# Optimal Control of Multiple Robot Systems with Friction using MPCC

Jufeng Peng
Mathematical Sciences
Rensselaer Polytechnic Institute
Troy, NY 12180
Email: pengj@rpi.edu

Mihai Anitescu
Mathematics and Computer Science
Argonne National Lab
Argonne, IL 60439
Email: anitescu@mcs.anl.gov

Srinivas Akella
Computer Science
Rensselaer Polytechnic Institute
Troy, NY 12180
Email: sakella@cs.rpi.edu

***Abstract*— This paper studies optimal control of multiple robot systems with frictional contact. The robots have nonlinear dynamics, which may arise from the robot body dynamics, friction between robot and environment, and friction between robot and robot. Nonpenetration constraints between robots are imposed, and the robots are assumed rigid. The problem is modelled as a mathematical program with complementarity constraints (MPCC). The MPCC model is solved using its elastic mode, which is a well-behaved nonlinear programming problem. Preliminary results with this approach are illustrated on example problems. The main contributions of this paper are: (1) a novel optimal control model that can deal with friction in the multiple robot system; (2) application of a new mathematical programming algorithm to solve the MPCC model effectively. The coordination model has potential applications in robot systems with friction, such as multi-finger manipulation, manipulation with ropes, and multi-robot pushing coordination.**

## I. Introduction

Consider a scenario where a robot hand with multiple fingers is above a table on which multiple coins are resting. The goal is to control the robot fingers so they can manipulate all the coins to specified target regions. Each finger may contact multiple coins, and the coins may contact each other during manipulation. Our long term goal is to automatically generate optimal controls for such tasks involving multiple objects, friction, and contact.

Coordinating multiple robots with simultaneous kinematic and dynamics constraints has applications in a variety of tasks. Examples include manipulation of multiple objects by multiple fingers [16], manipulation of objects by multiple disk-shaped robots [29] and even the manipulation of multiple objects by robots using ropes [5]; a rope can be modelled as a chain of rigid bodies that contact the objects and robots.

This paper addresses the following coordination problem: Given a system of $n$ robots, $\mathcal{A}_1, \ldots, \mathcal{A}_n$, each robot is required to move into a specified target region with a target velocity from its specified initial position and initial velocity. The optimal control problem is to generate the control inputs (the applied external forces) and robot paths so that a cost function, such as completion time or fuel consumption of the robot system, is minimized. The robots motions must satisfy dynamics constraints, including the friction constraints and control constraints, and kinematics constraints, such as collision avoidance or nonpenetration constraints.

We build the optimal control model using a mathematical program with complementarity constraints (MPCC) formulation. We then apply a newly developed mathematical programming algorithm [1]to solve the MPCC by transforming it to a well-behaved nonlinear programming problem known as its elastic mode.

To illustrate the approach, we consider a simplified model where the robots are circular disks with uniform density. We also assume that the control, which is an external force, can be applied at the center of mass of each robot. The translational support friction force is opposite to the velocity of the disk robot due to the symmetry of the bottom face of the disk robot. The rotational support friction of each robot can be modelled through a damper friction proportional to the size of the robot. In our paper, only minimal time control is used to illustrate our coordination strategy; however other objectives can be optimized by just changing the model objective.

Friction is an especially important factor in multiple robot control and manipulation. Contact friction can be described using complementarity constraints [10], [30], [28], [3]. We say that two vectors $a$ and $b$ are complementary ( $a \perp b$ ) if they satisfy the constraints

$$a \geq 0 \quad b \geq 0 \quad a^T b = 0$$

or equivalently

$$a \geq 0 \quad b \geq 0 \quad a^T b \leq 0$$

The modelled contact-friction problems are called (linear or nonlinear) complementarity problems and can be expressed in the form.

$$F(u, v, w) = 0; \quad u \geq 0, \quad v \geq 0, \quad u^T v \leq 0,$$

where $u, v$ are vectors.

A mathematical program with complementarity constraints (MPCC) formulation is an optimization problem in $u, v, w$ that has complementarity constraints:

$$\begin{array}{rlcl} \text{minimize} & \phi(u,v,w) & & \\ \text{subject to} & F(u,v,w) & = & 0 \\ & u & \geq & 0, \\ & v & \geq & 0, \\ & u^T v & \leq & 0. \end{array}$$

The MPCC is significantly more difficult than the complementarity problem, and solving MPCCs is an active area of research. Recent algorithms to solve MPCCs potentially provide new techniques to solve optimal control problems for multiple robot systems. As such, MPCCs can extend the class of solvable problems beyond the multibody dynamics simulations currently solved by complementarity formulations.

## II. Related Work

Related work falls into the categories of dynamics for the manipulation of multiple bodies, and motion planning and coordination of multiple robots. For an overview of recent research in robotic manipulation, including pushing, see [20].

Motion planning for multiple robots is a broad research area (see [17] for an overview). Recent efforts have focused on using probabilistic approaches [32], [24], [15], [18]. However, none of these papers addresses the system optimization with friction.

Dynamics problems with frictional contact have been extensively studied in the past couple of decades and complementarity constraints have been shown to be an effective way to model the contact friction [4], [10], [11], [21], [22], [30], [13], [3], [28], [27].

Dynamics problems with frictional contact have been done previously by solving the corresponding complementarity problems successfully[10], [30], [28], [3]; however due to a dramatic increase in the level of difficulty of solving an mathematical problem with complementarity constraints (MPCC) as opposed to solving a complementarity problem, the optimal control problem of a multiple robot system that involves intermittent contact friction has rarely been addressed.

Mathematical programs with complementarity constraints (MPCC), also known as mathematical programs with equilibrium constraints are an important class of constrained optimization problem arising from economics and engineering [19]. In general, MPCCs are difficult to solve due to the nonconvexity and the lack of constraint qualification [25]; however, recently it has been shown in [7], [6] and [1] that nonlinear programming solvers can solve a large set of MPCC successfully by its elastic mode, which lumps all complementarity constraints into one nonlinear constraint.

## III. Problem Formulation

To describe our multiple robot system, we introduce the notation in Table I, for the planar disk-robots $\mathcal{A}^i$ and $\mathcal{A}^j$ (Figure 1).

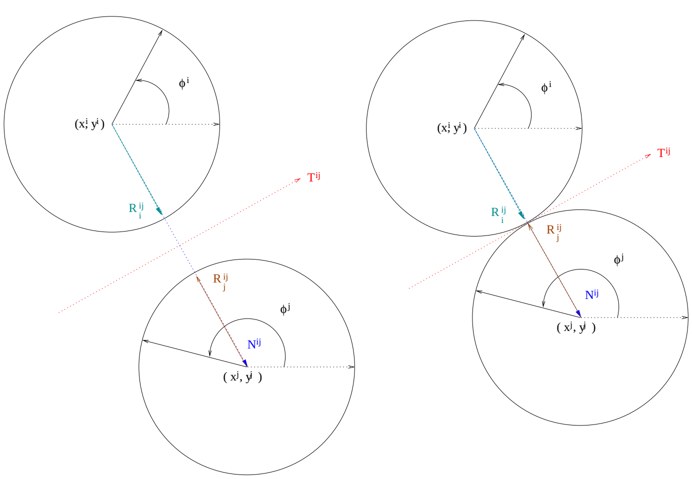


Fig. 1. two-robot system modes: noncontact and contact

TABLE I

NOTATION

| Notation | Description |
| --- | --- |
| $(\cdot)^i$ | property of robot $\mathcal{A}_i$ |
| $R$ | radius |
| $q = (x, y, \phi)^T$ | configuration of robot |
| $\vec{v}_t = (\dot{x}, \dot{y})^T = (v_x, v_y)^T$ | translational velocity |
| $v_r = \dot{\phi}$ | rotational velocity of robot |
| $\vec{v} = (\dot{x}, \dot{y}, \dot{\phi})^T = (v_t, v_r)^T$ | the velocity of robot |
| $\vec{N}^{ij} = \begin{pmatrix} x^j - x^i \\ y^j - y^i \end{pmatrix}$ | the vector from center of $\mathcal{A}^i$ to center of $\mathcal{A}^j$ |
| $\vec{T}^{ij} = \begin{pmatrix} -(y^j - y^i) \\ x^j - x^i \end{pmatrix}$ | the tangential vector, which is perpendicular to $\vec{N}^{ij}$ |
| $\vec{R}_i^{ij} = R^i \vec{N}^{ij}_{unit}$ | vector of the radius $i$ in the normal direction |
| $\vec{R}_j^{ij} = -R^j \vec{N}^{ij}_{unit}$ | vector of the radius $j$ in the normal direction |
| $o^{ij} = \lvert\lvert\vec{N}^{ij}\rvert\rvert$ | distance between center of $\mathcal{A}^i$ and center of $\mathcal{A}^j$ |
| $\rho^{ij} = o^{ij} - R^i - R^j$ | distance between the two disks |
| $c_n^{ij}$ | the scaled magnitude of the normal contact force |
| $\vec{f}_n^{ij} = c_n^{ij} \vec{N}^{ij}$ | contact force from $\mathcal{A}^i$ to $\mathcal{A}^j$, |
| $\vec{f}_T = \mu c_n \vec{T}^{ij}$ | tangential force |

### A. Dynamics between Robot and Environment

In this paper, the friction dynamics between each robot and environment is modelled as friction between robot and a horizontal floor. We decompose the translational friction and rotational friction, and approximate them separately. We can show that the translational friction force is opposite to the velocity of the disk robot due to the symmetry of the bottom face of the disk robot. The rotational friction of each robot can be described through a damper friction proportional to the size of the robot.

1) **Translational friction**
   Denote the weight of the robot $i$ by $G^i$, we have a tangential friction force acting between the body and the floor.

$$f_T^i = \begin{cases} -\mu^i G^i \frac{(\dot{x}^i, \dot{y}^i)^T}{||(\dot{x}^i, \dot{y}^i)||_2} & \text{if } \left|\left|(\dot{x}^i, \dot{y}^i)\right|\right|_2 \neq 0 \\ (\hat{\beta}_1^i, \hat{\beta}_2^i)^T, & \text{if } \left|\left|(\dot{x}^i, \dot{y}^i)\right|\right|_2 = 0. \\ \quad \text{here } \left|\left|(\hat{\beta}_1^i, \hat{\beta}_2^i)\right|\right|_2 \leq \mu^{(i)} G^{(i)} & \end{cases}$$

   $f_T^i$ satisfies, among others, a circular constraint that cannot be expressed by a finite number of linear constraints. Following the standard approach in the robotics literature [23], where the circle is approximated by a equilateral K-polygon $\mathcal{P}$, where $K$ is the number of vertices, whose half-diagonals are the vectors $\vec{l}_k^{if}$. The friction force between the robot and the floor can be described in the dynamics case using the maximum dissipation principle [28], [27]:

$$f_T^i = \sum_{k=1}^{K} \beta_k^{if} \vec{l}_k^{if}$$
$$\text{where } \left(\beta_k^{if}\right)_{k=1,2,\ldots,K} \text{ solve}$$
$$\text{minimize} \quad (\sum_{k=1}^{K} \beta_k^{if} \vec{l}_k^{if})^T \left(\dot{x}^i, \dot{y}^i\right)^T$$
$$\text{subject to: } \sum \beta_k^{if} \leq \mu^i G^i$$
$$\beta_k^{if} \geq 0 \quad \text{ for } k = 1, 2, \ldots, K$$

We use the Karush–Kuhn–Tucker optimality condition to represent the solution of the above minimization problem as a complementarity problem.

$$f_T^i = \sum_{k=1}^{K} \beta_k^{if} \vec{l}_k^{if}, \quad \text{where}$$
$$0 \leq \beta_k^{if} \quad \perp \quad \vec{l}_k^{if^T} \left(\dot{x}^i, \dot{y}^i\right)^T + \gamma^{if} \geq 0$$
$$\text{for } k = 1, 2, \ldots, K$$
$$0 \leq \mu^i G^i - \sum_{k=1}^{K} \beta_k^{if} \quad \perp \quad \gamma^{if} \geq 0$$

where $\gamma^{if}$ is the Lagrange multiplier of the optimality system.

2) **Rotational Friction**
The distributed friction forces will also generate a frictional torque which is the integral of all torques. Since the density of the friction force is proportional to $Mg$, it follows that the torque induced by the friction forces is proportional to the weight $Mg$. We approximately model this friction torque as damping friction,

$$f_r = -\sigma_d M g \dot{\phi}$$

### B. Kinematics and Dynamics between Robot and Robot

These constraints require that two bodies do not penetrate each other. For the case of two robots considered here, we have that

$$\rho^{ij} = \sqrt{(x^i - x^j)^2 + (y^i - y^j)^2} - R^i - R^j \geq 0$$

When there exists contact, penetration between bodies is prevented by a normal force $\vec{f}_n^{ij}$ along the vector $\vec{N}^{ij}$, with a multiplier, $c_n^{ij} \geq 0$. In other words, the normal force is $c_n^{ij} \vec{N}^{ij}$, although $\vec{N}^{ij}$ is not a unit vector. To completely quantify the contact configuration, and the fact that $c_n^{ij} = 0$ when there is no contact, we use the complementarity constraint

$$0 \leq \rho^{ij} \quad \perp \quad c_n^{ij} \geq 0.$$

In the local coordinate frame of the robot, the tangential force is $\vec{f}_T^{ij} = \alpha^{ij} \vec{T}^{ij}$, where $\vec{T}$ is the tangential vector and $\alpha^{ij}$ is the scaled magnitude of the friction force. Following the generalized coordinate notation in [14], the relative tangential velocity between body 1 and body 2 are

$$v_T^{ij} = (\Psi(\vec{T}_1^{ij}))^T \left(\dot{x}^j, \dot{y}^j, \dot{\phi}^j, \dot{x}^i, \dot{y}^i, \dot{\phi}^i\right)^T$$

where $\Psi(\vec{a}) = \begin{pmatrix} \vec{a} \\ \vec{R}_i^{ij} \times \vec{a} \\ -\vec{a} \\ \vec{R}_j^{ij} \times \vec{a} \end{pmatrix}$. Here the two-dimensional cross product $\times$ is defined as: $\vec{r} \times \vec{a} = det([r, a])$.

For the Coulomb friction model, the multiplier $\alpha^{ij}$ must satisfy

$$\alpha^{ij} = \begin{cases} \mu^{ij} c_n^{ij} & & v_T^{ij} < 0 \\ t, & t \in [-\mu^{ij} c_n^{ij}, \mu^{ij} c_n^{ij}] & v_T^{ij} = 0 \\ -\mu^{ij} c_n^{ij} & & v_T^{ij} > 0 \end{cases}$$

Here $\mu^{ij}$ is the friction coefficient between bodies $\mathcal{A}^i$ and $\mathcal{A}^j$.

The Coulomb friction model can be represented using complementarity constraints using the maximum dissipation principle [10], [30], [28], [3].

$$\begin{aligned} 0 \leq c_n^{ij} &\quad \perp \quad \rho^{ij} \geq 0 \\ 0 \leq \alpha_1^{ij} &\quad \perp \quad \Psi(\vec{T}_1^{ij})^T v^{ij} + \gamma^{ij} \geq 0 \\ 0 \leq \alpha_2^{ij} &\quad \perp \quad \Psi(\vec{T}_2^{ij})^T v^{ij} + \gamma^{ij} \geq 0 \\ 0 \leq \mu c_n^{ij} - \alpha_1^{ij} - \alpha_2^{ij} &\quad \perp \quad \gamma^{ij} \geq 0 \end{aligned}$$

where $\rho^{ij}$ is the distance function between the two bodies, $\left|\Psi(\vec{T}_1^{ij})^T v^{ij}\right|$ is the magnitude of the relative tangential velocity at the contact point, $\gamma^{ij}$ is the Lagrange multiplier, which will be equal to $\left|\Psi(\vec{T}_1^{ij})^T v^{ij}\right|$, except for the degenerate cases.

### C. The Dynamics Equations

We now write the dynamics equations for robot $\mathcal{A}_j$.

$$\begin{aligned} m^j \dot{v}_t^j &= \sum_{k=1}^{K} \beta_k^{jf} d_k^{jf} + \quad F^j \\ &+ \sum_{i \neq j} (\alpha_1^{ij} - \alpha_2^{ij}) \vec{T}^{ij} + \sum_{i \neq j} c_n^{ij} \vec{N}^{ij} \\ I^j \dot{\phi}^j &= \sum_{i \neq j} \vec{R}_j^{ij} \times (-f_n^{ij} - f_T^{ij}) - \sigma_d M^j \dot{\phi}^j \\ &= \sum_{i \neq j} -\alpha_1^{ij} R^j o^{ij} + \alpha_2^{ij} R^j o^{ij} - \sigma_d M^j \dot{\phi}^j \end{aligned}$$

In our approach, the external forces $F^i, i = 1 \ldots n$, represent the control variables. We assume the force on each robot can be applied individually, although the magnitude of the force cannot exceed a specified value $F_{max}$. The $\infty$-norm is used in our implementation, i.e., $|F_x^i| \leq F_{max}, \quad |F_y^i| \leq F_{max}$, although other norms are easy to accommodate.

### D. Mathematical Programs with Complementarity Constraints (MPCC) Formulation for Multi-robot System with Approximated Friction

The full minimal time optimal control model for multiple robot coordination is now obtained by combining the contents in the previous subsections. Let $\tilde{z} = (q, v, \gamma, \alpha, \beta, c_n, T_f)$, i.e.,

$\tilde{z}$ is a vector by stacking all variables and the unknown terminal time $T_f$. The optimal control model can be abbreviated as follows:

$$\begin{aligned}
&\text{minimize} \quad T_f\\
&\text{subject to}\\
&\qquad \tilde{M}\dot{\tilde{z}} = \tilde{H}(\tilde{z})\\
&\qquad \tilde{\mathcal{F}}_2(\tilde{z}) = 0\\
&\qquad \tilde{\mathcal{F}}_1(\tilde{z}) \le 0\\
&\qquad 0 \le \tilde{G}_1(\tilde{z}) \quad \perp \quad \tilde{G}_2(\tilde{z}) \ge 0
\end{aligned}$$

The dynamics, inequality, equality and complementarity constraints are described using appropriate vector functions.

We now use a semi-implicit finite difference discretization method to solve the dynamics equations numerically [3]. The discretized formulation then can be written as:

$$\begin{aligned}
&\text{minimize} \quad T_f\\
&\text{subject to}\\
&\qquad Mz = H(z)z\\
&\qquad \mathcal{F}_2(z) = 0\\
&\qquad \mathcal{F}_1(z) \le 0\\
&\qquad 0 \le G_1(z) \quad \perp \quad G_2(z) \ge 0
\end{aligned} \tag{1}$$

where $z$ is a vector by stacking together all $\tilde{z}^t$ at each time step $t = 1, \ldots, N_s$. The vector functions representing the dynamics, inequality, equality and complementarity constraints are updated correspondingly. We append the full discretized model in Appendix **A**.

## IV. Algorithms for Solving MPCCs

### *A. Elastic Mode for a Nonlinear Programming Problem*

Given a nonlinear programming problem,

$$\begin{aligned}
&\text{minimize} \quad \phi(z)\\
&\text{subject to}\\
&\qquad \mathcal{W}_1(z) \le 0 \qquad \mathcal{W}_2(z) = 0
\end{aligned} \tag{2}$$

where $z$ is the vector variable, $\phi(z)$ is the objective functions, and $\mathcal{W}_1, \mathcal{W}_2$ are the constraint functions. The elastic mode for a general nonlinear program via an $L_1$ penalty function can be presented as follows:

$$\begin{aligned}
&\text{minimize} \quad \phi(z) + \zeta e^T \epsilon\\
&\text{subject to}\\
&\qquad \mathcal{W}_1(z) \le \epsilon_1 \quad -\epsilon_2 \le \mathcal{W}_2(z) \le \epsilon_3\\
&\qquad \epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)^T
\end{aligned} \tag{3}$$

where $e = (1, 1, \ldots, 1)^T$ with appropriate dimension and $\zeta$ is a penalty parameter.

Solving an elastic mode of a nonlinear programming problem (3) instead of solving that nonlinear programming problem itself (2) is a penalty approach; the method has been proved successful for various nonlinear programming problems [19] [9], especially if the nonlinear programming problem (2) is not well-behaved, i.e., does not have constraint qualification, which is a necessary condition for convergence of many nonlinear programming algorithms.

### *B. Solving MPCC using its Elastic Mode*

The MPCC formulation of the discretized nonsmooth coordination problem (1) can be rewritten as:

$$\begin{aligned}
&\text{minimize} \quad T_f\\
&\text{subject to}\\
&\qquad Mz = H(z)z\\
&\qquad \mathcal{F}_2(z) = 0\\
&\qquad \mathcal{F}_1(z) \le 0\\
&\qquad G_1(z) \ge 0 \quad G_2(z) \ge 0\\
&\qquad G_1^T(z)G_2(z) \le 0
\end{aligned} \tag{4}$$

The above problem is considerably more difficult to solve than a typical nonlinear program. The difficulty originates in the complementarity constraints, which implies the feasible set does not have an interior. In other words, every feasible solution of the above MPCC is on the boundary of the solution set. This difficulty results in the fact that MPCC does not satisfy a constraint qualification. In practical terms, this implies that the linearization of the feasible set may be infeasible arbitrarily close to the solution, which may result in abnormal termination of most nonlinear programming algorithms, a fact that has been amply demonstrated in practice for many otherwise robust packages.

Nevertheless, it has been recently shown that certain approaches can handle this degeneracy quite well. Of particular interest here is the elastic mode of the MPCC. Let

$$\mathcal{W}_1(z) = \begin{pmatrix} \mathcal{F}_1(z) \le 0 \\ G_1(z) \ge 0 \\ G_2(z) \ge 0 \\ G_1^T(z)G_2(z) \le 0 \end{pmatrix}$$
$$\mathcal{W}_2(z) = \begin{pmatrix} Mz = H(z)z \\ \mathcal{F}_2(z) = 0 \end{pmatrix}$$

then MPCC (1) is transformed into the nonlinear programming form (2). Now we can solve the elastic mode of (2), which is a well-behaved problem (3), using standard nonlinear programming solvers.

It was shown in [1] that, under reasonable assumptions and for a sufficiently large but finite penalty parameter, the MPCC and its elastic mode (3) will have the same solution. The main advantage of the elastic mode (3), however, is that it can be shown that it has a nonempty interior of the feasible set and the main barrier in the path of most solvers is now removed.

Now we give some practical notes about solving the models. Finding an initial feasible solution is critical to the solver performance; a feasible initial solution for our MPCC model can be found by fixing a terminal time $T_f$ that is large enough for the solver to easily find a solution by solving a complementarity problem. Almost every nonlinear algorithm can be trapped in local minimum. We implement an inexact algorithm to search for the global optimal solution by reducing the objective. The *inexact* refers to the fact that the rate of the objective reduction is inexact and adaptive.

## V. Experimental Results

Several experiments were conducted on multiple robot coordination problems. The MPCC formulation and the inexact reducing-objective algorithm are implemented in AMPL [8]. The resulting models are solved using SNOPT 6.1 [9], which is a highly effective nonlinear programming solver that uses a smooth augmented Lagrangian merit function and makes explicit provision for infeasibility in the original problem and the QP subproblems. All runs were made on a Linux workstation with 2GB memory and four Intel(R) XEON(TM) 2.00GHz CPU. Table II defines the notation used in all tables of results in this section.

TABLE II

Description of headers of the result tables

| Symbol | Description |
|---|---|
| $Nr$ | Number of robots |
| $D_{ef}$ | Dimension of the external force(s) |
| $N_t$ | Number of time steps |
| $K$ | K-polygon approximation for circle |
| $N_v$ | Number of variables |
| $N_{tc}$ | Number of total constraints |
| $G_s$ | Global search solution |
| $LB$ | Lower bound of the optimal solution |
| $G_t$ | Global search time (seconds) |

The robots have specified initial positions and velocities, and each robot is required to move into a specified final region. In the following data tables, the lower bound solutions are obtained based on the bang-bang optimal solutions for individual robots without considering the collision avoidance or nonpenetration constraints. We only report the global search time in the tables; however the program may find a good local optimal solution using much less time. For some cases, even the first local solution is very close to the final search solution.

We have two test example sets A and B. For examples in set A, robot-floor friction exists, but robots are not permitted to contact each other. For examples in set B, both robot-robot contact friction and robot-floor friction exist.

1) Set A: Each pair of robots is required to have a minimal safe distance $d_s > 0$. In our experiments, $d_s = 0.01$. See Table III for the problems in set A and Table IV for computational results for set A. The snapshots of the coordination of six robots with collision avoidance are shown in Figure 2. We include two examples, A1 and A2, of system with only one robot to show that the computational results are very close to the theoretical results; indeed the computational results are bang-bang solutions. (A1 and A2 are the only two examples that have theoretical solutions in our test examples. )
2) Set B: Robot-robot contact is allowed in the test examples. See Table V for the problems in set B and Table VI for computational results for set B. The animation of three robots coordination with contact friction is shown in Figure 3.

TABLE III

Test examples in set A

| Problem | $N_r$ | $D_{ef}$ | $N_t$ | $K$ | $N_v$ | $N_{tc}$ |
|---|---|---|---|---|---|---|
| A1 | 1 | 2 | 100 | 16 | 4385 | 2693 |
| A2 | 1 | 1 | 100 | 16 | 4385 | 2693 |
| A3 | 4 | 2 | 50 | 8 | 11248 | 8877 |
| A4 | 6 | 2 | 50 | 4 | 20177 | 17220 |

TABLE IV

Computational results of examples in set A

| Problem | $G_s$ | $G_t$ (secs) | $LB$ |
|---|---|---|---|
| A1 | 9.03 | 5.8 | 8.94 |
| A2 | 9.85 | 98.45 | 9.75 |
| A3 | 9.60 | 20749.2 | 8.944 |
| A4 | 9.71 | 12605.14 | 8.94 |

TABLE V

Test examples in set B

| Problem | $N_r$ | $D_{ef}$ | $N_t$ | $K$ | $N_v$ | $N_{tc}$ |
|---|---|---|---|---|---|---|
| B1 | 2 | 2 | 100 | 16 | 10665 | 7080 |
| B2 | 3 | 2 | 50 | 16 | 9392 | 6564 |
| B3 | 3 | 2 | 50 | 16 | 9392 | 6564 |

TABLE VI

Computational results of examples in set B

| Problem | $G_s$ | $G_t$ | $LB$ |
|---|---|---|---|
| B1 | 9.78 | 3157.97 | 8.94 |
| B2 | 9.48 | 4116.81 | 8.94 |
| B3 | 9.39 | 4834.96 | 8.94 |

For all problems, SNOPT finished with an *Optimal solution found* message. This shows that a Karush-Kuhn-Tucker stationary feasible point has been found for all cases by using the elastic mode, as predicted in [1]. Of course, we have no way of guaranteeing that this point is also a global minimum of the problem.

However, from Tables IV and VI we note that the objective function gap between the point that was found by the solver and the lower bound is always smaller than 1 and is sometimes as small as .07. This shows that we have obtained a high quality solution point. It is true that Tables IV and VI also show that the final point was not cheap to compute. However, we note that in our experience the results are obtained much faster than using disjunctive programming or integer programming to compute a local solution point. Indeed, using disjunctive programming to replace the complementarity constraints will result in having to deal with thousands of binary variables for problem A4, which seems an impossible task for a computer like the one we used, and quite possibly for any current computing architecture.

## VI. Applicability of MPCC

The MPCC formulation is potentially applicable to a broader class of problems, as illustrated below.

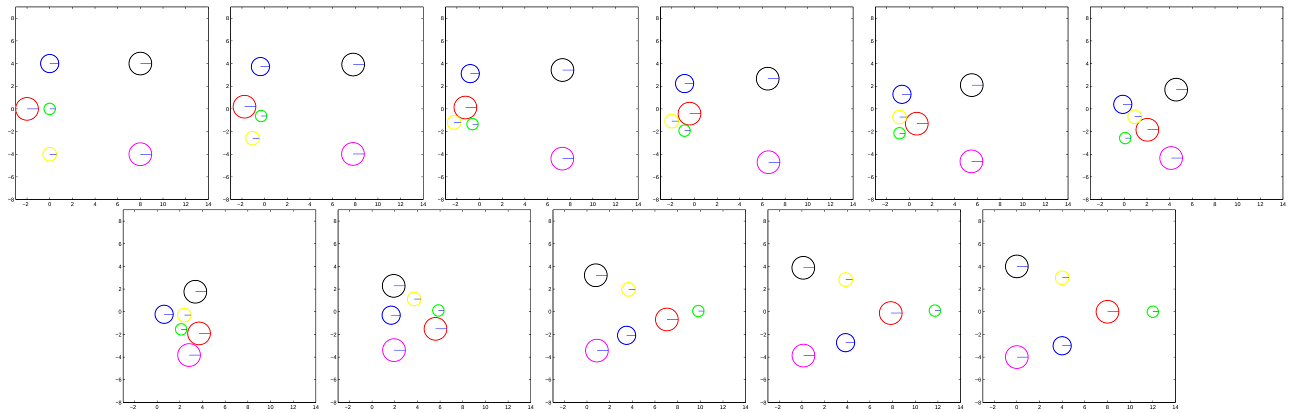

Fig. 2. Six robot coordination that has robot-fbor friction and a robot-robot minimal safe distance $d_s = 0.01$.

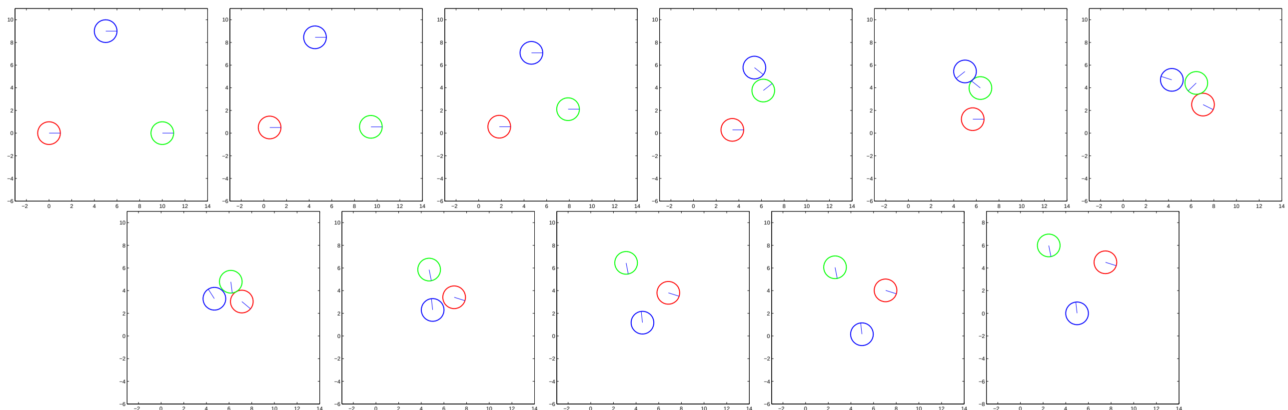

Fig. 3. Three robot coordination that has both robot-robot friction and robot-fbor friction: note the change in orientation arising from robot-robot contact

1) Motion planning problems in general have static and moving obstacles. Computing the distance between the each obstacle and robot involves solving a minimization problem, which in general can be written as complementarity constraints.
2) The high gear ratios of industrial manipulator actuators cause significant reductions of the effective joint torques due to friction. In [26], [12], the manipulator drive friction torque $\tau_q$ is given by

$$\tau_q(\beta, \tau) = \begin{cases} sign(\beta)\kappa_0 & \text{if } \beta \neq 0; \\ sign(\tau)\min(\kappa_0, |\tau|) & \text{if } \beta = 0; \end{cases}$$

where $\beta$ is the rotor velocity, $\tau$ is the rotor torque, $\kappa_0$ models the friction. $sign(\beta), sign(\tau)$ and $\min(\kappa_0, |\tau|)$ can be described using complementarity constraints [31].

## VII. Conclusions

We built a novel optimal control model for coordinating multiple robots with friction based on mathematical program with complementarity constraints (MPCC). A new mathematical programming algorithm is applied to solve the MPCC problem by solving the elastic mode of that MPCC problem. Numerous dynamics simulations of multibody systems with contact friction have been studied previously in robotics and mechanics. However, to the best of our knowledge, there has been no previous reported work on optimal control of multiple robot systems involving contact friction.

MPCC is a convenient way to represent the optimal control model for robot systems with friction. Our preliminary results show that solving MPCC is computationally challenging, though it still compares favorably to most alternatives, in particular disjunctive or integer programming approaches. Therefore we believe that an important future direction is improving the efficiency of algorithms to solve MPCCs.

The approach presented here can be immediately extended, in principle, to problems that have robots of other smooth shapes, for example elliptical. For such shapes, the depth of penetration function is smooth [2]. In addition, the depth of penetration can be represented by adding a few other constraints that involve the shape of the bodies. Extending the model to 3D is a natural next step, although we expect more computational challenges. Studying simpler problems, for example, where control can only be applied at the initial state, may give us theoretical guarantees of faster convergence. Additionally, it will be interesting to look at optimal control problems for hybrid systems that can be modelled as complementarity systems.

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## Appendix A: The Discretized Optimal Control Model for Multiple Robot Coordination using MPCC



$$\min \quad T_f$$

subject to : *(at each time step $t$)*

dynamics constraints of disk-robot $\mathcal{A}^i$ and fbor

$$0 \leq \beta_k^{if,t} \quad \perp \quad (d_k^{if})^T \cdot v^{if,t+1} + \gamma^{if,t} \geq 0 \quad k = 1 \ldots K$$

$$0 \leq (\mu g M^i - \sum_{k=1}^{K} \beta_k^{if,t}) \quad \perp \quad \gamma^{if,t} \geq 0$$

nonpenetration between $\mathcal{A}^i$ and $\mathcal{A}^j$:

$$\rho^{ij,t} = \sqrt{(x^{j,t} - x^{i,t})^2 + (y^{j,t} - y^{i,t})^2} - R^i - R^j \geq 0$$

dynamics related to $\mathcal{A}^i$ and $\mathcal{A}^j$

$$\vec{N}^{ij,t} = \begin{pmatrix} x^{j,t} - x^{i,t} \\ y^{j,t} - y^{i,t} \end{pmatrix}$$

$$o^{ij,t} = \sqrt{(x^{j,t} - x^{i,t})^2 + (y^{j,t} - y^{i,t})^2}$$

$$\rho^{ij,t} = o^{ij,t} - R^i - R^j$$

$$0 \leq c_n^{ij,t} \quad \perp \quad \rho^{ij,t}(q) \geq 0$$

$$\begin{aligned} \Psi(\vec{T}_1^{ij}) v^{ij,t+1} = & -(y^{j,t} - y^{i,t}) v_x^{j,t+1} + (x^{j,t} - x^{i,t}) v_y^{j,t+1} \\ & - R^j o^{ij,t} v_\phi^j + (y^{j,t} - y^{i,t}) v_x^{i,t+1} \\ & - (x^{j,t} - x^{i,t}) v_y^{i,t+1} - R^i o^{ij,t} v_\phi^i \end{aligned}$$

$$\begin{aligned} \Psi(\vec{T}_2^{ij}) v^{ij,t+1} = & (y^{j,t} - y^{i,t}) v_x^{j,t+1} - (x^{j,t} - x^{i,t}) v_y^{j,t+1} \\ & + R^j o^{ij,t} v_\phi^j - (y^{j,t} - y^{i,t}) v_x^{i,t+1} \\ & + (x^{j,t} - x^{i,t}) v_y^{i,t+1} + R^i o^{ij,t} v_\phi^i \end{aligned}$$

$$0 \leq \alpha_1^{ij,t} \quad \perp \quad \Psi(\vec{T}_1^{ij})^T \cdot v^{ij} + \gamma^{ij,t} \geq 0$$

$$0 \leq \alpha_2^{ij,t} \quad \perp \quad \Psi(\vec{T}_2^{ij})^T \cdot v^{ij} + \gamma^{ij,t} \geq 0$$

$$0 \leq \mu c_n^{ij,t} - \alpha_1^{ij} - \alpha_2^{ij,t} \qquad \perp \quad \gamma^{ij,t} \geq 0$$

time stepping

$$h = \frac{T_f}{N_s}$$

the Kinematics Equations

$$\begin{pmatrix} x^{j,t+1} - x^{j,t} \\ y^{j,t+1} - y^{j,t} \end{pmatrix} = v^{j,t+1} h$$

the Dynamics Equations

$$\begin{aligned} \frac{M^j (v_t^{j,t+1} - v_t^{j,t})}{h} = & \sum_{k=1}^{K} \beta_k^{jf,t} d_k^{jf,t} + \quad F^{j,t} \\ & + \sum_{i \neq j} (\alpha_1^{ij,t} - \alpha_2^{ij,t}) \vec{T}^{ij,t} + \sum_{i \neq j} c_n^{ij,t} \vec{N}^{ij,t} \end{aligned}$$

$$\begin{aligned} I^j \frac{(v_r^{j,t+1} - v_r^{j,t})}{h} & = \sum_{i \neq j} \vec{R}_j^{ij,t} \times (-f_n^{ij,t} - f_T^{ij,t}) - \sigma_d I^j \dot{\phi}^{j,t} \\ & = \sum_{i \neq j} -\alpha_1^{ij,t} R^j o^{ij,t} + \alpha_2^{ij,t} R^j o^{ij,t} - \sigma_d M^j v^{j,t} \end{aligned}$$

external force constraints

$$|F_x^{i,t}| \leq f_{max} \qquad |F_y^{i,t}| \leq F_{max}^i$$

the given initial values:

$$q^{i,0} = q_0^i \qquad v^{i,0} = v_0^i$$

the box constrained final values:

$$\begin{aligned} q_{low,end}^i \ & \leq \ q^i(T_f) \ \leq \ q_{upp,end}^i \\ v_{low,end}^i \ & \leq \ v^i(T_f) \ \leq \ v_{upp,end}^i \end{aligned}$$