\right\|$ made by projecting linearly is $\mathcal{O}(\varepsilon)$ and proportional to the curvature of the fiber (see also [17]).

For fast-slow systems, there is yet another way to linearly project initial conditions on the slow manifold. One projects along the approximate CSPF to the space $\mathcal{T}_{p} \mathcal{F}_{\varepsilon}$, where $p$ is the point on the CSPM that lies on the same $\varepsilon=0$ fiber as the initial condition. This type of projection is also consistent, in the sense that it yields an exact result for $\varepsilon=0$, but has an error of $\mathcal{O}(\varepsilon)$ for $\varepsilon>0$. Moreover, it is algorithmically simpler, since it does not involve a
search for the base point of the linearized fiber on which the initial conditions lie. However, it has the disadvantage that the projection is not exact in the special case that the fast fibers are (non-vertical) hyperplanes.

## A The Action of the $\mathcal{O}(1)$ Jacobian on $\mathcal{T}_{p} \mathcal{M}_{0}$

The spaces $\mathcal{T}_{p} \mathcal{F}_{\varepsilon}$ and $\mathcal{T}_{p} \mathcal{M}_{\varepsilon}$ depend, in general, on both the point $p \in \mathcal{M}_{\varepsilon}$ and $\varepsilon$. As a result, the basis $A$ also depends on $p$ and $\varepsilon$, and hence $A_{f}$ and $A_{s}$ possess formal asymptotic expansions in terms of $\varepsilon$,

$$
\begin{equation*}
A_{f}=\sum_{i=0} \varepsilon^{i} A_{f}^{i}, \quad A_{s}=\sum_{i=0} \varepsilon^{i} A_{s}^{i} . \tag{1.1}
\end{equation*}
$$

Next, we compute the action of the Jacobian on $A_{s}$ to leading order.

Lemma A. $1 \operatorname{Ker}(\operatorname{Dg}(p))_{0}=\mathcal{T}_{p} \mathcal{M}_{0}$, for $p \in \mathcal{M}_{0}$. In particular, $(D g)_{0} A_{s}^{0}=0$.

Proof. The Jacobian is a linear operator, so it suffices to show that every column vector of a basis for $\mathcal{T}_{p} \mathcal{M}_{0}$ vanishes under the left action of the Jacobian. We choose this basis to be the matrix $\binom{I_{m}}{D_{y} h_{0}}$.

We compute

$$
D g_{0}\binom{I_{m}}{D_{y} h_{0}}=\left(\begin{array}{cc}
0 & 0  \tag{1.2}\\
D_{y} g_{2} & D_{z} g_{2}
\end{array}\right)\binom{I_{m}}{D_{y} h_{0}}=\binom{0}{D_{y} g_{2}+D_{z} g_{2} D_{y} h_{0}} .
$$

Differentiating both members of the $\mathcal{O}(1)$ invariance equation $g_{2}\left(y, h_{0}(y), 0\right)=$ 0 with respect to $y$, we obtain

$$
\begin{equation*}
D_{y} g_{2}\left(y, h_{0}(y), 0\right)+D_{z} g_{2}\left(y, h_{0}(y), 0\right) D_{y} h_{0}(y)=0 \tag{1.3}
\end{equation*}
$$

Equations (1.2) and (1.3) yield the desired result

$$
\begin{equation*}
D g_{0}\binom{I_{m}}{D_{y} h_{0}}=\binom{0}{0}, \quad \text { on } \mathcal{M}_{0} \tag{1.4}
\end{equation*}
$$

Finally, the identity $(D g)_{0} A_{s}^{0}=0$ follows from the fact that $A_{s}^{0}$ spans $\mathcal{T}_{p} \mathcal{M}_{0}$, since $A_{s}^{0}=\left.A_{s}\right|_{\varepsilon=0}$ by Eq. (1.1).

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## References

[1] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, Modern Geometry Methods and Applications, Vol. 2, Graduate Texts in Mathematics, 104, Springer-Verlag, New York, 1985
[2] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, J. Diff. Eq. 31 (1979) 53-98
[3] D. A. Goussis and S. H. Lam, A study of homogeneous methanol oxidation kinetics using CSP, in: Twenty-Fourth Symposium (International) on Combustion, The University of Sydney, Sydney, Australia, July 5-10, 1992, The Combustion Institute, Pittsburgh, 1992, pp. 113-120
[4] M. Hadjinicolaou and D. A. Goussis, Asymptotic solutions of stiff PDEs with the CSP method: The reaction diffusion equation, SIAM J. Sci. Comput. 20 (1999) 781-810
[5] C. K. R. T. Jones, Geometric singular perturbation theory, in: Dynamical Systems, Montecatini Terme, L. Arnold, Lecture Notes in Mathematics, 1609, Springer-Verlag, Berlin, 1994, pp. 44-118
[6] S. H. Lam, Using CSP to understand complex chemical kinetics, Combust. Sci. Tech. 89 (1993) 375-404
[7] S. H. Lam and D. A. Goussis, Understanding complex chemical kinetics with computational singular perturbation, in Twenty-Second Symposium (International) on Combustion, The University of Washington, Seattle, Washington, August 14-19, 1988, The Combustion Institute, Pittsburgh, 1988, pp. 931-941
[8] S. H. Lam and D. A. Goussis, Conventional asymptotics and computational singular perturbation theory for simplified kinetics modeling, in $R e-$ duced Kinetic Mechanisms and Asymptotic Approximations for MethaneAir Flames, M. Smooke, ed., Lecture Notes in Physics 384, SpringerVerlag, New York, 1991, Chapter 10
[9] S. H. Lam and D. A. Goussis, The CSP method for simplifying kinetics, Internat. J. Chem. Kin. 26 (1994) 461-486
[10] T. F. Lu, Y. G. Ju, and C. K. Law, Complex CSP for chemistry reduction and analysis, Combustion and Flame 126 (2001) 1445-1455
[11] A. Massias, D. Diamantis, E. Mastorakos, and D. Goussis, Global reduced mechanisms for methane and hydrogen combustion with nitric oxide formation constructed with CSP data, Combust. Theory Modelling 3 (1999) 233-257
[12] A. Massias and D. A. Goussis, On the manifold of stiff reaction-diffusion PDE's: The effects of diffusion, preprint (2001)
[13] K. D. Mease, Geometry of computational singular perturbations, in Nonlinear Control System Design, vol. 2, A. J. Kerner and D. Q. M. Mayne, editors, Pergamon Press, Oxford, U.K., 1996, pp. 855-861
[14] P. J. Olver, Applications of Lie Groups to Differential Equations, Graduate Texts in Mathematics, 107, Springer-Verlag, New York, 1986
[15] B. O. Palsson, On the dynamics of the irreversible Michaelis-Menten reaction mechanism, Chem. Eng. Sci. 42 (1987) 447-458
[16] B. O. Palsson and E. N. Lightfoot, Mathematical modelling of dynamics and control in metabolic networks. I. On Michaelis-Menten kinetics, J. theor. Bio. 111 (1984) 273-302
[17] A. J. Roberts, Computer algebra derives correct initial conditions for lowdimensional dynamical systems, arXiv: chao-dyn/9901010
[18] M. Valorani and D. A. Goussis, Explicit time-scale splitting algorithm for stiff problems: auto-ignition of gaseous mixtures behind a steady shock, J. Comp. Phys. 169 (2001) 44-79
[19] M. Valorani, H. M. Najm, and D. A. Goussis, CSP analysis of a transient flame-vortex interaction: time scales and manifolds, Combustion and Flame 134 (2003) 35-53
[20] A. Zagaris, H. G. Kaper, and T. J. Kaper, Analysis of the Computational Singular Perturbation reduction method for chemical kinetics, J. Nonlin. Sci. (to appear); also available at arXiv: math.DS/0305355

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