# A linearly implicit trapezoidal method for integrating stiff multibody dynamics with contact, joints, and friction 

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#### Abstract

We present a hard constraint, linear complementarity based, method for the simulation of stiff multibody dynamics with contact, joints and friction. The approach uses a linearization of the modified trapezoidal method, incorporates a Poisson restitution model at collision, and solves only one linear complementarity problem per time step when no collisions are encountered. We prove that the method has order two, a fact that is also demonstrated by our numerical simulations. When we use a special approximation of the Jacobian matrix for the case where the stiff forces originate in springs and dampers attached to two points in the system, we prove that the linear complementarity problem can be solved for any value of the time step and that the method is stiffly stable. In particular, the numerical solution converges to the numerical solution of the constraint problem where the stiff springs and dampers have been replaced by rigid links, a fact that we also demonstrate by numerical simulations. The method was implemented in UMBRA, an industrialgrade virtual prototyping software.


Keywords: Multibody Dynamics, Rigid Bodies, Coulomb Friction, Stiff Methods, Linear Complementarity Problems, Linearly Implicitly Methods.

AMS Subject Classification: 65K10, 90C33.

## 1 Introduction

Mechanical systems composed of nominally rigid bodies interacting through frictional contacts are pervasive, and in fact, are responsible for the high levels of productivity and living standard enjoyed by citizens of modern industrialized countries. The most obvious example of such mechanical systems is the internal combustion engine, but many human-powered devices fall into this category too. Spurred by global economic competition and dwindling resources, companies have begun to use analysis and simulation software to optimize existing systems and to design highly efficient new systems. However, when the proper operation of the device relies on the formation and loss of frictional contacts at possibly unpredictable locations, and the stick-slip behavior of those contacts, the available software is neither efficient nor robust enough to be useful. Compelling applications that fall into this category range

[^0]from autonomous robots that explore and clean hazardous environments (see the left-hand picture in Figure 6), to intelligent prosthetic hands that can deftly manipulate objects based on high-level requests, to the simulations of surgical procedures for preoperative planning and risk-free training of surgeons, to the automation of manufacturing processes, including automatic fixturing (see Figure 6) or part-feeding systems [25].

When analyzing mechanical systems that experience intermittent contacts with dry friction, the rigid body assumption considerably simplifies the mathematical model. The deformable body assumption naturally leads to a system of partial differential equations (PDEs) which depend on material and geometric properties that are not easy to obtain, but are important for reasonably accurate system response, especially when collisions occur. Even without collisions, the modeling of the contact interfaces (which are distributed when the bodies are deformable) is extremely challenging. In fact, one of the key research areas in the ASCI (Advanced Scientific Computation Initiative) project at Sandia National Laboratories is the modeling and numerical analysis of contact interfaces to extend the capability of their suite of PDE solvers. Moreover, the numerical solution of these PDEs is extremely time consuming, so much so as to make design optimization on widely available computers impractical. Under the rigid body assumption, the dynamic equations and constraints of a multibody system can be formulated as a differential complementarity problem (DCP). The resulting model consists of a system of ordinary differential equations (ODE) subject to complementarity constraints (defined in Section 4). When bilateral constraints are also present, then the ODE is replaced by a system of mixed differential algebraic equations (DAE).

DCP formulations of dynamic mechanical system are advantageous in two main ways: first, they depends on a relatively small number of physical properties, and second, their forms are easily exploited in numerical integration schemes. However, when formulated in terms of body accelerations and forces ${ }^{1}$, these approaches also have the problem that the DCP may not have a solution, as has been pointed out by P. Painlevé [20] 1895. The example found by Painlevé shows in fact that the Coulomb friction model and the equations of classical rigid body dynamics are incompatible in the sense that models based on them may lead to continuous problems without solution in the classical sense. However, Mason and Wang were able to resolve the inconsistency by expanding the solution space to allow impulsive forces to act at times other than those when impacts occurred [16]. After a century of debate among scientists, it appears that the continuous problem of rigid multibody dynamics is better understood in terms of measure differential inclusions (see, e.g., [18]) and measure differential equations rather then in terms of ordinary differential equations. A comprehensive survey of rigid multibody dynamics from this point of view can be found in the book by Brogliato [6] and the review article of Stewart [31].

Over the past decade another way of dealing with the rigid multibody dynamics problem has come to the forefront of research. Instead of developing a continuous-time model that admits generalized solutions (in the sense of differential inclusions or measure differential equations), several researchers have built discrete-time models based on time-stepping formulations of the problem $[1,2,15,17,18,27$, 28]. Generally speaking, these time-stepping formulations are based on the idea of using time integrals of the forces (i.e., impulses) over each time-step, rather than trying to find the forces at each instant. This approach implicitly allows impulsive forces to act at any time during contact, not just at the moment an impact occurs. Formally they can be regarded as some sort of numerical integrator applied to the continuous formulation and a generalization of the method used to Mason and Wang to resolve Painlevé's Paradox. With regard to the former point, the methods of Anitescu and Potra [1] and Stewart and Trinkle [28] are based on a semi-implicit Euler scheme, while the model of [2] is based on a linearly implicit Euler scheme. All three formulations require the solution of one linear complementarity problem at each time step. The key difference between the methods presented in [1] and [28] are that in the latter, the non-penetration constraint is written in terms of positions and therefore includes a constraint

[^1]stabilization term. The method from [1] was recently modified to include constraint stabilization terms as well [4]. In the latter reference, it is proved that constraint stabilization is achieved and the energy stays bounded as the time step goes to zero, even if only one linear complementarity problem is solved per fixed time step, without assuming that the feasible set is an intersection of half-spaces, as was the case in [28].

A physically correct time-stepping formulation has to be dissipative, since the creation of energy in a system is physically inconsistent. At each step of the time-stepping scheme in [1], the kinetic energy cannot exceed that obtained in the same configuration with no contacts enforced. As a consequence, the resulting velocities are uniformly bounded on any finite time interval. This property is essential in proving the convergence of the numerical solution given by the time-stepping scheme as the step size goes to zero. The strongest convergence result to date was obtained by Stewart [26] who showed that under some restrictive assumptions (which are general enough to be satisfied by Painlevé's example and other problems of interest) the numerical trajectories produced by the time-stepping scheme converge, in some suitable sense, as the step size tends to zero, and that the corresponding limits satisfy all the conditions required in the instantaneous-time problem. More precisely, it is shown that for a subsequence, the positions converge uniformly, while the velocities converge pointwise. The impulses are used to construct measures for which there are subsequences that converge weak* to measures that are solutions to the corresponding measure differential inclusion.

The time-stepping scheme of [1] proved to be very robust in numerical simulations. However, in the presence of stiff springs and dampers attached to points of the multibody system, a prohibitively small time step had to be chosen to accommodate the related numerical stiffness. The time-stepping formulation of [2] was designed to properly handle such stiffness. Numerical experiments showed that this scheme could use relatively large time steps for systems with very stiff dampers and/or springs. It was shown that as the stiffness of the springs and dampers increase to infinity, the time-stepping formulation of [2] converges to a time-stepping formulation where the corresponding springs and dampers are replaced by rigid bodies.

The time-stepping formulations mentioned above are of first order. Since the trajectories of multibody systems with contact and friction are piecewise smooth, the accuracy of the formulation can be improved by using higher-order methods, provided a reliable collision detection procedure is available, thereby allowing larger steps to be taken by the algorithm. In the present paper, we present a timestepping formulation based on a linearly implicit trapezoidal method that provides new discrete-time LCP model of second order. Only one LCP has to be solved at each time step, as long as there are no collisions occurring. In this work, we do not address the stabilization of the nonlinear geometrical constraints. For this algorithm, constraint stabilization that does not significantly increase the energy, even when stiff forces are present, can be achieved by a projection technique, much as in [2].

Since it involves only one linear complementarity problem per step, the second-order model can be implemented with the same computational effort per time step as the first-order models described in $[1,2,28]$. We prove that the second model has the same properties concerning the kinetic energy as the first model of [1], that it can accommodate moderately stiff forces, and that it can accommodate arbitrarily stiff forces that originate in springs and dampers, as in [2], at the expense of falling back to an order one scheme. Our numerical experiments show that it can efficiently handle stiff problems. To our knowledge this the first time-stepping scheme of second order that requires the solution of only one LCP per time step. We mention that Tzitzouris [32] derived a time-stepping scheme based on the fully implicit trapezoidal method that requires the solution of nonlinear complementarity problem at each time step. The latter is solved by means of a nonsmooth Newton method, so that the computational complexity of Tzitzouris' scheme depends on the number of Newton iterations required at each time step. Moreover there are no results about the kinetic energy corresponding to the time-stepping scheme from [32].

Conventions. We denote by $\mathbb{N}$ the set of all nonnegative integers. $\mathbb{R}, \mathbb{R}_{+}, \mathbb{R}_{++}$denote the set of real, nonnegative real, and positive real numbers respectively. If $\|$.$\| is a vector norm on \mathbb{R}^{n}$ and $A$ is a matrix, then the operator norm induced by $\|$.$\| is defined by \|A\|=\max \{\|A x\| ;\|x\|=1\}$. If $x^{1}, x^{2}, \ldots, x^{m} \in \mathbb{R}^{n}$, then the column vector $z \in \mathbb{R}^{m n}$ obtained by concatenating the vectors $x^{k}$ will be denoted by $\left\lceil x^{1}, x^{2}, \ldots, x^{m}\right\rfloor$, i.e.,

$$
\begin{equation*}
z=\left\lceil x^{1}, x^{2}, \ldots, x^{m}\right\rfloor=\left[x^{1 T}, x^{2 T}, \ldots, x^{m T}\right]^{T} . \tag{1.1}
\end{equation*}
$$

If $x, y \in \mathbb{R}_{+}^{n}$ are two nonnegative vectors such that $x^{T} y=0$, then we say that $x$ and $y$ are complementary and we write $0 \leq x \perp y \geq 0$. The symbol $e$ represents the vector of all ones, $e=\lceil 1,1, \ldots, 1\rfloor$, with dimension given by the context.

If $y$ is a vector depending on a positive parameter $h$ then we write $y=O(h)$ to indicate that there is a constant $\bar{\alpha}>0$ such that $\|y\| \leq \bar{\alpha} h$. If $\gamma$ is a scalar depending on $h$ then we write $\gamma=\Omega(h)$ to indicate that there is a constant $\underline{\alpha}>0$ such that $\gamma \geq \underline{\alpha} h$.

If $f \in \mathbb{R}^{n}$ and $x \in \mathbb{R}^{m}$, then the Jacobian of $f$ with respect to $x$ is denoted by $f_{x}$. This is an $n \times m$ matrix. We will also use the notations $J_{x} f=f_{x}, \nabla_{x} f=f_{x}^{T}$. In case $n=1$, the column vector $\nabla_{x} f$ is the usual gradient of $f$.

## 2 The linearly implicit trapezoidal method

Take a generic differential equation

$$
\begin{equation*}
\frac{d y}{d t}=f(t, y) \tag{2.1}
\end{equation*}
$$

where $y \in \mathbb{R}^{n}, f: \mathbb{R} \times \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$. Given an approximation $y^{l}$ of the solution at time $t_{l}$, the classical trapezoidal method [12] defines an approximate solution at time $t_{l+1}=t_{l}+h$ as the solution of the following nonlinear system

$$
\begin{equation*}
y^{l+1}=y^{l}+\frac{h}{2}\left(f\left(t^{l}, y^{l}\right)+f\left(t^{l+1}, y^{l+1}\right)\right) . \tag{2.2}
\end{equation*}
$$

Let us consider the case of the dynamics equations in the absence of constraints

$$
\begin{equation*}
\frac{d q}{d t}=v ; \quad M(t, q) \frac{d v}{d t}=k(t, q, v) \tag{2.3}
\end{equation*}
$$

and assume that an approximation $z^{l}=\left\lceil q^{l}, v^{l}\right\rfloor$ of the solution at time step $t_{l}$ is given. In order to apply the trapezoidal method to (2.3), we first multiply the second equation with $M^{-1}(t, q)$ in order to obtain an equation of the form (2.1) and then we apply (2.2) to obtain an approximation $z_{T}^{l+1}=\left\lceil q_{T}^{l+1}, v_{T}^{l+1}\right\rfloor$ of the solution at time step $t_{l+1}$ as the solution of the nonlinear system

$$
\begin{align*}
q_{T}^{l+1} & =q^{l}+\frac{h}{2}\left(v^{l}+v_{T}^{l+1}\right)  \tag{2.4}\\
v_{T}^{l+1} & =v^{l}+\frac{h}{2}\left(M^{-1}\left(t_{l}, q^{l}\right) k\left(t_{l}, q^{l}, v^{l}\right)+M^{-1}\left(t_{l+1}, q_{T}^{l+1}\right) k\left(t_{l+1}, q_{T}^{l+1}, v_{T}^{l+1}\right)\right)
\end{align*}
$$

In what follows we will consider the following more convenient variant of the trapezoidal method

$$
\begin{align*}
q_{M T}^{l+1} & =q^{l}+\frac{h}{2}\left(v^{l}+v_{M T}^{l+1}\right)  \tag{2.5}\\
\left(M\left(t_{l}, q^{l}\right)+M\left(t_{l+1}, q_{M T}^{l+1}\right)\right)\left(v_{M T}^{l+1}-v^{l}\right) & =h\left(k\left(t_{l}, q^{l}, v^{l}\right)+k\left(t_{l+1}, q_{M T}^{l+1}, v_{M T}^{l+1}\right)\right)
\end{align*}
$$

We have denoted by the subscript $M T$ quantities connected to the modified trapezoidal method (2.5).

If the mass matrix $M$ is constant then (2.4) and (2.5) coincide. It is known that the trapezoidal method is a second order method [12]. This follows from the fact that if $z^{l}=z\left(t_{l}\right)$, where $z(t)=$ $\lceil q(t), v(t)\rfloor$ is the exact solution of $(2.3)$, then the truncation error $z_{T}^{l+1}-z\left(t_{l+1}\right)$ satisfies

$$
z_{T}^{l+1}-z\left(t_{l+1}\right)=O\left(h^{3}\right)
$$

In the next section we will prove that

$$
z_{M T}^{l+1}-z_{T}^{l+1}=O\left(h^{3}\right)
$$

which shows that the modified trapezoidal method (2.5) is also a second order method. The Jacobian of the nonlinear system (2.5) is easier to compute than that of of the nonlinear system (2.4). One Newton step with starting point $z^{l}=\left\lceil q^{l}, v^{l}\right\rfloor$ for (2.5) produces a point $z_{L T}^{l+1}=\left\lceil q_{L T}^{l+1}, v_{L T}^{l+1}\right\rfloor$ as the solution of the following linear system:

$$
\begin{align*}
q_{L T}^{l+1} & =q^{l}+\frac{h}{2}\left(v^{l}+v_{L T}^{l+1}\right)  \tag{2.6}\\
\widehat{M} v_{L T}^{l+1} & =\widehat{M} v^{l}+h \widehat{k}
\end{align*}
$$

where,

$$
\begin{align*}
\widehat{M} & =\frac{1}{2}\left(M\left(t_{l}, q^{l}\right)+M\left(t_{l+1}, q^{l}\right)-h k_{v}\left(t_{l+1}, q^{l}, v^{l}\right)-\frac{h^{2}}{2} k_{q}\left(t_{l+1}, q^{l}, v^{l}\right)\right)  \tag{2.7}\\
\widehat{k} & =\frac{1}{2}\left(k\left(t_{l}, q^{l}, v^{l}\right)+k\left(t_{l+1}, q^{l}, v^{l}\right)+h k_{q}\left(t_{l+1}, q^{l}, v^{l}\right) v^{l}\right) \tag{2.8}
\end{align*}
$$

The numerical integration method (2.6) is called the linearly implicit trapezoidal method. We have denoted by the subscript $L_{T}$ quantities connected to the modified trapezoidal method (2.6).

In the next section we will prove that

$$
z_{L T}^{l+1}=z_{M T}^{l+1}+ \begin{cases}O\left(h^{3}\right) & \text { if } M \text { is constant }  \tag{2.9}\\ O\left(h^{2}\right) & \text { in general }\end{cases}
$$

We deduce that the linearly implicit trapezoidal is also a second order method in the constant mass matrix case. Since Newton's method solves linear equations exactly in one step, it follows that it has the same linear stability properties as the classical trapezoidal method.

If constraints are added to the dynamics equations (2.3) then a linear complementarity problem has to be solved at each time-step, instead of a linear system. In this case, it will be essential to have a positive definite matrix $\widehat{M}$. Since $M$ is positive definite it follows that $\widehat{M}$ will be positive definite for sufficiently small $h$. However it is important that $\widehat{M}$ be positive definite for moderate values of $h$ so that larger integration steps can be taken. In order to ensure the positive definiteness of $\widehat{M}$, we will consider appropriate approximations of $\widetilde{k}_{q}^{l}$ and $\widetilde{k}_{v}^{l}$ of $k_{q}\left(t_{l+1}, q^{l}, v^{l}\right)$ and $k_{v}\left(t_{l+1}, q^{l}, v^{l}\right)$ which will lead to an approximate linearly implicit method of the form

$$
\begin{align*}
q^{l+1} & =q^{l}+\frac{h}{2}\left(v^{l}+v^{l+1}\right)  \tag{2.10}\\
\widetilde{M} v^{l+1} & =\widetilde{M} v^{l}+h \widetilde{k}
\end{align*}
$$

where,

$$
\begin{align*}
\widetilde{M} & =\frac{1}{2}\left(M\left(t_{l}, q^{l}\right)+M\left(t_{l+1}, q^{l}\right)-h \widetilde{k}_{v}^{l}-\frac{h^{2}}{2} \widetilde{k}_{q}^{l}\right)  \tag{2.11}\\
\widetilde{k} & =\frac{1}{2}\left(k\left(t_{l}, q^{l}, v^{l}\right)+k\left(t_{l+1}, q^{l}, v^{l}\right)+h \widetilde{k}_{q}^{l} v^{l}\right) \tag{2.12}
\end{align*}
$$

This integration scheme can be interpreted as the result of a Newton-like step applied to (2.5) where the Jacobian of (2.5) is approximated by

$$
A=\left[\begin{array}{cc}
I & -0.5 h I  \tag{2.13}\\
-h \widetilde{k}_{q}^{l} & M\left(t_{l}, q^{l}\right)+M\left(t_{l+1}, q^{l}\right)-h \widetilde{k}_{v}^{l}
\end{array}\right] .
$$

Let us denote by $z^{l+1}=\left\lceil q^{l+1}, v^{l+1}\right\rfloor$ the point produced by (2.10). From (2.6)-(2.8) and (2.10)-(2.12) we deduce that

$$
\begin{align*}
\left\|v^{l+1}-v_{L T}^{l+1}\right\| & =h\left\|\widetilde{M}^{-1} \widetilde{k}^{l}-\widehat{M}^{-1} \widehat{k}\right\| \\
& \leq h\left\|\widetilde{M}^{-1}\right\|\|\widetilde{k}-\widehat{k}\|+h\left\|\widetilde{M}^{-1}-\widehat{M}^{-1}\right\|\|\widehat{k}\|  \tag{2.14}\\
& =\left(\left\|\widetilde{k}_{q}^{l}-k_{q}\left(t_{l+1}, q^{l}, v^{l}\right)\right\|+\left\|\widetilde{k}_{v}^{l}-k_{v}\left(t_{l+1}, q^{l}, v^{l}\right)\right\|\right) O\left(h^{2}\right) \\
\left\|q^{l+1}-q_{L T}^{l+1}\right\| & =\frac{h}{2}\left\|v^{l+1}-v_{L T}^{l+1}\right\|  \tag{2.15}\\
& =\left(\left\|\widetilde{k}_{q}^{l}-k_{q}\left(t_{l+1}, q^{l}, v^{l}\right)\right\|+\left\|\widetilde{k}_{v}^{l}-k_{v}\left(t_{l+1}, q^{l}, v^{l}\right)\right\|\right) O\left(h^{3}\right)
\end{align*}
$$

## 3 The truncation error

In this section we will prove that the modified trapezoidal method (2.5) is a second order method in general, while the linearly implicit trapezoidal method (2.6) is a second order method in the constant mass matrix case, and only a first order method otherwise. We will also investigate the order of the approximate linearly implicit trapezoidal method (2.10). Our analysis is based on the celebrated Kantorovich Theorem [14], which gives sufficient conditions for the existence and uniqueness of solutions of a nonlinear operator equation and shows than under those conditions the sequences generated by Newton's method and the simplified Newton method converge to the solution.

Let $F: D \subset \mathbb{R}^{p} \rightarrow \mathbb{R}^{p}$ be a nonlinear operator defined on a domain $D$ of the p-dimensional linear space $\mathbb{R}^{p}$ with values in $\mathbb{R}^{p}$. Let $\|$.$\| be a given norm on \mathbb{R}^{p}$, and let $x^{0}$ be a point of $D$ such that the closed ball of radius $\rho$ centered at $x^{0}$,

$$
\begin{equation*}
\bar{B}\left(x^{0}, \rho\right):=\left\{x \in \mathbb{R}^{p}:\left\|x-x^{0}\right\| \leq \rho\right\} \tag{3.1}
\end{equation*}
$$

is included in $D$, i.e.,

$$
\begin{equation*}
\bar{B}\left(x^{0}, \rho\right) \subset D \tag{3.2}
\end{equation*}
$$

We assume that the Jacobian $F^{\prime}\left(x^{0}\right)$ is nonsingular and we denote

$$
\begin{equation*}
\sigma:=\left\|F^{\prime}\left(x^{0}\right)^{-1}\right\| \tag{3.3}
\end{equation*}
$$

We also assume that the Jacobian satisfies a Lipschitz condition of the form

$$
\begin{equation*}
\left.\| F^{\prime}(y)-F^{\prime}(\bar{y})\right)\|\leq \omega\| y-\bar{y} \|, \forall y, \bar{y} \in \bar{B}\left(x^{0}, \rho\right) \tag{3.4}
\end{equation*}
$$

The Kantorovich Theorem essentially states that if the quantity

$$
\begin{equation*}
\alpha:=\left\|F^{\prime}\left(x^{0}\right)^{-1} F\left(x^{0}\right)\right\| \tag{3.5}
\end{equation*}
$$

is small enough, in the sense that

$$
\begin{equation*}
\kappa:=\alpha \sigma \omega \leq \frac{1}{2} \tag{3.6}
\end{equation*}
$$

then there is a $x^{*}$ with $F\left(x^{*}\right)=0$ and the sequences produced by Newton's method

$$
\begin{equation*}
x^{k+1}=x^{k}-F^{\prime}\left(x^{k}\right)^{-1} F\left(x^{k}\right), k=0,1, \ldots \tag{3.7}
\end{equation*}
$$

and the simplified Newton method

$$
\begin{equation*}
x_{\mathcal{S}}^{0}=x^{0}, \quad x_{\mathcal{S}}^{k+1}=x_{\mathcal{S}}^{k}-F^{\prime}\left(x^{0}\right)^{-1} F\left(x_{\mathcal{S}}^{k}\right), k=0,1, \ldots \tag{3.8}
\end{equation*}
$$

are well defined and converge to $x^{*}$. More precisely we have
Theorem 3.1 (The Kantorovich Theorem) Let $F: D \subset \mathbb{R}^{p} \rightarrow \mathbb{R}^{p}$ be a differentiable mapping and let $z \in D$ be such that $F^{\prime}\left(x^{0}\right)$ is nonsingular. Assume that conditions (3.2), (3.4), (3.6) are satisfied and that the radius $\rho$ is large enough in the sense that

$$
\begin{equation*}
\widehat{\rho}:=\frac{2 \alpha \sigma}{1+\sqrt{1-2 \kappa}} \leq \rho . \tag{3.9}
\end{equation*}
$$

Then:

1. $F$ has a zero $x^{*}$ in the closed ball $\bar{B}\left(x^{0}, \widehat{\rho}\right)$;
2. The open ball $B\left(x^{0}, \check{\rho}\right)$ with radius

$$
\begin{equation*}
\check{\rho}:=\frac{2 \alpha \sigma}{1-\sqrt{1-2 \kappa}} \tag{3.10}
\end{equation*}
$$

does not contain any zero of $F$ different from $x^{*}$;
3. The iterative procedures (3.7) and (3.8) produce sequences included into the open ball $B\left(x^{0}, \widehat{\rho}\right)$ that converge to $x^{*}$;
4. If $\kappa<1 / 2$ then we have the following error bounds

$$
\begin{align*}
\left\|x^{k}-x^{*}\right\| & \leq \frac{2 \beta \lambda^{2^{k}}}{1-\lambda^{2^{k}}}  \tag{3.11}\\
\left\|z_{\mathcal{S}}^{k}-z^{*}\right\| & \leq \frac{2 \beta \lambda^{2}}{1-\lambda^{2}} \xi^{k-1} \tag{3.12}
\end{align*}
$$

with

$$
\begin{equation*}
\beta=\frac{\sqrt{1-2 \kappa}}{\sigma \omega}, \quad \lambda=\frac{2 \kappa}{(1+\sqrt{1-2 \kappa})^{2}}, \quad \xi=\frac{2 \kappa}{1+\sqrt{1-2 \kappa}} \tag{3.13}
\end{equation*}
$$

For a proof of the estimates (3.11) see [11] or [24], and for a proof the estimates (3.12) see [22].
The modified trapezoidal method (2.5) defines the point $z_{M T}^{l+1}=\left\lceil q_{M T}^{l+1}, v_{M T}^{l+1}\right\rfloor$ as the solution of the nonlinear system $F(z)=0$, where $F$ is the nonlinear operator

$$
z=\left[\begin{array}{l}
q  \tag{3.14}\\
v
\end{array}\right] \rightarrow F(z)=\left[\begin{array}{c}
q-q^{l}-h\left(v^{l}+v\right) / 2 \\
\left(M^{l}+M\left(t_{l+1}, q\right)\right)\left(v-v^{l}\right)-h\left(k^{l}+k\left(t_{l+1}, q, v\right)\right)
\end{array}\right]
$$

with

$$
M^{l}=M\left(t_{l}, q^{l}\right), \quad k^{l}=k\left(t_{l}, q^{l}, v^{l}\right) .
$$

The Jacobian of $F$ is given by

$$
F^{\prime}(z)=\left[\begin{array}{cc}
I & -0.5 h I \\
M_{q}\left(t_{l+1}, q\right) & \left(v-v^{l}\right)-h k_{q}\left(t_{l+1}, q, v\right)
\end{array} M^{l}+M\left(t_{l+1}, q\right)-h k_{v}\left(t_{l+1}, q, v\right) . .\right.
$$

In particular at the point $z^{l}=\left\lceil q^{l}, v^{l}\right\rfloor$ we have

$$
F^{\prime}\left(z^{l}\right)=\left[\begin{array}{cc}
I & -0.5 h I  \tag{3.15}\\
-h k_{q}\left(t_{l+1}, q^{l}, v^{l}\right) & M^{l}+M\left(t_{l+1}, q^{l}\right)-h k_{v}\left(t_{l+1}, q^{l}, v^{l}\right)
\end{array}\right] .
$$

Since $M^{l}$ is positive definite it follows that $F^{\prime}\left(z^{l}\right)$ is invertible for all sufficiently small $h$ and we have

$$
F^{\prime}\left(z^{l}\right)^{-1}=\left[\begin{array}{cc}
I & 0 \\
0 & \left(2 M^{l}\right)^{-1}
\end{array}\right]+O(h) .
$$

It follows that there is a constant $\bar{\sigma}$ such that

$$
\begin{equation*}
\left\|F^{\prime}\left(z^{l}\right)^{-1}\right\| \leq \bar{\sigma} \quad \text { for all sufficiently small } h \tag{3.16}
\end{equation*}
$$

Let us consider a ball centered at $z^{l}$ and with radius a given multiple of $h$.

$$
B=B\left(z^{l}, \rho\right), \quad \rho=O(h)
$$

Under standard smoothness assumptions it follows that there is a Lipschitz constant

$$
\omega= \begin{cases}O(h) & \text { if } M \text { is constant }  \tag{3.17}\\ O(1) & \text { in general }\end{cases}
$$

such that

$$
\begin{equation*}
\left\|F^{\prime}(z)-F^{\prime}(\bar{z})\right\| \leq \omega\|z-\bar{z}\|, \quad \text { for all } z, \bar{z} \in B \tag{3.18}
\end{equation*}
$$

We deduce that there is a constant $\check{\sigma}$ such that

$$
\begin{equation*}
\left\|F^{\prime}(z)\right\| \leq \check{\sigma},\left\|F^{\prime}(z)^{-1}\right\| \leq 2 \bar{\sigma} \quad \text { for all } z \in B \text { and all sufficiently small } h \tag{3.19}
\end{equation*}
$$

We have now all the ingredients for applying the Kantorovich Theorem to the integration methods presented in the previous section.

Theorem 3.2 Let $z_{T}^{l+1}=\left\lceil q_{T}^{l+1}, v_{T}^{l+1}\right\rfloor$ be a solution of the nonlinear system (2.4) defining the classical trapezoidal method. Then the nonlinear system (2.5) has a solution $z_{M T}^{l+1}=\left\lceil q_{M T}^{l+1}, v_{M T}^{l+1}\right\rfloor$ satisfying

$$
z_{M T}^{l+1}=z_{T}^{l+1}+O\left(h^{3}\right)
$$

Moreover $z_{M T}^{l+1}$ is the unique solution of (2.5) in an open ball centered at $z_{T}^{l+1}$ with radius

$$
\rho_{1}= \begin{cases}1 / \Omega(h) & \text { if } M \text { is constant }  \tag{3.20}\\ \Omega(1) & \text { in general }\end{cases}
$$

If $M$ is constant then $z_{M T}^{l+1}=z_{T}^{l+1}$.
Proof. If $z=z_{T}^{l+1}$ is the solution of the nonlinear equation (2.4), then the first component of $F(z)$ in (3.14) vanishes and the second component becomes

$$
\begin{aligned}
& \left(M^{l}+M\left(t_{l+1}, q\right)\right)\left(v-v^{l}\right)-h\left(k^{l}+k\left(t_{l+1}, q, v\right)\right) \\
= & \frac{h}{2}\left(M^{l}+M\left(t_{l+1}, q\right)\right)\left(\left(M^{l}\right)^{-1} k^{l}+M^{-1}\left(t_{l+1}, q\right) k\left(t_{l+1}, q, v\right)\right)-h\left(k^{l}+k\left(t_{l+1}, q, v\right)\right) \\
= & \frac{h}{2}\left(M^{l} M^{-1}\left(t_{l+1}, q\right)-I\right) k\left(t_{l+1}, q, v\right)+\frac{h}{2}\left(M\left(t_{l+1}, q\right)\left(M^{l}\right)^{-1}-I\right) k^{l}
\end{aligned}
$$

Since

$$
M\left(t_{l+1}, q\right)=M^{l}+A+O\left(h^{2}\right), \text { where A is a matrix such that }\|A\|=O(h)
$$

we have

$$
\begin{aligned}
& \left(M^{l} M^{-1}\left(t_{l+1}, q\right)-I\right) k\left(t_{l+1}, q, v\right)+\left(M\left(t_{l+1}, q\right)\left(M^{l}\right)^{-1}-I\right) k^{l} \\
= & A\left(\left(M^{l}\right)^{-1} k^{l}-M^{-1}\left(t_{l+1}, q\right) k\left(t_{l+1}, q, v\right)\right)+O\left(h^{2}\right) .
\end{aligned}
$$

Using the fact that $z_{T}^{l+1}=z^{l}+O(h)$ we deduce that

$$
\begin{equation*}
\left\|F\left(z_{T}^{l+1}\right)\right\|=O\left(h^{3}\right) \tag{3.21}
\end{equation*}
$$

It follows that the hypothesis of Kantorovich Theorem with $x^{0}=z_{T}^{l+1}$ is satisfied because according to (3.16), (3.17), (3.18), and (3.21) we have

$$
\kappa=\left\{\begin{array}{ll}
O\left(h^{4}\right) & \text { if } M \text { is constant } \\
O\left(h^{3}\right) & \text { in general }
\end{array} \quad<\frac{1}{2} \quad \text { for } h\right. \text { sufficiently small. }
$$

This implies that the nonlinear system (2.4) has a solution $z_{M T}^{l+1}$ such that

$$
\left\|z_{T}^{l+1}-z_{M T}^{l+1}\right\| \leq \widehat{\rho}=O\left(h^{3}\right)
$$

This solution is unique in the open ball $B\left(z_{T}^{l+1}, \check{\rho}\right)$ where according to (3.10) we have

$$
\check{\rho} \geq 2 \frac{1+\sqrt{1-2 \kappa}}{\omega}
$$

and the estimate (3.20) follows from (3.17).
Since the classical trapezoidal is an order two method so is the modified trapezoidal.
Theorem 3.3 If $z_{L T}^{l+1}=\left\lceil q_{L T}^{l+1}, v_{L T}^{l+1}\right\rfloor$ is given by the linearly implicit trapezoidal method (2.6) then the estimate (2.9) holds.

Proof. We apply the Kantorovich Theorem for $F(z)=0$ with $x^{0}=z^{l}$. In this case we have

$$
\begin{equation*}
\alpha \leq \bar{\sigma}\left\|F\left(z^{l}\right)\right\|=O(h) \tag{3.22}
\end{equation*}
$$

so that

$$
\kappa=\left\{\begin{array}{ll}
O\left(h^{2}\right) & \text { if } M \text { is constant }  \tag{3.23}\\
O(h) & \text { in general }
\end{array} \quad<\frac{1}{2} \quad \text { for } h\right. \text { sufficiently small }
$$

and the hypothesis of the Kantorovich Theorem is satisfied. One Newton step produces the point $x^{1}=z_{L T}^{l+1}$. According to (3.11) we have

$$
\left\|z_{L T}^{l+1}-z_{M T}^{l+1}\right\| \leq \frac{2 \beta \lambda^{2}}{1-\lambda^{2}}=\frac{4 \alpha \lambda \sqrt{1-2 \kappa}}{(1+\sqrt{1-2 \kappa})^{2}\left(1-\lambda^{2}\right)}= \begin{cases}O\left(h^{3}\right) & \text { if } M \text { is constant } \\ O\left(h^{2}\right) & \text { in general }\end{cases}
$$

Theorem 3.4 Let $z^{l+1}=\left\lceil q^{l+1}, v^{l+1}\right\rfloor$ be given by the approximate linearly implicit trapezoidal method (2.10).
i) If $\left\lceil\widetilde{k}_{q}^{l}, \widetilde{k}_{v}^{l}\right\rfloor=\left\lceil k_{q}\left(t_{l+1}, q^{l}, v^{l}\right), k_{v}\left(t_{l+1}, q^{l}, v^{l}\right)\right\rfloor+O(1)$, then

$$
z^{l+1}=z_{M T}^{l+1}+O\left(h^{2}\right)
$$

ii) If $\left\lceil\widetilde{k}_{q}^{l}, \widetilde{k}_{v}^{l}\right\rfloor=\left\lceil k_{q}\left(t_{l+1}, q^{l}, v^{l}\right), k_{v}\left(t_{l+1}, q^{l}, v^{l}\right)\right\rfloor+O(1)$, and the mass matrix $M$ is constant, then

$$
q^{l+1}=q_{M T}^{l+1}+O\left(h^{3}\right), \quad v^{l+1}=v_{M T}^{l+1}+O\left(h^{2}\right)
$$

iii) If $\left\lceil\widetilde{k}_{q}^{l}, \widetilde{k}_{v}^{l}\right\rfloor=\left\lceil k_{q}\left(t_{l+1}, q^{l}, v^{l}\right), k_{v}\left(t_{l+1}, q^{l}, v^{l}\right)\right\rfloor+O(h)$ and the mass matrix $M$ is constant, then

$$
z^{l+1}=z_{M T}^{l+1}+O\left(h^{3}\right)
$$

Proof. The estimates follow immediately from (2.9), (2.14), and (2.15).

## 4 The constraints

In this section we will use the approximate linearly implicit trapezoidal method (2.10) to construct a time-stepping method for stiff rigid-multi-body systems with contact and friction. Our construction will be done along the lines of $[1,2]$. We assume that the state of the system of rigid bodies can be described by a generalized position vector $q \in \mathbb{R}^{s}$ and a generalized velocity vector $v \in \mathbb{R}^{s}$. We assume that the system is subject to equality, noninterpenetration, and frictional constraints.

The equality constraints arise usually in the presence of joints [13] and they can be described by equations of the form

$$
\begin{equation*}
\Theta^{(i)}(q)=0, i=1,2, \ldots, m \tag{4.1}
\end{equation*}
$$

where $\Theta^{(i)}$ are sufficiently smooth functions. The force exerted by joint (i) on the system is $c_{\nu}^{(i)} \nu^{(i)}(q)$, where $\nu^{(i)}(q)=\nabla_{q} \Theta^{(i)}(q)$ is the gradient of $\Theta^{(i)}(q)$ and $c_{\nu}^{(i)}$ is the appropriate Lagrange multiplier [13].

The noninterpenetration constraints are generated by the rigid body hypothesis according to which the bodies comprising the system cannot penetrate each other. We assume that for any pair of bodies we can define a continuous signed distance function $\Phi_{j}(q)$ so that the noninterpenetration constraints can be written as

$$
\begin{equation*}
\Phi^{(j)}(q) \geq 0, \quad j=1,2, \ldots, p \tag{4.2}
\end{equation*}
$$

where $p$ is the number of pairs of bodies of the system that could get in contact, which in most applications is substantially smaller than the number of all possible pairs of bodies. Although such continuous functions cannot be determined in the most general case, under some weak assumptions it is possible to define them at least in a neighborhood of all contact configurations $[3,7,4]$, which is sufficient for our developments.

The contact constraints are complementarity constraints. If two bodies are in contact then $\Phi^{(j)}(q)=0$ for some index $j$, and a "normal" force $c_{n}^{(j)} n^{(j)}(q)$ (where $n^{(j)}(q)=\nabla_{q} \Phi^{(j)}(q)$ is the gradient of $\Phi^{(j)}(q)$ ) will act at the contact. The force can be only a compression force, which means that $c_{n}^{(j)} \geq 0$. If the two bodies are not in contact, i.e., if $\Phi^{(j)}(q)>0$, then there is no normal force at contact $j$, so that $c_{n}^{(j)}=0$. Therefore the complementarity constraints become

$$
\begin{equation*}
\Phi^{(j)}(q) \geq 0, \quad c_{n}^{(j)} \geq 0, \quad \Phi^{(j)}(q) c_{n}^{(j)}=0, \quad j=1,2, \ldots, p \tag{4.3}
\end{equation*}
$$

By ignoring the superscript $(j)$, the above conditions can be written simply as

$$
\begin{equation*}
0 \leq \Phi(q) \perp c_{n} \geq 0 \tag{4.4}
\end{equation*}
$$

The frictional constraints connect the tangential force, the normal force, and the velocity at contacts. In what follows we adopt the description of frictional constraints from [29]. They are imposed at each
contact $(j)$. To simplify notation we omit the superscript $(j)$, although all the quantities refer to the $(j)$ th contact.

The set of possible friction forces is given by

$$
F C_{0}(q)=\left\{\bar{D}(q) \bar{\beta} \mid \bar{\beta} \in \mathbb{R}^{d}, \psi(\bar{\beta}) \leq \mu\right\},
$$

where $\bar{D}(q)$ is a given $d \times s$ matrix, $\psi(\bar{\beta})$ is a convex, positively homogeneous, coercive function, and $\mu$ is a nonnegative friction coefficient. The total force at the contact belongs to the friction cone

$$
F C(q)=c_{n}\left(n(q)+F C_{0}(q)\right)=\left\{c_{n} n(q)+\bar{D}(q) \bar{\beta} \mid \bar{\beta} \in \mathbb{R}^{d}, \psi(\bar{\beta}) \leq \mu c_{n}\right\} .
$$

If $\bar{D}(q)$ spans the friction plane and $\psi(\bar{\beta})=\|\bar{\beta}\|_{2}$, then $F C(q)$ becomes the classical circular friction cone [19]. The current representation, however, also covers the representation in global coordinates, where $n(q)$ is not necessarily orthogonal to $\bar{D}(q)$ [3].

According to the maximal dissipation principle we choose $\bar{\beta}$ to maximize the dissipation rate $-v^{T} \bar{D}(q) \bar{\beta}$ over $\bar{D}(q) \bar{\beta} \in c_{n} F C_{0}(q)$, which defines $\bar{\beta}$ as the solution of the following optimization problem:

$$
\begin{equation*}
\min _{\bar{\beta} \in \mathbb{R}^{d}} v^{T} \bar{D}(q) \bar{\beta} \quad \text { subject to } \quad \psi(\bar{\beta}) \leq \mu c_{n} \tag{4.5}
\end{equation*}
$$

Since our purpose is to construct a time-stepping scheme that requires the solution of a linear complementarity problem at each time step we use a polyhedral approximation of the friction cone $[1,30,29]$. This approximation is generated by the set

$$
\left\{n(q)+d_{i}(q), i=1,2, \ldots, m_{C}\right\}, \text { with } D(q)=\left[d_{1}(q), d_{2}(q), \ldots, d_{m_{C}}(q)\right]
$$

where $d_{i}(q)$ is a collection of direction vectors in $F C_{0}(q)$. If we denote the vector of tangential forces by $\beta=\left(\beta_{1}, \beta_{2}, \ldots, \beta_{m_{C}}\right)^{T}$, then the total tangential force can be written as $D(q) \beta[1,30]$. In terms of these variables, the frictional constraints, including the maximum dissipation principle (4.5), can be expressed in terms of the following complementarity conditions [1, 30, 29]:

$$
0 \leq D(q)^{T} v+\lambda e \perp \beta \geq 0, \quad 0 \leq \mu c_{n}-e^{T} \beta \perp \lambda \geq 0
$$

If there is relative motion at the contact, $\left\|D(q)^{T} v\right\| \neq 0$, then $\lambda$ is approximately equal to the norm of the tangential velocity at the contact, $[5,30]$.

When using the maximum dissipation principle in a time stepping scheme, the issue is what is the value of the velocity that should be used in the complementarity constraints. Choosing $v=v^{l}$, the current value of the velocity, results in an explicit scheme that is unstable in many interesting situations. In the recently developed time stepping scheme that use a linear complementarity subproblem $[1,2,30,4]$, the choice is $v=v^{(l+1)}$ (the future value of the velocity) that makes the treatment of friction implicit. The good stability properties of such a choice has been demonstrated both theoretically and practically. In this work, since we intend to develop a linearized trapezoidal scheme, the appropriate choice is $v=\frac{v^{(l+1)}+v^{(l)}}{2}$. When matrix $D(q)$ is constant, this can be shown to correspond to a trapezoidal discretization of the integral formulation of the time stepping scheme, which makes our frictional approach well suited for working with the trapezoidal discretization of the Newton-Euler equation.

## 5 The time-stepping scheme

If we combine the Newton-Euler equation of dynamics with the constraints described in the previous section we obtain the following differential complementarity problem (DCP):

$$
\begin{array}{r}
M(q) \frac{d^{2} q}{d t^{2}}-\sum_{i=1}^{m} \nu^{(i)}(q) c_{\nu}^{(i)}-\sum_{j=1}^{p}\left(n^{(j)}(q) c_{n}^{(j)}+D^{(j)}(q) \beta^{(j)}\right)=k\left(t, q, \frac{d q}{d t}\right) \\
\Theta^{(i)}(q)=0, \quad i=1,2, \ldots, m \\
0 \leq \Phi^{(j)}(q) \perp c_{n}^{(j)} \geq 0, \quad j=1,2, \ldots, p  \tag{5.1}\\
0 \leq D^{(j)}(q)^{T} v+\lambda^{(j)} e^{(j)} \perp \beta^{(j)} \geq 0, \quad j=1,2, \ldots, p \\
0 \leq \mu^{(j)} c_{n}^{(j)}-e^{(j)^{T}} \beta^{(j)} \perp \lambda^{(j)} \geq 0, \quad j=1,2, \ldots, p
\end{array}
$$

where $k\left(t, q, \frac{d q}{d t}\right)$ is the external force. The mass matrix $M(q)$ is considered to be symmetric and uniformly positive definite.

As mentioned in the introduction there are examples for which this DCP does not have a solution in the classical sense. However by considering integrals of the forces appearing in the DCP over a small time interval of length $h$ it is possible to obtain time-stepping schemes that have solutions under general assumptions. The time-stepping schemes from [1] and [2] are based on the explicit and the linearly implicit Euler methods respectively. In what follows we will construct a new time-stepping scheme by using the approximate linearly implicit trapezoidal method (2.10) that requires the numerical solution of a linear complementarity problem at each time step. In order to arrive at a linear complementarity problem, we need to linearize the constraints, a step that is related to the index reduction step in differential algebraic equations. For simplicity, in this paper we have analyzed the order of convergence of the method for the ODE case. The same type of analysis can be extended for a larger class of linearly implicit methods used in DAE integration schemes, [23]. It can be shown that the order of the method is preserved if we replace the nonlinear equality constraints $\Theta^{(i)}\left(q^{(l+1)}\right)=0, i=1,2, \ldots m$ by the linearizations:

$$
\begin{equation*}
\nu^{(i)}\left(q^{(l)}+\frac{h}{2} v^{(l)}\right)^{T} \frac{v^{(l+1)}+v^{(l)}}{2}=0, i=1,2, \ldots m \tag{5.2}
\end{equation*}
$$

For practical reasons, it is useful to think about a larger active contact constraint set. During the integration procedure it is possible that while a contact constraint $(j)$ should be theoretically active, the value of $\Phi^{(j)}$ will not be zero because of numerical error. Such a contact will be considered active. Also, some bodies may collide, generating additional active constraints. In this work we will just assume that the active set $\mathcal{A}$ is provided, and we will not discuss the methods for updating the active set unless this has an immediate consequence for the dynamics resolution problem. Using the same strategy as for the nonlinear equality constraints, we replace the noninterpenetration constraints $\Phi^{(j)}\left(q^{(l+1)}\right) \geq 0$ by $n^{(j)}\left(q^{(l)}\right)^{T} \frac{v^{(l+1)}+v^{(l)}}{2} \geq 0$, whenever $j \in \mathcal{A}$.

By applying the approximate linearly implicit trapezoidal method (2.10) to the differential equation ((5.1)) and by using the above approximations for the joint constraints and the active noninterpenetration constraints we obtain the following time-stepping scheme:

$$
\begin{align*}
q^{l+1} & =q^{(l)}+\frac{h}{2}\left(v^{l}+v^{l+1}\right)  \tag{5.3}\\
\widetilde{M} v^{l+1}-\sum_{i=1}^{m} \nu^{(i)} c_{\nu}^{(i)}-\sum_{j \in \mathcal{A}}\left(n^{(j)} c_{n}^{(j)}+D^{(j)} \beta^{(j)}\right) & =\widetilde{M} v^{l}+\widetilde{k}  \tag{5.4}\\
\nu^{(i) T} \frac{v^{(l+1)}+v^{(l)}}{2} & =0, i=1,2, \ldots m  \tag{5.5}\\
0 \leq \rho^{(j)}:=n^{(j) T} \frac{v^{(l+1)}+v^{(l)}}{2} & \perp \quad c_{n}^{(j)} \geq 0, \quad j \in \mathcal{A} \tag{5.6}
\end{align*}
$$

$$
\begin{align*}
& 0 \leq \sigma^{(j)}:=\lambda^{(j)} e^{(j)}+D^{(j) T} \frac{v^{(l+1)}+v^{(l)}}{2} \quad \perp \quad \beta^{(j)} \geq 0, \quad j \in \mathcal{A}  \tag{5.7}\\
& 0 \leq \zeta^{(j)}:=\mu^{(j)} c_{n}^{(j)}-e^{(j)^{T}} \beta^{(j)} \perp \quad \lambda^{(j)} \geq 0, \quad j \in \mathcal{A} \tag{5.8}
\end{align*}
$$

where $\nu^{(i)}=\nu^{(i)}\left(q^{l}+\frac{h}{2} v^{l}\right), n^{(j)}=n^{(j)}\left(q^{l}\right)$,

$$
\begin{equation*}
\widetilde{M}=\left(M\left(q^{l}\right)-\frac{h}{2} \widetilde{k}_{v}^{l}-\frac{h^{2}}{4} \widetilde{k}_{q}^{l}\right), \widetilde{k}=\frac{h}{2}\left(k\left(t_{l}, q^{l}, v^{l}\right)+k\left(t_{l+1}, q^{l}, v^{l}\right)+h \widetilde{k}_{q}^{l} v^{l}\right), \tag{5.9}
\end{equation*}
$$

and $\widetilde{k}_{q}^{l}, \widetilde{k}_{v}^{l}$ are some approximations of $k_{q}\left(t_{l+1}, q^{l}, v^{l}\right), k_{v}\left(t_{l+1}, q^{l}, v^{l}\right)$ respectively.
We notice that equations (5.4)-(5.8) represent a mixed linear complementarity problem. If at timestep $l$ the index set of active contact constraints is given by $\mathcal{A}=\left\{j_{1}, j_{2}, \ldots, j_{s}\right\}$, and if we denote

$$
\begin{array}{lll}
\widetilde{\nu}=\left[\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(m)}\right], & \widetilde{c}_{\nu}=\left\lceil c_{\nu}^{(1)}, c_{\nu}^{(2)}, \ldots, c_{\nu}^{(m)}\right], & \widetilde{D}=\left[D^{\left(j_{1}\right)}, D^{\left(j_{2}\right)}, \ldots, D^{\left(j_{s}\right)}\right] \\
\widetilde{c}_{n}=\left\lceil c_{n}^{\left(j_{1}\right)}, c_{n}^{\left(j_{2}\right)}, \ldots, c_{n}^{\left(j_{s}\right)}\right], & \widetilde{n}=\left[n^{\left(j_{1}\right)}, n^{\left(j_{1}\right)}, \ldots, n^{\left(j_{s}\right)}\right], & \widetilde{\beta}=\left\lceil\beta^{\left(j_{1}\right)}, \beta^{\left(j_{2}\right)}, \ldots, \beta^{\left(j_{s}\right)}\right]^{T} \\
\widetilde{\lambda}=\left[\lambda^{\left(j_{1}\right)}, \lambda^{\left(j_{2}\right)}, \ldots, \lambda^{\left(j_{s}\right)}\right], & \widetilde{\zeta}=\left[\zeta^{\left(j_{1}\right)}, \zeta^{\left(j_{2}\right)}, \ldots, \zeta^{\left(j_{s}\right)}\right], & \widetilde{E}=\operatorname{diag}\left(e^{\left(j_{1}\right)}, e^{\left(j_{2}\right)}, \ldots, e^{\left(j_{s}\right)}\right) \\
\widetilde{\sigma}=\left[\sigma^{\left(j_{1}\right)}, \sigma^{\left(j_{2}\right)}, \ldots, \sigma^{\left(j_{s}\right)}\right], & \widetilde{\rho}=\left[\rho^{\left(j_{1}\right)}, \rho^{\left(j_{2}\right)}, \ldots, \rho^{\left(j_{s}\right)}\right], & \widetilde{\mu}=\operatorname{diag}\left(\mu^{\left(j_{1}\right)}, \mu^{\left(j_{2}\right)}, \ldots, \mu^{\left(j_{s}\right)}\right),
\end{array}
$$

then the mixed LCP (5.4)-(5.8) can be written under the matrix form

$$
\begin{gather*}
{\left[\begin{array}{ccccc}
\widetilde{M} & -\widetilde{\nu} & -\widetilde{n} & -\widetilde{D} & 0 \\
\widetilde{\nu}^{T} & 0 & 0 & 0 & 0 \\
\widetilde{n}^{T} & 0 & 0 & 0 & 0 \\
\widetilde{D}^{T} & 0 & 0 & 0 & \widetilde{E} \\
0 & 0 & \widetilde{\mu} & -\widetilde{E}^{T} & 0
\end{array}\right]\left[\begin{array}{c}
v^{(l+1)}+v^{(l)} \\
\widetilde{c}_{\nu} \\
\widetilde{c}_{n} \\
\widetilde{\beta} \\
\widetilde{\lambda}
\end{array}\right]-\left[\begin{array}{c}
2 \widetilde{M} v^{l}+\widetilde{k} \\
0 \\
0 \\
0 \\
0
\end{array}\right]=\left[\begin{array}{c}
0 \\
0 \\
\widetilde{\rho} \\
\widetilde{\sigma} \\
\widetilde{\zeta}
\end{array}\right]}  \tag{5.10}\\
0 \leq\left\lceil\widetilde{c}_{n}, \widetilde{\beta}, \widetilde{\lambda}\right\rfloor \perp\lceil\widetilde{\rho}, \widetilde{\sigma}, \widetilde{\zeta}\rfloor \geq 0 \tag{5.11}
\end{gather*}
$$

The above LCP has the same structure as the ones considered in [1, 2], so that by using the same analysis we obtain the following result,

Theorem 5.1 If the matrix $\widetilde{M}$ given by (5.9) is positive definite, then the solution set $\mathcal{L}^{T}\left(q^{l}, v^{l}, h, \widetilde{k}\right)$ of LCP (5.10)-(5.11) is nonempty, and Lemke's algorithm can be used to compute an element of this set.

Since $M\left(q^{l}\right)$ is positive definite, $\widetilde{M}$ is guaranteed to be positive definite for sufficiently small $h$. In order to be able to take large time steps for integration, we have to construct approximations $\widetilde{k}_{q}^{l}$ and $\widetilde{k}_{v}^{l}$ of $k_{q}\left(t_{l+1}, q^{l}, v^{l}\right)$ and $k_{v}\left(t_{l+1}, q^{l}, v^{l}\right)$ such that $\widetilde{M}$ is positive definite for the desired time step range. Such approximations are discussed in Section 9.

## 6 The collision model

In what follows, we incorporate the partially elastic collision model with friction from [1] in our timestepping scheme. The list $\mathcal{A}$ of active constraints is updated at each time step by setting

$$
\begin{equation*}
\mathcal{A}=\left\{j: \min \left\{\Phi^{(j)}(q), \Phi^{(j)}(q+h v)\right\} \leq \epsilon_{a}\right\} \tag{6.1}
\end{equation*}
$$

where $q, v$ are the current position and velocity, and $\epsilon_{a}>0$ is a fixed, user-defined, parameter. In this way we make sure that all constraints that are likely to be active at the present time step or are likely to become active during the next time step are accounted for. After updating $\mathcal{A}$ we compute a solution
$\left(\bar{v}, \widetilde{c}_{\nu}, \widetilde{c}_{n}, \widetilde{\beta}, \widetilde{\lambda}\right) \in \mathcal{L}^{T}(q, v, h, \widetilde{k})$ and set $\bar{q}=q+h(v+\bar{v}) / 2$. If all of the noninterpenetration constraints (4.2) are satisfied then we continue with the next time step. Otherwise, there is at least one $j$ such that $\Phi^{(j)}(\bar{q})<0$, which means that a collision has taken place in the time interval $(t, t+h)$. In order to maintain the order of the numerical solution we have to determine the collision time with sufficient accuracy. This can be done by using the unique cubic interpolant $\widehat{q}$ satisfying

$$
\widehat{q}(t)=q, \quad \widehat{q}(t+h)=\bar{q}, \quad \widehat{q}^{\prime}(t)=v, \quad \widehat{q}^{\prime}(t+h)=\bar{v}
$$

and finding the smallest root $t^{*}$ of the scalar nonlinear equation $\Phi^{(j)}(\widehat{q}(\tau))=0$ in the interval $(t, t+h)$. If multiple collisions are detected then we consider the earliest collision time

$$
\begin{equation*}
t^{*}=\min _{j}\left\{\min _{\tau}\left\{\tau \in(t, t+h): \Phi^{(j)}(\widehat{q}(\tau))=0\right\}: \Phi^{(j)}(\bar{q})<0\right\} \tag{6.2}
\end{equation*}
$$

and we add the corresponding contact, $j^{*}$, to $\mathcal{A}$. The pre-collision position and velocity are defined as

$$
\begin{equation*}
q^{-}=\widehat{q}\left(t^{*}\right), \quad v^{-}=\widehat{q}^{\prime}\left(t^{*}\right) \tag{6.3}
\end{equation*}
$$

Our collision model has two phases: compression and decompression. In the former phase interpenetration is prevented by normal compression contact impulses, while in the latter a fraction of each normal compression contact impulse is restituted to the system (the Poisson hypothesis [9]).

### 6.1 The compression phase

In the compression phase, the dynamic system will respond with constraint impulses generated by joints, contacts, and friction. We denote the impulses by the same symbols as before but with superscript $c$. Let $v^{c}$ be the velocity at the end of the compression phase. At the end of this phase, each contact from the list is either maintained $c_{n}^{c} \geq 0, n^{T} v^{c}=0$, or is breaking, $c_{n}^{c}=0, n^{T} v^{c} \geq 0$. Therefore we recover the same complementarity conditions as in [1]. Conservation of momentum requires

$$
\begin{equation*}
M\left(v^{c}-v^{-}\right)-\sum_{i=1}^{m} \nu^{(i)} c_{\nu}^{c(i)}-\sum_{j=1}^{p}\left(n^{(j)} c_{n}^{c(j)}+D^{(j)} \beta^{c(j)}\right)=0 \tag{6.4}
\end{equation*}
$$

The conditions on joints and friction with the appropriate complementarity conditions used to solve the compression phase are the same with the ones used in [1]. It follows that

$$
\begin{equation*}
\left(v^{c}, \widetilde{c}_{\nu}^{c}, \widetilde{c}_{n}^{c}, \widetilde{\beta}^{c}, \widetilde{\lambda}^{c}\right) \in \mathcal{L}^{c}\left(q^{-}, v^{-}, 0,0\right) \tag{6.5}
\end{equation*}
$$

where by $\mathcal{L}^{c}(q, v, h, k)$ we have denoted the solution set of the mixed LCP formulated in [1].

### 6.2 The decompression phase

During the decompression phase each active contact generates a decompression impulse, dependent on its coefficient of restitution $e_{j}$. For instance, contact $j$ generates an impulse $c_{n}^{d(j)}=e_{j} c_{n}^{c(j)}+c_{n}^{x(j)}$. The additional impulse $c_{n}^{x(j)} \geq 0$ is necessary to prevent interpenetration. Let $v^{+}$be the velocity after the decompression phase, or the post-collision velocity. At the end of the decompression phase contact $j$ either breaks, and then $n^{(j) T} v^{+} \geq 0, c_{n}^{x(j)}=0$, or it is maintained and then $n^{(j) T} v^{+}=0, c_{n}^{x(j)} \geq 0$. Therefore the following complementarity condition

$$
0 \leq n^{(j) T} v^{+} \perp c_{n}^{x(j)} \geq 0
$$

is generated for each contact. The conditions on joints and friction remain the same as in [1]. The conservation of momentum requires

$$
\begin{equation*}
M\left(v^{+}-v^{c}\right)-\sum_{i=1}^{m} \nu^{(i)} c_{\nu}^{x(i)}-\sum_{j=1}^{p}\left(n^{(j)} c_{n}^{x(j)}+D^{(j)} \beta^{x(j)}\right)=F^{r} \tag{6.6}
\end{equation*}
$$

where $F^{r}$ is the restitution impulse,

$$
\begin{equation*}
F^{r}=\sum_{j=1}^{p} e_{j} n^{(j)} c_{n}^{c(j)} \tag{6.7}
\end{equation*}
$$

Hence, according to an analogue of Theorem 5.1 from [1], the post-collision velocity $v^{+}$can be found by Lemke's algorithm since

$$
\begin{equation*}
\left(v^{+}, \widetilde{c}_{\nu}^{x}, \widetilde{c}_{n}^{x}, \widetilde{\beta}^{x}, \widetilde{\lambda^{x}}\right) \in \mathcal{L}^{c}\left(q^{-}, v^{c}, 0, F^{r}\right) \tag{6.8}
\end{equation*}
$$

While approaches based on the Poisson hypothesis are widely used in the robotics and mechanics literature [21], it cannot be guaranteed in general that their application results in a decrease of energy after the two collision phases [5]. Although this raises several questions about the appropriateness of the Poisson model, in this work we are concerned with the numerical implications of this observation.

To prevent arbitrarily large increases of energy following the Poisson restitution model, we will use an assumption that plays an important role in insuring convergence to a measure differential inclusion as the time step goes to zero [26] as well as constraint stabilization [4].

Definition (pointed friction cone) We say that the total friction cone

$$
\widehat{F C}(q)=\left\{f \mid f=\sum_{i=1}^{m} \bar{\nu}^{(i)} \bar{c}_{\nu}^{(i)}+\sum_{j \in \mathcal{A}}\left(n^{(j)} \bar{c}_{n}^{(j)}+D^{(j)} \bar{\beta}^{(j)}\right) ; \quad \bar{c}_{n}^{(j)}-e^{(j)^{T}} \bar{\beta}^{(j)} \geq 0, \bar{c}_{n}^{(j)} \geq 0, \bar{\beta}^{(j)} \geq 0, j \in \mathcal{A}\right\}
$$

is pointed if it satisfies the following, equivalent, conditions:
1.

$$
\begin{aligned}
0= & \sum_{i=1}^{m} \bar{\nu}^{(i)} \bar{c}_{\nu}^{(i)}+\sum_{j \in \mathcal{A}}\left(n^{(j)} \bar{c}_{n}^{(j)}+D^{(j)} \bar{\beta}^{(j)}\right), \quad \bar{c}_{n}^{(j)}-e^{(j)^{T}} \bar{\beta}^{(j)} \geq 0, \bar{c}_{n}^{(j)} \geq 0, \bar{\beta}^{(j)} \geq 0, j \in \mathcal{A} \\
& \Longrightarrow \begin{cases}\bar{c}_{\nu}^{(i)}=0, & i=1,2, \ldots, m \\
\bar{c}_{n}^{(j)}=0, \bar{\beta}^{(j)}=0, & j \in \mathcal{A}\end{cases}
\end{aligned}
$$

2. There exists a parameter $c_{F C}>0$ such that [4]

$$
\left.\begin{array}{l}
\left.\sum_{i=1}^{m} \bar{\nu}^{(i)} \bar{c}_{\nu}^{(i)}+\sum_{j \in \mathcal{A}}{ }^{(n}{ }^{(j)} \bar{c}_{n}^{(j)}+D^{(j)} \bar{\beta}^{(j)}\right)=f \\
\bar{c}_{n}^{(j)}-e^{(j)^{T}} \bar{\beta}^{(j)} \geq 0, \bar{c}_{n}^{(j)} \geq 0, \bar{\beta}^{(j)} \geq 0, j \in \mathcal{A}
\end{array}\right\} \Rightarrow\left\|\left(\bar{c}_{\nu}, \bar{c}_{n}, \bar{\beta},\right)\right\| \leq c_{F C}\|f\|
$$

The name "pointed friction cone" originates in the fact that, when no joints are present, the definition above is equivalent to requiring that the cone contain no nontrivial linear subspace [31].

When the friction cone is pointed, we can say more about the possible increase in kinetic energy following a Poisson collision resolution. ¿From the compression phase rule, it immediately follows that $v^{c^{T}} M v^{c} \leq v^{-^{T}} M v^{-}$and thus, since the restitution coefficients are bounded above by 1 , it follows that there is a parameter $c_{c}$ that depends only on $c_{F C}$ and on the number of generators used for the friction cone such that $v^{+^{T}} M v^{+} \leq c_{c} v^{-^{T}} M v^{-}$.

Therefore, if the parameter $c_{F C}$ is uniformly upper bounded with respect to all possible choices of the position vector $q$, it follows that the parameter $c_{c}$ is itself uniformly upper bounded. We call such a situation uniform pointedness of the friction cone [4] and it results in the fractional increase in kinetic energy following a collision being uniformly upper bounded.

## 7 The algorithm

By combining the time-stepping scheme from Section 5 with the collision resolution described in Section 6 we obtain the following algorithm:

Input:

- $q^{0}, v^{0}$ - consistent initial position and velocity;
- $T$ - length of simulation time interval;
- $h_{s}$ - standard step-size;
- $\epsilon_{a}$ - positive parameter.

Set $l=0, t_{l}=0$;
while $\left(t_{l}<T\right)$

1. Set $q=q^{l}, v=v^{l}, t=t_{l}, h=h_{s}$;
2. Determine list of active contacts by (6.1);
3. Compute $\left(\bar{v}, \widetilde{c}_{\nu}, \widetilde{c}_{n}, \widetilde{\beta}, \widetilde{\lambda}\right) \in \mathcal{L}^{T}(q, v, h, \widetilde{k})$;
4. Set $\bar{q}=h(v+\bar{v}) / 2$;
if (no collision detected between $t$ and $t+h$ )
Set $t_{l+1}=t_{l}+h_{s}, h=h_{s}, q^{l+1}=\bar{q}, v^{l=1}=\bar{v}$; Set $l \leftarrow l+1$;
else
Estimate the collision time $t^{*}$ from (6.2);
Add contact $j^{*}$ to the list of active contacts;
Determine pre-collision position and velocity $q^{-}$and $v^{-}$from (6.3);
Compute $\left(v^{c}, \widetilde{c}_{\nu}^{c}, \widetilde{c}_{n}^{c}, \widetilde{\beta}^{c}, \widetilde{\lambda^{c}}\right) \in \mathcal{L}^{c}\left(q^{-}, v^{-}, 0,0\right)$;
Compute $\left(v^{+}, \widetilde{c}_{\nu}^{x}, \widetilde{c}_{n}^{x}, \widetilde{\beta}^{x}, \lambda^{x}\right) \in \mathcal{L}^{c}\left(q^{-}, v^{c}, 0, F^{r}\right)$; Set $t=t *, h=t_{l}+h_{s}-t^{*}, q=q^{-}, v=v^{+}$and go to 3 .
end if
end while

## 8 Boundedness of kinetic energy

In this section we prove that the kinetic energy of the discrete model described in this paper stays bounded on any given finite interval $[0, T]$ as the step-size $h$ tends to zero. To simplify analysis we consider the case where the mass matrix $M$ is constant and the force $k$ is of the form

$$
\begin{equation*}
k(t, q, v)=f(v)+k_{1}(t, q, v) \tag{8.1}
\end{equation*}
$$

where $f(v)$ is the Coriolis force and $k_{1}(t, q, v)$ are external forces satisfying the following growth condition

$$
\begin{equation*}
\left\|k_{1}(t, q, v)\right\| \leq c_{1}+c_{2}\|q\|+c_{3}\|v\| \tag{8.2}
\end{equation*}
$$

where the constants $c_{1}, c_{2}, c_{3}$ depend only on the length $T$ of the given time interval. We also assume that the Coriolis force is given by a bilinear operator

$$
[f(v)]_{i}=\sum_{j k} f_{i j k} v_{j} v_{k}
$$

This is certainly true in case the system is described by Newton-Euler equations in body coordinates [19, Section 2.4] where the matrix $F(v)$ of entries

$$
[F(v)]_{i j}=\sum_{k} f_{i j k} v_{k}
$$

is antisymmetric in the sense that

$$
u^{T} F(v) u=0, \quad \forall u
$$

Without loss of generality we may assume that the tensor $\left(f_{i j k}\right)$ is symmetric in the sense that $f_{i j k}=$ $f_{i k j}$. In this case it is easily seen that

$$
f(v)=F(v) v, \quad f^{\prime}(v)=2 F(v)
$$

where $f^{\prime}$ is the Fréchet derivative of $f$. In our time-stepping scheme we take

$$
\widetilde{k}_{q}^{l}=\widetilde{k}_{1 q}^{l}, \quad \widetilde{k}_{v}^{l}=F\left(v^{l}\right)+\widetilde{k}_{1 v}^{l}
$$

where $\widetilde{k}_{1 q}^{l}$ and $\widetilde{k}_{1 v}^{l}$ are approximations of $k_{1 q}\left(t_{l+1}, q^{l}, v^{l}\right)$ and $k_{1 v}\left(t_{l+1}, q^{l}, v^{l}\right)$ respectively, that are bounded in the sense that there are constant $c_{4}, c_{5}$ such that

$$
\begin{equation*}
\left\|\widetilde{k}_{1 q}^{l}\right\| \leq c_{4}, \quad\left\|\widetilde{k}_{1 v}^{l}\right\| \leq c_{5}, \quad \forall l \tag{8.3}
\end{equation*}
$$

For the following, we will assume that the mass matrix $M(q)$ is constant. This is not an exceedingly restrictive assumption and it holds when we use the Newton-Euler formulation in body coordinates to describe our problem [19].

Theorem 8.1 Assume that the algorithm from Section 7 solves a number of collisions in the time interval $[0, T]$ that is uniformly upper bounded as $h \rightarrow 0$. Suppose that the friction cone is uniformly pointed for the possible choices of the position vector $q$. Suppose that the mass matrix $M$ is a constant symmetric positive definite matrix and that conditions (8.2), (8.3) are satisfied. Then there is a constant c such that

$$
\left(v^{l}\right)^{T} M v^{l} \leq \max \left\{\left(v^{0}\right)^{T} M v^{0},\left\|q^{0}\right\|+1\right\} e^{c t_{l}}, \quad l=0,1, \ldots,\lfloor T / h\rfloor
$$

for all sufficiently small $h$.

## Proof.

Since the friction cone is uniformly pointed it follows that the fractional increase in kinetic energy following a collision is uniformly upper bounded. From the properties of the Hermite interpolation and from the definition of our scheme, it follows that the velocity before collision has an uniformly upper bounded fractional increase when compared to the maximum of the velocity before a collision and the velocity after a collision.

Therefore, since the number of collisions is uniformly upper bounded, it is sufficient to prove the conclusion of this theorem for the case where all collisions are totally plastic (that is, the restitution coefficient is zero). Formally, this assumption is the same as assuming that all time steps are computed using just the time stepping LCP (5.3-5.9), except that some of the time steps may be equal to 0 [5]. Since this does not result in kinetic energy increase, it is sufficient to prove the result for a uniform time step.

Suppose that no collisions are detected in the interval $\left[t_{l}, t_{l+1}\right]$. The new velocity $v^{(l+1)}$ will be determined by solving the LCP (5.4)-(5.8).

Left multiplying (5.4) by $\left(v^{(l+1)}+v^{(l)}\right)^{T}$ we get that

$$
\begin{align*}
&\left(v^{(l+1)}+v^{(l)}\right)^{T} \widetilde{M^{(l)}} v^{(l+1)}=\sum_{i=1}^{m}\left(v^{(i)^{T}}\left(v^{(l+1)}+v^{(l)}\right)\right) c_{\nu}^{(i)}+\sum_{j \in \mathcal{A}}\left(n^{(j)^{T}}\left(v^{(l+1)}+v^{(l)}\right)\right) c_{n}^{(j)}+ \\
& D^{(j)^{T}}\left(v^{(l+1)}+v^{(l)}\right) \beta^{(j)}+\widetilde{k}^{(l)^{T}}\left(v^{(l+1)}+v^{(l)}\right)+\left(v^{(l+1)}+v^{(l)}\right)^{T} M v^{(l)} . \tag{8.4}
\end{align*}
$$

Using (5.5), we deduce that $\nu^{(i)^{T}}\left(v^{(l+1)}+v^{(l)}\right)=0, i=1,2, \ldots, m$. Also, using the contact constraints (5.6), we obtain $c_{n}^{(j)} n^{(j)^{T}}\left(v^{(l+1)}+v^{(l)}\right)=0, j \in \mathcal{A}$. Finally, from the frictional constraints (5.7) and (5.8) we get that

$$
D^{(j)^{T}}\left(v^{(l+1)}+v^{(l)}\right) \beta^{(j)}=-\lambda^{(j)} \beta^{(j)^{T}} e^{(j)}=-\mu^{(j)} c_{n}^{(j)} \lambda^{(j)} \leq 0, \forall j \in \mathcal{A} .
$$

Then (8.4) implies

$$
\begin{equation*}
\left(v^{(l+1)}+v^{(l)}\right)^{T} \widetilde{M}^{(l)} v^{(l+1)} \leq\left(v^{(l+1)}+v^{(l)}\right)^{T} \widetilde{M}^{(l)} v^{(l)}+\left(v^{(l+1)}+v^{(l)}\right)^{T} \widetilde{k}^{(l)^{T}} . \tag{8.5}
\end{equation*}
$$

By expanding the left and right-hand sides of the above inequality and using (8.2) and (8.3), it follows that there is a constant $c_{6}$ such that for sufficiently small $h$ we have

$$
\begin{aligned}
\left(v^{(l+1)}+v^{(l)}\right)^{T} \widetilde{M} v^{l+1} & =\left(v^{l+1}\right)^{T} M v^{l+1}-\frac{h}{2}\left(v^{l+1}\right)^{T}\left(\widetilde{k}_{1 v}^{l}+\frac{h}{2} \widetilde{k}_{1 q}^{l}\right) v^{l+1}-\frac{h}{2}\left(v^{l}\right)^{T}\left(\widetilde{k}_{1 v}^{l}+\frac{h}{2} \widetilde{k}_{1 q}^{l}\right) v^{l+1} \\
& +v^{(l)^{T}} M v^{(l+1)}-\frac{h}{2} v^{(l)^{T}} F\left(v^{(l)}\right) v^{(l+1)}
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\left(v^{(l+1)}+v^{(l)}\right)^{T} \widetilde{M} v^{l+1} & \geq\left(1-c_{6} h\right)\left\|M^{1 / 2} v^{(l+1)}\right\|^{2}-c_{6} h\left\|M^{1 / 2} v^{(l)}\right\|\left\|M^{1 / 2} v^{(l+1)}\right\|+v^{(l)^{T}} M v^{(l+1)} \\
& -\frac{h}{2} v^{(l)^{T}} F\left(v^{(l)}\right) v^{(l+1)} .
\end{aligned}
$$

For the right-hand side we use the same procedure:

$$
\begin{aligned}
\left(v^{(l+1)}+v^{(l)}\right)^{T}\left(\widetilde{M} v^{(l)}+\widetilde{k}\right) & =v^{(l)^{T}} M v^{(l)}-\frac{h}{2} v^{(l)^{T}}\left(\widetilde{k}_{1 v}^{l}+\frac{h}{2} \widetilde{k}_{1 q}^{l}\right) v^{(l)}-\frac{h}{2}\left(v^{(l+1)}\right)^{T}\left(\widetilde{k}_{1 v}^{l}+\frac{h}{2} \widetilde{k}_{1 q}^{l}\right) v^{(l)} \\
& +\frac{h}{2}\left(v^{(l+1)}+v^{(l)}\right)^{T}\left(k_{1}\left(t_{l}, q^{l}, v^{l}\right)+k_{1}\left(t_{l+1}, q^{l}, v^{l}\right)\right)+\frac{h^{2}}{2} v^{(l)^{T}} \widetilde{k}_{1 q} v^{(l)} \\
& +\frac{h^{2}}{2} v^{(l+1)^{T}} \widetilde{k}_{1 q} v^{(l)}+v^{(l+1)^{T}} M v^{(l)}+\frac{h}{2} v^{(l+1)^{T}} F\left(v^{(l)}\right) v^{(l)}
\end{aligned}
$$

leading to

$$
\begin{aligned}
\left(v^{(l+1)}+v^{(l)}\right)^{T}\left(\widetilde{M} v^{(l)}+\widetilde{k}\right) & \leq\left(1+c_{6} h\right)\left\|M^{1 / 2} v^{(l)}\right\|^{2}+c_{6} h\left\|M^{1 / 2} v^{(l)}\right\|\left\|M^{1 / 2} v^{(l+1)}\right\|+\frac{h}{2} v^{(l+1)^{T}} F\left(v^{(l)}\right) v^{(l)} \\
& +v^{(l+1)^{T}} M v^{(l)}+c_{6} h\left(\left\|M^{1 / 2} v^{(l+1)}\right\|+\left\|M^{1 / 2} v^{(l)}\right\|\right)\left(1+\left\|q^{(l)}\right\|+\left\|M^{1 / 2} v^{(l)}\right\|\right) .
\end{aligned}
$$

Let us denote

$$
\rho_{l}=\left\|M^{1 / 2} v^{l}\right\|, \quad \sigma_{l}=\left\|q^{l}\right\|+1 .
$$

Using the symmetry of the mass matrix $M$, the antisymmetry of $F\left(v^{(l)}\right)$ as well as the above estimates and notation, it turns out that there exists a constant $c_{7}$ such that

$$
\begin{equation*}
\left(1-c_{7} h\right) \rho_{l+1}^{2} \leq\left(1+c_{7} h\right) \rho_{l}^{2}+c_{7} h \rho_{l} \rho_{l+1}+c_{7} h \sigma_{l}\left(\rho_{l}+\rho_{l+1}\right) \tag{8.6}
\end{equation*}
$$

for all sufficiently small $h$. In what follows we analyze the two possibilities $\rho_{l}<\rho_{l+1}$ and $\rho_{l} \geq \rho_{l+1}$.

If $\rho_{l}<\rho_{l+1}$, the inequality (8.6) implies that there exists a constant $c_{8}$ such that:

$$
\begin{equation*}
\rho_{l+1} \leq\left(1+c_{8} h\right) \rho_{l}+c_{8} h \sigma_{l} \tag{8.7}
\end{equation*}
$$

for all sufficiently small $h$. On the other hand from (5.3) we have

$$
\begin{equation*}
\left\|q^{l+1}\right\| \leq\left\|q^{l}\right\|+\frac{h}{2}\left\|M^{-1 / 2}\right\|\left(\left\|M^{1 / 2} v^{l}\right\|+\left\|M^{1 / 2} v^{l+1}\right\|\right) \tag{8.8}
\end{equation*}
$$

Substituting the over-estimate for $\rho_{l+1}$ into (8.8) gives

$$
\sigma_{l+1} \leq \frac{h}{2}\left\|M^{-1 / 2}\right\|\left(2+c_{8} h\right) \rho_{l}+\left(1+c_{8} \frac{h^{2}}{2}\left\|M^{-1 / 2}\right\|\right) \sigma_{l}
$$

It follows that there is a constant $c_{9}$ such that

$$
\begin{align*}
& \rho_{l+1} \leq\left(1+c_{9} h\right) \rho_{l}+c_{9} h \sigma_{l}  \tag{8.9}\\
& \sigma_{l+1} \leq c_{9} h \rho_{l}+\left(1+c_{9} h\right) \sigma_{l}
\end{align*}
$$

for all sufficiently small $h$.
In the case when $\rho_{l+1} \leq \rho_{l}$, it is straightforward to see that the equivalent of (8.9) is

$$
\begin{aligned}
& \rho_{l+1} \leq \rho_{l} \\
& \sigma_{l+1} \leq \sigma_{l}+h\left\|M^{-1 / 2}\right\| \rho_{l}
\end{aligned}
$$

If the pair $\left(\rho_{l+1}, \rho_{l}\right)$ satisfies the above set of inequalities, then it satisfies also (8.9) for sufficiently small values of $h$ (i.e., one can obviously choose $c_{9}>\left\|M^{-1 / 2}\right\|$ ). Therefore, in both cases the inequalities (8.9) are satisfied for all sufficiently small $h$ and by taking $c=.5 c_{9}$, we have

$$
\begin{aligned}
\left\|\left[\begin{array}{c}
\rho_{l} \\
\sigma_{l}
\end{array}\right]\right\|_{\infty} & \leq\left\|\left[\begin{array}{cc}
1+c_{9} h & c_{9} h \\
c_{9} h & 1+c_{9} h
\end{array}\right]\right\|_{\infty}^{l}\left\|\left[\begin{array}{l}
\rho_{0} \\
\sigma_{0}
\end{array}\right]\right\|_{\infty}=(1+c h)^{l}\left\|\left[\begin{array}{l}
\rho_{0} \\
\sigma_{0}
\end{array}\right]\right\|_{\infty} \\
& \leq e^{c l h}\left\|\left[\begin{array}{c}
\rho_{0} \\
\sigma_{0}
\end{array}\right]\right\|_{\infty}=e^{c t_{l}}\left\|\left[\begin{array}{c}
\rho_{0} \\
\sigma_{0}
\end{array}\right]\right\|_{\infty}
\end{aligned}
$$

which proves our theorem.

## $9 \quad$ Stiff stability

In this section we consider the case where there are several springs and dampers between pairs of points of the system, in addition to a nonstiff force. In this case the force $k_{1}(t, q, v)$ from (8.1) can be written as (see [2]):

$$
\begin{align*}
k_{1}(t, q, v)=- & \sum_{i=1}^{n_{\gamma}} \gamma_{i} \phi^{(i)}(q) \nabla_{q} \phi^{(i)}(q)-\sum_{j=1}^{n_{\delta}} \delta_{j} \nabla_{q} \psi^{(j)}(q)\left(\nabla_{q} \psi^{(j)}(q)^{T} v\right)- \\
& \sum_{k=1}^{n_{\delta \gamma}}\left(\bar{\gamma}_{k} \bar{\phi}^{(k)}(q)+\bar{\delta}_{k}\left(\nabla_{q} \bar{\phi}^{(k)}(q)^{T} v\right)\right) \nabla_{q} \bar{\phi}^{(k)}(q)+k_{2}(t, q, v) . \tag{9.1}
\end{align*}
$$

Here $\gamma_{i}, i=1,2, \ldots, n_{\gamma}$ and $\bar{\gamma}_{k}, k=1,2, \ldots, n_{\gamma \delta}$, are spring parameters, and $\delta_{j}, j=1,2, \ldots, n_{\delta}$, $\bar{\delta}_{k}, k=1,2, \ldots, n_{\gamma \delta}$, are damping parameters. The functions $\phi^{(i)}(q)$ and $\psi^{(j)}(q)$ are related to the distances between the points where the springs and the dampers are attached. They are normalized in such a way that they vanish if the springs and/or dampers are at equilibrium. The functions $\bar{\phi}^{(k)}(q), k=1,2, \ldots, n_{\delta \gamma}$, are associated with pairs of points between which there are both springs and dampers. We assume that the coordinates of the system vary in a region where $\phi^{(i)}(q)$ and $\psi^{(j)}(q)$ are differentiable. Here, the term $k_{2}(t, q, v)$ denotes the nonstiff forces.

We are interested in the behavior of our discrete model when the stiffness parameters tend to infinity. In particular we investigate the case where the kinetic energy at the next time step remains bounded when the stiffness parameters tend to infinity.This is achieved by our time-stepping scheme when we start with the springs at equilibrium, and it will be one of the main ingredients used in proving the limit behavior. We analyze the time-stepping scheme (5.4)-(5.8) with

$$
\begin{array}{r}
\widetilde{k}_{q}^{(l)}=-\sum_{i=1}^{n_{\gamma}} \gamma_{i} \nabla_{q} \phi^{(i)}\left(q^{(l)}\right) \nabla_{q} \phi^{(i)^{T}}\left(q^{(l)}\right)-\sum_{k=1}^{n_{\delta \gamma}} \bar{\gamma}_{k} \nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right) \nabla_{q} \bar{\phi}^{(k)^{T}}\left(q^{(l)}\right), \\
\widetilde{k}_{v}^{(l)}=F\left(v^{(l)}\right)-\sum_{i=1}^{n_{\delta}} \delta_{j} \nabla_{q} \psi^{(j)}\left(q^{(l)}\right) \nabla_{q} \psi^{(j)^{T}}\left(q^{(l)}\right)-\sum_{k=1}^{n_{\delta \gamma}} \bar{\delta}_{k} \nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right) \nabla_{q} \bar{\phi}^{(k)^{T}}\left(q^{(l)}\right) . \tag{9.3}
\end{array}
$$

We note that $\widetilde{M}$ is positive definite for any value of the time step. We can use Theorem 5.1 to guarantee that Lemke's method will return a solution of the linear complementarity problem ((5.10)(5.11)) that needs to be solved to obtain the next value of the velocity.

Following Theorem 3.3, it follows that using the above approximations for the Jacobian matrix, the scheme can no longer have the order 2 , unless the functions $\psi, \phi, \bar{\psi}$ are linear. However, for the case where there are stiff external forces, if we use the exact Jacobian when we are linearizing (9.1), it is possible to obtain a matrix $\widetilde{M}$ in (5.4) that is not positive definite even for relatively small values of the time step. Since the solvability of the LCP $(5.4-5.9)$ depends on $\widetilde{M}$, this could create serious difficulties especially when some of the functions $\psi, \phi, \bar{\psi}$ are highly nonconvex. Since the order of the method is an asymptotic property as $h \rightarrow 0$, one could easily derived a scheme that switches to the classical linearized trapezoidal when either the time step is sufficiently small or $\widetilde{M}$ is still positive definite when the exact Jacobian matrix is used. Therefore, of interest in this section are the stability properties of the scheme that uses the approximation to the Jacobian matrix described above.

Theorem 9.1 Consider the time-stepping scheme (5.4)-(5.8), where the matrices $\widetilde{M}^{(l)}$ and $\widetilde{k}^{(l)}$ are defined by (5.9). The velocity solution of (5.4)-(5.8) satisfies

$$
\begin{array}{r}
w_{1}^{T} M w_{1}+\sum_{i=1}^{n_{\gamma}} \gamma_{i}\left(\phi^{(i)}\left(q^{(l)}\right)+\frac{h}{2} \nabla_{q} \phi^{(i)^{T}}\left(q^{(l)}\right)\left(v^{(l+1)}+v^{(l)}\right)\right)^{2}+ \\
\sum_{k=1}^{n_{\delta \gamma}} \bar{\gamma}_{k}\left(\bar{\phi}^{(k)}\left(q^{(l)}\right)+\frac{h}{2} \nabla_{q} \bar{\phi}^{(k)^{T}}\left(q^{(l)}\right)\left(v^{(l+1)}+v^{(l)}\right)\right)^{2} \leq w_{2}^{T} M w_{2}+\sum_{i=1}^{n_{\gamma}} \gamma_{i}\left(\phi^{(i)}\left(q^{(l)}\right)\right)^{2}+\sum_{k=1}^{n_{\delta \gamma}} \bar{\gamma}_{k}\left(\bar{\phi}^{(k)}\left(q^{(l)}\right)\right)^{2}
\end{array}
$$

where the expressions for $w_{1}$ and $w_{2}$ are $w_{1}=v^{(l+1)}-\frac{h}{4} M^{-1}\left(k_{2}\left(t_{l}, q^{(l)}, v^{(l)}\right)+k_{2}\left(t_{l+1}, q^{(l)}, v^{(l)}\right)\right)$, and $w_{2}=v^{(l)}+\frac{h}{4} M^{-1}\left(k_{2}\left(t_{l}, q^{(l)}, v^{(l)}\right)+k_{2}\left(t_{l+1}, q^{(l)}, v^{(l)}\right)\right)$.

Proof Left multiplying (5.4) by $\left(v^{(l+1)}+v^{(l)}\right)^{T}$ and using the same algebra as in the beginning of the proof of Theorem 8.1, we obtain the inequality

$$
\begin{equation*}
\left(v^{(l+1)}+v^{(l)}\right)^{T} \widetilde{M}^{(l)} v^{(l+1)} \leq\left(v^{(l+1)}+v^{(l)}\right)^{T} \widetilde{M}^{(l)} v^{(l)}+\left(v^{(l+1)}+v^{(l)}\right)^{T} \widetilde{k}^{(l)^{T}} \tag{9.4}
\end{equation*}
$$

We now use the definitions (5.9) of $\widetilde{M}^{(l)}$ and $\widetilde{k}^{(l)}$; the identity $(a+b)(a-b)+2 b(a+b)=(a+b)(a+b)$ where $a$ is one of $\nabla_{q} \psi^{(j)^{T}} v^{(l+1)}, \nabla_{q} \phi^{(j)^{T}} v^{(l+1)}$, or $\nabla_{q} \bar{\psi}^{(j)^{T}} v^{(l+1)}$, and $b$ is one of $\nabla_{q} \psi^{(j)^{T}} v^{(l)}, \nabla_{q} \phi^{(j)^{T}} v^{(l)}$, or $\nabla_{q} \bar{\psi}^{(j)^{T}} v^{(l)}$; , as well as the fact that the matrix $F\left(v^{(l)}\right)$ appearing in the definition of the Coriolis
force is antisymmetric, to obtain that

$$
\begin{aligned}
& v^{(l+1)^{T}} M v^{(l+1)}+\frac{h}{2} \sum_{j=1}^{n_{\delta}} \delta_{j}\left(\nabla_{q} \psi^{(j)}\left(q^{(l)}\right)^{T}\left(v^{(l+1)}+v^{(l)}\right)\right)^{2}+\frac{h}{2} \sum_{k=1}^{n_{\delta \gamma}} \bar{\delta}_{k}\left(\nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right)^{T}\left(v^{(l+1)}+v^{(l)}\right)\right)^{2}+ \\
& \frac{h^{2}}{4} \sum_{i=1}^{n_{\gamma}} \gamma_{i}\left(\nabla_{q} \phi^{(j)}\left(q^{(l)}\right)^{T}\left(v^{(l+1)}+v^{(l)}\right)\right)^{2}+\frac{h^{2}}{4} \sum_{k=1}^{n_{\gamma_{\delta}}} \bar{\gamma}_{k}\left(\nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right)^{T}\left(v^{(l+1)}+v^{(l)}\right)\right)^{2}+ \\
& h \sum_{i=1}^{n_{\gamma}} \gamma_{i} \phi^{(i)}\left(q^{(l)}\right) \nabla_{q} \phi^{(i)}\left(q^{(l)}\right)^{T}\left(v^{(l+1)}+v^{(l)}\right)+h \sum_{k=1}^{n_{\gamma_{\delta}}} \bar{\gamma}_{k} \bar{\phi}^{(k)}\left(q^{(l)}\right) \nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right)^{T}\left(v^{(l+1)}+v^{(l)}\right) \\
& \leq v^{(l)}{ }^{T} M v^{(l)}+\frac{h}{2}\left(v^{(l+1)}+v^{(l)}\right)^{T}\left(k_{2}\left(t_{l}, q^{(l)}, v^{(l)}\right)+k_{2}\left(t_{l+1}, q^{(l)}, v^{(l)}\right)\right) .
\end{aligned}
$$

From the above inequality it is obvious that

$$
\begin{align*}
& v^{(l+1)^{T}} M v^{(l+1)}+\frac{h^{2}}{4} \sum_{i=1}^{n_{\gamma}} \gamma_{i}\left(\nabla_{q} \phi^{(j)}\left(q^{(l)}\right)^{T}\left(v^{(l+1)}+v^{(l)}\right)\right)^{2}+\frac{h^{2}}{4} \sum_{k=1}^{n_{\gamma_{\delta}}} \bar{\gamma}_{k}\left(\nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right)^{T}\left(v^{(l+1)}+v^{(l)}\right)\right)^{2}+ \\
& \quad h \sum_{i}^{n_{\gamma}} \gamma_{i} \phi^{(i)}\left(q^{(l)}\right) \nabla_{q} \phi^{(i)}\left(q^{(l)}\right)^{T}\left(v^{(l+1)}+v^{(l)}\right)+h \sum_{k}^{n_{\gamma_{\delta}}} \overline{\gamma k}^{(k)}\left(q^{(l)}\right) \nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right)^{T}\left(v^{(l+1)}+v^{(l)}\right) \\
& \leq v^{(l)^{T}} M v^{(l)}+\frac{h}{2}\left(v^{(l+1)}+v^{(l)}\right)^{T}\left(k_{2}\left(t_{l}, q^{(l)}, v^{(l)}\right)+k_{2}\left(t_{l+1}, q^{(l)}, v^{(l)}\right)\right) . \tag{9.5}
\end{align*}
$$

Adding to both sides of the inequality (9.5) the quantity

$$
\sum_{i=1}^{n_{\gamma}} \gamma_{i}\left(\phi^{(i)}\left(q^{(l)}\right)\right)^{2}+\sum_{k=1}^{n_{\gamma_{\delta}}} \bar{\gamma}_{k}\left(\phi^{(k)}\left(q^{(l)}\right)\right)^{2}+\left\|\frac{h}{4} M^{-1}\left(k_{2}\left(t_{l}, q^{(l)}, v^{(l)}\right)+k_{2}\left(t_{l+1}, q^{(l)}, v^{(l)}\right)\right)\right\|
$$

and grouping the corresponding terms as well as using the expressions for $w_{1}$ and $w_{2}$ gives the desired result.

Note that if the system is isolated and the functions $\phi, \bar{\phi}, \psi$ are linear, then the previous Theorem guarantees that the total energy of the system will decrease and thus the numerical scheme is stable. This observation is analogous to the absolute stability characterization of a numerical scheme for ordinary differential equations. That characterization is also done on a model problem obtained by linearization, on which the scheme is proved stable. So in that sense, we can argue that our scheme is absolutely stable.

Corollary 9.2 Assume that $q^{(l)}$ is a point where the springs in the stiff force (9.1) are at equilibrium, or $\phi^{(i)}(q)=0, i=1,2, \ldots, n_{\gamma}, \bar{\phi}^{(k)}(q)=0, k=1,2, \ldots, n_{\delta \gamma}$. Then the velocity at the new step $v^{(l+1)}$ is bounded uniformly with respect to the stiffness parameters, and the following inequality holds

$$
w_{1}^{T} M w_{1} \leq w_{2}^{T} M w_{2}
$$

where $w_{1}$ and $w_{2}$ are the same with the ones used in Theorem 9.1.
Proof The proof follows immediately by replacing $\phi^{(i)}(q)=0, i=1,2, \ldots, n_{\gamma}, \bar{\phi}^{(k)}(q)=0, k=$ $1,2, \ldots, n_{\delta \gamma}$ in the conclusion of Theorem 9.1.

### 9.1 The limit system

In what follows we will analyze the behavior of the time-stepping scheme as the stiffness parameters increase to infinity. The expected physical behavior is such that when the stiffness parameters go to infinity the system behaves as if the springs and dampers were replaced by rigid links. in the following we prove that the numerical scheme follows the expected physical behavior.

Therefore, in addition to the original bilateral constraints, the limit system includes the additional constraints:

$$
\begin{array}{ll}
\phi^{(i)}(q)=0, & i=1,2, \ldots, n_{\gamma} \\
\psi^{(j)}(q)=0, & j=1,2, \ldots, n_{\delta} \\
\bar{\phi}^{(k)}(q)=0, & k=1,2, \ldots, n_{\delta \gamma}
\end{array}
$$

In the time-stepping scheme, the above constraints will be replaced by the same approximations used when modeling the original joint constraints. For example, $\phi^{(i)}\left(q^{(l+1)}\right)=0$ is replaced by $\nabla_{q} \phi^{(i)^{T}}\left(q^{(l)}\right)\left(v^{(l+1)}+v^{(l)}\right)=0$, for $i=1,2, \ldots, n_{\gamma}$. Adding such linear equality constraints corresponding to the limit case of the stiff force (9.1) to (5.3)-(5.8), we obtain the following time-stepping LCP:

$$
q^{(l+1)}=q^{(l)}+\frac{h}{2}\left(\bar{v}^{(l+1)}+v^{(l)}\right)
$$

where $\bar{v}^{(l+1)}$ is a solution of the mixed linear complementarity problem:

$$
\begin{array}{rrc}
\left(M-\frac{h}{2} F\left(v^{(l)}\right)\right) \bar{v}^{(l+1)}-\sum_{i=1}^{m} \bar{\nu}^{(i)} \bar{c}_{\nu}^{(i)} & -\sum_{j \in \mathcal{A}} & \left(n^{(j)} \bar{c}_{n}^{(j)}+D^{(j)} \bar{\beta}^{(j)}\right)-\bar{K}_{s}= \\
\nu^{(i)^{T}}\left(\bar{v}^{(l+1)}+v^{(l)}\right)=0, & & \bar{k}+M v^{(l)} \\
\nabla_{q} \phi^{(j)^{T}}\left(\bar{v}^{(l+1)}+v^{(l)}\right)=0, & i=1,2, \ldots, m \\
\nabla_{q} \psi^{(j)^{T}}\left(\bar{v}^{(l+1)}+v^{(l)}\right)=0, & i=1,2, \ldots, n_{\delta} \\
\nabla_{q} \bar{\phi}^{(k)^{T}}\left(\bar{v}^{(l+1)}+v^{(l)}\right)=0, & & j=1,2, \ldots, n_{\delta}  \tag{9.6}\\
\bar{\rho}^{(j)}=n^{(j)^{T}}\left(\bar{v}^{(l+1)}+v^{(l)}\right) \geq 0, & \perp & \bar{c}_{n}^{(j)} \geq 0, \quad j \in \mathcal{A} \\
\bar{\sigma}^{(j)}=\bar{\lambda}^{(j)} e^{(j)}+D^{(j) T}\left(\bar{v}^{(l+1)}+v^{(l)}\right) \geq 0, & \perp & \bar{\beta}^{(j)} \geq 0, \quad j \in \mathcal{A} \\
\bar{\zeta}^{(j)}=\mu^{(j)} \bar{c}_{n}^{(j)}-e^{(j)^{T}} \bar{\beta}^{(j)} \geq 0, & \perp & \bar{\lambda}^{(j)} \geq 0, \quad j \in \mathcal{A},
\end{array}
$$

with

$$
\begin{equation*}
\bar{K}_{s}=\sum_{i=1}^{n_{\gamma}} \bar{c}_{\gamma}^{(j)} \nabla_{q} \phi^{(i)^{T}}\left(q^{(l)}\right)+\sum_{j=1}^{n_{\delta}} \bar{c}_{\delta}^{(j)} \nabla_{q} \psi^{(j)^{T}}\left(q^{(l)}\right)+\sum_{k=1}^{n_{\delta \gamma}} \bar{c}_{\delta \gamma}^{(k)} \nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right) . \tag{9.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{k}=\frac{h}{2}\left(k_{2}\left(t_{l}, q^{(l)}, v^{(l)}\right)+k_{2}\left(t_{l+1}, q^{(l)}, v^{(l)}\right)+f\left(v^{(l)}\right)\right) . \tag{9.8}
\end{equation*}
$$

Given the antisymmetry property of the matrix $F\left(v^{(l)}\right)$ and the positive definiteness of the mass matrix $M$ the LCP (9.6)-(9.8) will always have a solution as a result of Theorem 5.1. We denote a solution of this complementarity problem by

$$
\begin{equation*}
\mathcal{U}=\left(\bar{v}^{(l+1)}, \widetilde{\bar{c}}_{\nu}, \widetilde{\bar{c}}_{n}, \widetilde{\bar{\beta}}, \widetilde{\bar{c}}_{\gamma}, \widetilde{\bar{c}}_{\delta}, \widetilde{\bar{c}}_{\delta \gamma}\right) . \tag{9.9}
\end{equation*}
$$

The symbol ${ }^{\sim}$ is used here to denote aggregate quantities with the same base symbol, such as $\widetilde{\bar{c}}_{\delta}=$ $\left(\bar{c}_{\delta}^{(1)}, \bar{c}_{\delta}^{(2)}, \ldots \bar{c}_{\delta}^{\left(n_{\delta}\right)}\right)$. In analyzing the limit behavior of the time-stepping (5.3)-(5.8), we will assume that the friction cone of the limit system (9.6)-(9.8) is pointed. Inspecting the time-stepping scheme (9.6)-(9.8), we obtain the following expression for the total friction cone of the limit problem:

$$
\begin{array}{r}
\widehat{F C}(q)=\left\{f \mid f=\sum_{i=1}^{m} \bar{\nu}^{(i)} \bar{c}_{\nu}^{(i)}+\sum_{j \in \mathcal{A}}\left(n^{(j)} \bar{c}_{n}^{(j)}+D^{(j)} \bar{\beta}^{(j)}\right)+\sum_{i=1}^{n_{\gamma}} \bar{c}_{\gamma}^{(i)} \nabla_{q} \phi^{(i)}(q)+\right. \\
\left.\sum_{j=1}^{n_{\delta}} \bar{c}_{\delta}^{(j)} \nabla_{q} \psi^{(j)^{T}}(q)+\sum_{k=1}^{n_{\delta \gamma}} \bar{c}_{\delta \gamma}^{(k)} \nabla_{q} \bar{\phi}^{(k)}(q) \quad \mid \quad \bar{c}_{n}^{(j)} \geq 0, \bar{\beta}^{(j)} \geq 0, \mu^{(j)} \bar{c}_{n}^{(j)}-e^{(j)^{T}} \bar{\beta}^{(j)} \geq 0, j \in \mathcal{A}\right\} . \tag{9.10}
\end{array}
$$

If the friction cone $\widehat{F C}(q)$ is pointed, then, by definition, there exists a parameter $c_{F C}$ such that, with the notations in (9.10), we have

$$
\begin{equation*}
f \in \widehat{F C}(q) \Rightarrow\left\|\left(\widetilde{\bar{c}}_{\nu}, \widetilde{\bar{c}}_{n}, \widetilde{\bar{\beta}}, \widetilde{\bar{c}}_{\gamma}, \widetilde{\bar{c}}_{\delta}, \widetilde{\bar{c}}_{\delta \gamma}\right)\right\| \leq c_{F C}\|f\| \tag{9.11}
\end{equation*}
$$

### 9.2 A stability result

We now analyze the accumulation points of the solution of (5.3)-(5.8) as we keep the step-size $h$ constant and as we increase the stiffness parameters to infinity. We will show that under conditions that ensure that our energy results from section 8 apply, such accumulation points will be solutions $\mathcal{U}$ of (9.6). In other words, in the limit, our linearly implicit LCP scheme will behave as would a similar scheme applied to a system with additional joint constraints in place of the dampers and the coupled damper-spring mechanisms.

Theorem 9.3 Assume that the total friction cone $\widehat{F C}(q)$ (9.10) of the limit LCP integration step (9.6) is pointed. Let $q^{(l)}$ be a position vector point where $\bar{\phi}^{(k)}(q)=0, k=1,2, \ldots, n_{\delta \gamma}$. Let $\mathcal{U}^{\Gamma}=$ $\left(v^{(l+1)^{T \Gamma}}, \widetilde{c}_{\nu}^{\Gamma}, \widetilde{c}_{n}^{\Gamma}, \widetilde{\beta}^{\Gamma}, \tilde{\lambda}^{\Gamma}\right)$ be a solution of (5.10)-(5.11) where the external force is defined by (9.1) and the matrices $\widetilde{M}^{(l)}$ and $\widetilde{k^{l}}$ are defined by (5.9) using the approximations (9.3), for a particular choice of the stiffness parameters $\Gamma=\left(\gamma_{1}, \gamma_{2}, \ldots, \gamma_{n_{\gamma}}, \delta_{1}, \delta_{2}, \ldots, \delta_{n_{\delta}}, \bar{\gamma}_{1}, \bar{\gamma}_{2}, \ldots, \bar{\gamma}_{n_{\gamma \delta}}, \bar{\delta}_{1}, \bar{\delta}_{2}, \ldots, \bar{\delta}_{n_{\gamma \delta}}\right)$ Let $\Gamma_{n}$ be a sequence of stiffness parameters such that, as $n \rightarrow \infty$, all its components increase toward infinity. Then the sequence $\mathcal{U}^{\Gamma_{n}}$ is uniformly bounded, and any limit point, together with appropriate multipliers for the additional joint constraints, is a solution of (9.6): the system that is obtained by replacing the stiff forces by rigid links.

Proof We introduce the following notations:

$$
\begin{aligned}
\widetilde{c}_{\gamma}^{\Gamma} & =-\frac{h^{2}}{4}\left(\gamma_{1} \nabla_{q} \phi^{(1)^{T}}\left(q^{(l)}\right)\left(v^{(l+1)^{\Gamma}}+v^{(l)}\right), \ldots, \gamma_{n_{\gamma}} \nabla_{q} \phi^{\left(n_{\gamma}\right)^{T}}\left(q^{(l)}\right)\left(v^{(l+1)^{\Gamma}}+v^{(l)}\right)\right) \\
\widetilde{c}_{\delta}^{\Gamma} & =-\frac{h}{2}\left(\delta_{1} \nabla_{q} \psi^{(1)^{T}}\left(q^{(l)}\right)\left(v^{(l+1)^{\Gamma}}+v^{(l)}\right), \ldots, \delta_{n_{\delta}} \nabla_{q} \psi^{\left(n_{\delta}\right)^{T}}\left(q^{(l)}\right)\left(v^{(l+1)^{\Gamma}}+v^{(l)}\right)\right) \\
\widetilde{c}_{\delta \gamma}^{\Gamma} & =-\frac{h}{2}\left(\left(\frac{h}{2} \bar{\gamma}_{1}+\bar{\delta}_{1}\right) \nabla_{q} \bar{\phi}^{(1)^{T}}\left(q^{(l)}\right)\left(v^{(l+1)^{\Gamma}}+v^{(l)}\right), \ldots,\left(\bar{\gamma}_{n_{\delta \gamma}}+h \bar{\delta}_{n_{\delta \gamma}}\right) \nabla_{q} \phi^{\left(n_{\delta \gamma}\right)^{T}}\left(q^{(l)}\right)\left(v^{(l+1)^{\Gamma}}+v^{(l)}\right)\right) \\
\widetilde{\nu}_{\gamma} & =\left(\nabla_{q} \phi^{(1)}\left(q^{(l)}\right), \nabla_{q} \phi^{(2)}\left(q^{(l)}\right), \ldots, \nabla_{q} \phi^{\left(n_{\gamma}\right)}\left(q^{(l)}\right)\right) \\
\widetilde{\nu}_{\delta} & =\left(\nabla_{q} \psi^{(1)}\left(q^{(l)}\right), \nabla_{q} \psi^{(2)}\left(q^{(l)}\right), \ldots, \nabla_{q} \psi^{\left(n_{\delta}\right)}\left(q^{(l)}\right)\right) \\
\widetilde{\nu}_{\gamma \delta} & =\left(\nabla_{q} \bar{\phi}^{(1)}\left(q^{(l)}\right), \nabla_{q} \bar{\phi}^{(2)}\left(q^{(l)}\right), \ldots, \nabla_{q} \bar{\phi}^{\left(n_{\gamma \delta}\right)}\left(q^{(l)}\right)\right)
\end{aligned}
$$

Using the fact that $\mathcal{U}^{\Gamma}$ satisfies (5.3)-(5.8), the assumptions of the theorem, the definition of $\widetilde{M}$, the approximations (9.3), and that

$$
\begin{aligned}
\widetilde{k} & =h F\left(v^{l}\right) v^{l}-\frac{h^{2}}{2} \sum_{i=1}^{n_{\gamma}} \gamma_{i}\left(\nabla_{q} \phi^{(i)^{T}}\left(q^{(l)}\right) v^{(l)}\right) \nabla_{q} \phi^{(i)}\left(q^{(l)}\right)-\frac{h^{2}}{2} \sum_{k=1}^{n_{\gamma_{\delta}}} \bar{\gamma}_{k}\left(\nabla_{q} \bar{\phi}^{(k)^{T}}\left(q^{(l)}\right) v^{(l)}\right) \nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right) \\
& -h \sum_{j=1}^{n_{\delta}} \delta_{j}\left(\nabla_{q} \psi^{(j)}\left(q^{l}\right)^{T} v^{l}\right) \nabla_{q} \psi^{(j)}\left(q^{l}\right)-h \sum_{k=1}^{n_{\delta \gamma}} \bar{\delta}_{k}\left(\nabla_{q} \bar{\phi}^{(k)}\left(q^{l}\right)^{T} v^{l}\right) \nabla_{q} \bar{\phi}^{(k)}\left(q^{l}\right) \\
& +\frac{h}{2}\left(k_{2}\left(t_{l}, q^{l}, v^{l}\right)+k_{2}\left(t_{l+1}, q^{l}, v^{l}\right)\right)
\end{aligned}
$$

we obtain

$$
\begin{align*}
&\left(M-h F\left(v^{(l)}\right)\right) v^{(l+1)^{\Gamma}}-\widetilde{\nu} \widetilde{c}_{\nu}^{\Gamma}-\widetilde{n} \widetilde{c}_{n}^{\Gamma}-\widetilde{D}_{\beta^{\Gamma}}-\widetilde{\nu}_{\gamma} \widetilde{c}_{\gamma}^{\Gamma}-\widetilde{\nu}_{\delta} \widetilde{c}_{\delta}^{\Gamma}-\widetilde{\nu}_{\gamma \delta} \widetilde{c}_{\gamma \delta}^{\Gamma}=  \tag{9.12}\\
& M v^{(l)}+\frac{h}{2}\left(k_{2}\left(t_{l}, q^{(l)}, v^{(l)}\right)+k_{2}\left(t_{l+1}, q^{(l)}, v^{(l)}\right)+f\left(v^{(l)}\right)\right)
\end{align*}
$$

Corollary 9.2 ensures that $v^{(l+1) \Gamma}$ is bounded uniformly with respect to $\Gamma$, and thus there exists a constant $K_{F C}$ independent of $\Gamma$ such that

$$
\left\|\left(M-\frac{h}{2} F\left(v^{l}\right)\right) v^{(l+1)^{\Gamma}}-M v^{(l)}-\frac{h}{2}\left(k_{2}\left(t_{l}, q^{(l)}, v^{(l)}\right)+k_{2}\left(t_{l+1}, q^{(l)}, v^{(l)}\right)+f\left(v^{(l)}\right)\right)\right\| \leq K_{F C}
$$

Using now the characterization (9.11) of the pointed cone $\widehat{F C}\left(q^{(l)}\right)$, we obtain that, for all $\Gamma$,

$$
\begin{equation*}
\left\|\left(\widetilde{c}_{\nu}^{\Gamma}, \widetilde{c}_{n}^{\Gamma}, \widetilde{\beta}^{\Gamma}, \widetilde{c}_{\gamma}^{\Gamma}, \widetilde{c}_{\delta}^{\Gamma}, \widetilde{c}_{\gamma \delta}^{\Gamma}\right)\right\| \leq c_{F C} K_{F C} . \tag{9.13}
\end{equation*}
$$

Now by taking the sequence $\Gamma_{n}$ and using the previous inequality we deduce that the sequence

$$
\mathcal{W}^{n}=\left(v^{(l+1)^{\Gamma_{n}}}, \widetilde{c}_{\nu}^{\Gamma_{n}}, \widetilde{c}_{n}^{\Gamma_{n}}, \widetilde{\beta}^{\Gamma_{n}}, \widetilde{c}_{\gamma}^{\Gamma_{n}}, \widetilde{c}_{\delta}^{\Gamma_{n}}, \widetilde{c}_{\gamma \delta}^{\Gamma_{n}}\right)
$$

is uniformly bounded with respect to the sequence $\Gamma_{n}$. The uniform boundedness of the above sequence implies the existence of an accumulation point. Let

$$
\overline{\mathcal{W}}=\left(\bar{v}^{(l+1)}, \widetilde{\bar{c}}_{\nu}, \widetilde{\bar{c}}_{n}, \widetilde{\bar{\beta}}, \widetilde{\bar{c}}_{\delta}, \widetilde{\bar{c}}_{\gamma \delta}\right)
$$

be such an accumulation point. For purposes of this proof we will assume, without loss of generality that $\lim _{n \rightarrow \infty} \mathcal{W}^{n}=\overline{\mathcal{W}}$. In particular, we must have that $\lim _{n \rightarrow \infty} v^{(l+1)^{n}}=\bar{v}^{(l+1)}$. Also, by using the definitions of $\widetilde{c}_{\delta}^{\Gamma}, \widetilde{c}_{\gamma \delta}^{\Gamma}$, and (9.13), we obtain

$$
\begin{align*}
\frac{h^{2}}{4} \gamma_{i}^{n}\left|\nabla_{q} \phi^{(i)}\left(q^{(l)}\right)^{T}\left(v^{\left.(l+1)^{\Gamma_{n}}+v^{(l)}\right)}\right)\right| \leq c_{F C} K_{F C}, & i=1,2, \ldots, n_{\gamma} \\
h \delta_{j}^{n} \mid \nabla_{q} \psi^{(j)}\left(q^{(l)}\right)^{T}\left(v^{\left.(l+1)^{\Gamma_{n}}+v^{(l)}\right)}\right) \leq c_{F C} K_{F C}, & j=1,2, \ldots, n_{\delta}  \tag{9.14}\\
h \bar{\delta}_{k}^{n} \mid \nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right)^{T}\left(v^{\left.(l+1)^{\Gamma_{n}}+v^{(l)}\right)} \mid \leq c_{F C} K_{F C},\right. & k=1,2, \ldots, n_{\gamma \delta}
\end{align*}
$$

Now, dividing both inequalities in (9.14) by the corresponding stiffness parameters, and taking the limit as $n \rightarrow \infty$ we get

$$
\begin{align*}
\nabla_{q} \phi^{(i)}\left(q^{(l)}\right)^{T}\left(\bar{v}^{(l+1)}+v^{(l)}\right) & =0 & & i=1,2, \ldots, n_{\gamma} \\
\nabla_{q} \psi^{(j)}\left(q^{(l)}\right)^{T}\left(\bar{v}^{(l+1)}+v^{(l)}\right) & =0 & & j=1,2, \ldots, n_{\delta}  \tag{9.15}\\
\nabla_{q} \bar{\phi}^{(k)}\left(q^{(l)}\right)^{T}\left(\bar{v}^{(l+1)}+v^{(l)}\right) & =0 & & k=1,2, \ldots, n_{\gamma \delta}
\end{align*}
$$

Since $\mathcal{W}^{n}$ satisfies (9.12), all inequalities and complementarity relations of (5.3)-(5.8), which are homogeneous, so will $\overline{\mathcal{W}}$. Since, in addition, $\bar{v}^{(l+1)}$ satisfies (9.15), we infer that $\overline{\mathcal{W}}$ is indeed a solution of (9.6) for the stiff force $k 1(t, q, v)$ given by (9.1), which proves the claim.

## 10 Numerical Simulations

To verify the theoretical results described above we have implemented a Matlab version of the algorithm. The appropriate LCPs were solved using the PATH package [8]. The numerical experiments performed were concerned with the accuracy of the method and its stability in the presence of stiff forces originating in springs and dampers. To maintain the accuracy of the scheme one must use an appropriate model for estimating the collision data. Assume that following the strategy presented in section 6 , we have detected that the earliest collision is given by $\phi^{(\bar{j})}(q)$. For simplicity we will omit the superscript $\bar{j}$. The next step will be to construct a cubic interpolation polynomial using the data $\phi(q(t)), \phi_{q}(q(t))^{T} v(t)$, $\phi\left(q(t+h)\right.$ ), and $\phi_{q}(q(t+h))^{T} v(t+h)$. The collision time $t^{*}$ will be found as the solution of the interpolating polynomial in the considered time interval. The position as well as the pre-collision velocity can be computed now by evaluating $\widetilde{q}$ and $\widetilde{q}^{\prime}$ at $t^{*}$ as explained in section 6 .

### 10.1 Numerical validation of the order of convergence

The rigid-body system used in the error analysis consists of a double pendulum constrained by a vertical wall, [32] as illustrated in Figure 1. The wall is situated at the position $x=0$. The masses of the two bobs are $m_{1}=m_{2}=1$, and the lengths of the massless rods are taken to be $L_{1}=L_{2}=1$.


Figure 1: A double pendulum with a long wall.
In modeling this system we have used the Cartesian coordinates of the point masses $m_{1}$ and $m_{2}$, denoted by $\left(x_{1}(t), y_{1}(t)\right)$ and $\left(x_{2}(t), y_{2}(t)\right)$ respectively. The initial position is given by

$$
q(0)=\left(x_{1}(0), y_{1}(0), x_{2}(0), y_{2}(0)\right)=\left(\sin \frac{\pi}{3},-\cos \frac{\pi}{3}, \sin \frac{\pi}{3}+\sin \frac{\pi}{5},-\cos \frac{\pi}{3}-\cos \frac{\pi}{5}\right)
$$

and the initial velocity is taken to be zero. The restitution coefficients were chosen both to be 0.1. The system was simulated for an interval of $2.5(\mathrm{~s})$, for different values of the step size $h$. The $x$-components of $m_{1}$ and $m_{2}$ together with the total energy of the system are plotted in Figure 2.


Figure 2: (a) The $x$-components of the positions of $m_{1}$ and $m_{2}$, (b) Total energy of the system.
To measure the error we have computed a reference solution $\widetilde{q}^{*}$ obtained by running the code with
$h=2^{-20}$. The error $E_{h}$ has been calculated as the 2-norm of the difference in positions at the final time $T$, or more precisely $E_{h}=\left\|q_{h}(T)-\widetilde{q}^{*}(T)\right\|_{2}$. The results are presented in Table 10.1.

| $h$ | $E_{h}$ | $R_{0, h}$ | $E_{1, h}$ | $R_{1, h}$ | $E_{2, h}$ | $R_{2, h}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $2^{-5}$ | $2.38 \mathrm{e}-003$ | $\mathrm{~N} / \mathrm{A}$ | $-1.27 \mathrm{e}-003$ | $\mathrm{~N} / \mathrm{A}$ | $-3.18 \mathrm{e}-003$ | N/A |
| $2^{-6}$ | $6.14 \mathrm{e}-004$ | $3.88 \mathrm{e}+000$ | $-3.14 \mathrm{e}-004$ | $4.04 \mathrm{e}+000$ | $-8.29 \mathrm{e}-004$ | $3.84 \mathrm{e}+000$ |
| $2^{-7}$ | $1.54 \mathrm{e}-004$ | $3.97 \mathrm{e}+000$ | $-7.82 \mathrm{e}-005$ | $4.01 \mathrm{e}+000$ | $-2.14 \mathrm{e}-004$ | $3.88 \mathrm{e}+000$ |
| $2^{-8}$ | $3.95 \mathrm{e}-005$ | $3.91 \mathrm{e}+000$ | $-1.95 \mathrm{e}-005$ | $4.00 \mathrm{e}+000$ | $-5.48 \mathrm{e}-005$ | $3.90 \mathrm{e}+000$ |
| $2^{-9}$ | $1.01 \mathrm{e}-005$ | $3.92 \mathrm{e}+000$ | $-4.88 \mathrm{e}-006$ | $4.00 \mathrm{e}+000$ | $-1.37 \mathrm{e}-005$ | $3.99 \mathrm{e}+000$ |
| $2^{-10}$ | $2.42 \mathrm{e}-006$ | $4.17 \mathrm{e}+000$ | $-1.22 \mathrm{e}-006$ | $4.00 \mathrm{e}+000$ | $-3.44 \mathrm{e}-006$ | $3.99 \mathrm{e}+000$ |
| $2^{-11}$ | $6.15 \mathrm{e}-007$ | $3.93 \mathrm{e}+000$ | $-3.05 \mathrm{e}-007$ | $4.01 \mathrm{e}+000$ | $-8.60 \mathrm{e}-007$ | $4.00 \mathrm{e}+000$ |

Table 10.1: The error results for the double pendulum.
The first column of Table 10.1 lists the values of the time-step used to obtain the numerical solution. The second column gives the error $E_{h}$ as explained above while the fourth and the sixth columns represent the errors in the joints. More precisely, if we denote by $\Theta^{(1)}(q(t)):=x_{1}(t)^{2}+x_{2}()^{2}-1$, and $\Theta^{(2)}(q(t)):=\left(x_{1}(t)-x_{2}(t)\right)^{2}+\left(y_{1}(t)-y_{2}(t)\right)^{2}-1$ the bilateral constraints, then columns 4 and 6 represent the errors of $\Theta^{(1)}\left(q_{h}(t)\right)$ and $\Theta^{(2)}\left(q_{h}(t)\right)$, respectively. The ratios $R_{j, h}, j=1,2$, will then measure the factor of decrease in these errors, i.e.,

$$
R_{j, h}=\left|\frac{\Theta^{(j)}\left(q_{h}\right)}{\Theta^{(j)}\left(q_{h / 2}\right)}\right|, \quad j=1,2 .
$$

These ratios are represented as columns 5 and 7 in Table 10.1. In the same way one defines the ratio represented by the second column, which gives the decrease in the calculated error. As we can see from Table 10.1, by halving the step size we achieve an error reduced by approximately a factor of 4 , which shows the second order convergence of the method applied to this system. The same type of behavior was observed in the error accumulated in the joints.

### 10.2 Numerical validation of the stability of the scheme

The second numerical experiment is concerned with the stability analysis of the proposed scheme when the system is subjected to stiff forces that originate in springs and/or dampers. The behavior predicted by the theoretical results presented in this work will be that, for dampers and isolated springs, when the stiffness parameters are increased to infinity, the system will behave as if the corresponding springs and/or dampers were to be replaced by rigid joints.

The mechanical system considered here consists of two carts of unit mass sliding on a flat surface, $[6,32]$. The left-most cart is connected by a linear spring with elasticity $\gamma$ to an immovable wall. A stopper acting on the left-most cart is placed at $x=0$. The spring is at equilibrium when the position of the left cart is $x=0.2$. The two carts are connected by a linear damper of stiffness $\delta$. In all the examples presented here, the original position of the second cart will be within 5 units from the left-most cart, and the friction coefficient between the carts and the floor is 0.05 . A graphical representation of this system is given in Figure 3. All the simulations are done with a time-step $h=0.01$ and the restitution coefficient between the left-most cart and the stopper is taken to be $e=0.3$.

The generalized position of the system is given by the $x$-coordinates of the carts: $x_{1}(t)$ represents the position of the left most cart and $x_{2}(t)$ the position of the right cart. We are interested in the behavior of the system when the stiffness parameters $\gamma$ or $\delta$ tend to infinity. We conducted the following experiments:

- Increasing the damping coefficient $(\delta \rightarrow \infty)$. The initial position is $q(0)=(0.1,5.1)$ and the initial velocity $v(0)=(-2,-2)$. The system is simulated for a total of 4 (s). The elasticity


Figure 3: Two carts and a stopper. The wheels are drawn for aesthetic reasons, they are assumed to be immobile.
parameter $\gamma$ is fixed at 100. Figure 4 depicts on the left, the distance between the two carts and on the right, the total energy of the system, for $\delta=10^{2}, 10^{3}$ and $10^{6}$. We can see that as the stiffness increases to infinity the variation in the distance between the carts approaches zero, which means that the dynamics of the system approaches the dynamics of the system where the damper is replaced by a rigid link, as predicted by Theorem 9.3. We also notice that the total energy of the system bounded even if the stiffness goes to infinity, which is also consistent with our theory.



Figure 4: Results for increasing values of $\delta$. On the left we have plotted the variation of the distance between the carts, $x_{2}(t)-x_{1}(t)$, and on the right the total energies.

- Increasing the elasticity parameter $(\gamma \rightarrow \infty)$. We start with the spring in equilibrium, so the initial position is $q(0)=(0.2,5.2)$, while the initial velocity is $v(0)=(-3,-3)$. The system is simulated for a total of $2(\mathrm{~s})$, with the stiffness of the damper fixed to $\delta=10$ and the elasticity constant of the spring taking the values $\gamma=10^{2}, 10^{4}$ and $10^{6}$. Figure 5 depicts on the left, the distance between the left-most cart and the equilibrium position of the spring, and on the right the total energy of the system for the values of $\gamma$ discussed above. Note that as we increase the elasticity constant of the spring the distance between the left-most cart and its equilibrium position approaches zero, which means that the dynamics of the system approaches the dynamics of the system where the spring is replaced with a rigid link, as predicted by Theorem 9.3. As in the previous case the total energy of the system is bounded with respect to the stiff parameter.


Figure 5: Results for increasing values of $\gamma$. On the left we have plotted the the distance between the left-most cart and the equilibrium position of the spring and on the right the total energies.

### 10.3 The use of the linearized trapezoidal method in a virtual prototyping environment

The time-stepping scheme from this work was implemented in an industrial-grade virtual prototyping simulation package, UMBRA [10], by a group at Sandia National Laboratories that included two of the authors of this paper (Trinkle and Potra). Constraint stabilization is achieved by including in the right hand side of our LCP a term that is connected to the linearization of the constraints [4].

Two of the applications simulated by this package are presented in Figure 6. The image on the left depicts a four-wheeled rover with a robot arm traversing an unknown landscape. With a terrain map and a traction model, one could plan optimal exploration sortees. The image on the right show a pawl ( 2.3 mm in length) from a micro-mechanical machine being pushed into an unusually shaped hole as a step in an assembly operation. Rigid body dynamics was used to design the best shape of the hole and cantilever beam (below pawl) to make insertion easy. Movies of these systems can be viewed at http://www.cs.rpi.edu/~trink/.

## 11 Conclusions and future work

We have presented a linearized trapezoidal scheme for the simulation of rigid multibody dynamics with contact and friction. The scheme solves only one linear complementarity problem per time step while achieving second order convergence, a fact that is both proven analytically and demonstrated by numerical simulations. For the case where the scheme must accommodate stiff forces that originate in springs and dampers attached between two points of the system, we show that using an approximate Jacobian as in [2] results in the scheme still being solvable and stable. In addition, as the stiffness parameters approach infinity, the numerical sequence does not blow up, instead it approaches the one that is generated by replacing the stiff springs and dampers by rigid links, a fact that is also both proven analytically and demonstrated by numerical simulations. To our knowledge, this is the first presentation of a scheme with these properties. In addition, the scheme has been successfully implemented in and industrial grade rigid multibody dynamics simulating system, UMBRA [10].

Some of the issues that still need to be resolved are achieving constraint stabilization without using any additional projection [4], maintaining second order convergence while working with a convenient


Figure 6: Examples of UMBRA simulations, that uses the time-stepping scheme developed in this work
approximate Jacobian, eventually by switching to an exact Jacobian when the time step is sufficiently small, as well as establishing the convergence of the scheme to the a solution of a measure differential inclusion as the time step goes to zero [26]. Although none of these issues are completely addressed here, this work contains much of the key results which have allowed to prove or demonstrate the similar conclusions for the linearized backward Euler scheme [2, 30, 26, 4] which leads us to believe that these issues will likely be successfully resolved.

## Acknowledgments

The authors are grateful to Todd Munson and Michael Ferris for providing PATH [8], a solver for the general linear complementarity problem and to Eric Gottlieb and Patrick Xavier for implementing the rigid body time-stepping and collision/proximity detection algorithms in UMBRA. Florian Potra, Bogdan Gavrea and Jeff Trinkle have been supported by the National Science Foundation through Grant DMS-0139701. Mihai Anitescu was supported by the Mathematical, Information, and Computational Sciences Division subprogram of the Office of Advanced Scientific Computing Research, Office of Science, U.S. Department of Energy, under Contract W-31-109-ENG-38.

## References

[1] M. Anitescu and F. A. Potra. Formulating dynamic multi-rigid-body contact problems with friction as solvable linear complementarity problems. Nonlinear Dynamics, 14:231-247, 1997.
[2] M. Anitescu and F. A. Potra. A time-stepping method for stiff multibody dynamics with contact and friction. International Journal for Numerical Methods in Engineering, 55(7):753-784, 2002.
[3] Mihai Anitescu, James F. Cremer, and Florian A. Potra. Formulating 3d contact dynamics problems. Mechanics of Structures and Machines, 24(4):405-437, 1996.
[4] Mihai Anitescu and Gary D. Hart. A constraint-stabilized time-stepping approach for rigid multibody dynamics with joints, contact and friction. International Journal for Numerical Methods in Engineering, page to be determined by the publisher, 2004.
[5] Mihai Anitescu and Florian A. Potra. Formulating dynamic multi-rigid-body contact problems with friction as solvable linear complementarity problems. Nonlinear Dynamics, 14:231-247, 1997.
[6] B. Brogliato. Nonsmooth impact mechanics. Springer-Verlag London Ltd., London, 1996. Models, dynamics and control.
[7] J. C. and G. Vaněček. Isaac: Building simulations for virtual environments. In Workshop IFIP IC 5 WG 5.10 on Virtual Environments, Coimbra, Portugal, 1994.
[8] Michael Ferris and Todd Munson. Interfaces to PATH 3.0: Design, implementation and usage. Computational Optimization and Applications, 12:207-227, 1999.
[9] Christian Glocker and Friedrich Pfeiffer. Multiple impacts with friction in rigid multi-body systems. Nonlinear Dynamics, 7:471-497, 1995.
[10] Eric J. Gottlieb, Michael J. McDonald, Fred J. Oppel, J. Brian Rigdon, and Patrick G. Xavier. The umbra simulation framework as applied to building hla federates. In Proceedings of the 2002 Winter Simulation Conference, pages 981-989, San Diego, California, 2002.
[11] W. B. Gragg and R. A. Tapia. Optimal error bounds for the Newton-Kantorovich theorem. SIAM J. Numer. Anal., 11:10-13, 1974.
[12] E. Hairer and G. Wanner. Solving ordinary differential equations. II. Springer-Verlag, Berlin, second edition, 1996. Stiff and differential-algebraic problems.
[13] E.J. Haug. Computer Aided Kinematics and Dynamics of Mechanical Systems. Allyn and Bacon, Boston, 1989.
[14] L. Kantorovich. On Newton's method for functional equations (Russian). Dokl. Akad. Nauk. SSSR, 59:1237-1240, 1948.
[15] M. D. P.Monteiro Marques. Differential Inclusions in Nonsmooth Mechanical Problems: Shocks and Dry Friction, volume 9 of Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser Verlag, Basel, Boston, Berlin, 1993.
[16] M. T. Mason and Y. Wang. On the inconsistency of rigid-body frictional planar mechanics. In Proceedings, IEEE International Conference on Robotics and Automation, pages 524-528, April 1988.
[17] J.-J. Moreau. Standard inelastic shocks and the dynamics of unilateral constraints. In G. del Piero and F. Maceri, editors, Unilateral Problems in Structural Mechanics, volume 288 of C.I.S.M. Courses and Lectures, pages 173-221, Vienna, New York, 1985. Springer-Verlag.
[18] J.-J. Moreau. Bounded variation in time. In Topics in Nonsmooth Mechanics, pages 1-74. Birkhäuser, Basel-Boston, MA, 1988.
[19] Richard M. Murray, Zexiang Li, and S. Shankar Sastry. A mathematical introduction to robotic manipulation. CRC Press, Boca Raton, FL, 1993.
[20] P. Painlevé. Sur le lois du frottement de glissemment. Comptes Rendus Acad. Sci. Paris, 121:112115, 1895. Following articles under the same title appeared in this journal, vol. 141, pp. 401-405 and 546-552 (1905).
[21] F. Pfeiffer and Ch. Glocker. Multibody dynamics with unilateral contacts. Wiley Series in Nonlinear Science. John Wiley \& Sons Inc., New York, 1996. A Wiley-Interscience Publication.
[22] F. A. Potra. The Kantorovich Theorem and interior point methods. Math. Programming, (Ser. A, electronic):DOI 10.1007/s10107-003-0501-8, 2004.
[23] F. A. Potra. On a class of linearly implicit methods for integrating the Euler-Lagrange equations of motion. preprint, UMBC, May 2004.
[24] F. A. Potra and V. Pták. Nondiscrete induction and iterative processes. Number 103 in Research Notes in Mathematics. John Wiley \& Sons, Boston-London-Melbourne, 1984.
[25] P. Song, J.C. Trinkle, V. Kumar, and J.-S. Pang. Design of part feeding and assembly processes with dynamics. In Proceedings, IEEE International Conference on Robotics and Automation, April 2004.
[26] D. E. Stewart. Convergence of a time-stepping scheme for rigid body dynamics and resolution of Painlevé's problems. Archive Rational Mechanics and Analysis, 145(3):215-260, 1998.
[27] D. E. Stewart and J. C. Trinkle. Dynamics, friction, and complementarity problems. In Complementarity and variational problems (Baltimore, MD, 1995), pages 425-439. SIAM, Philadelphia, PA, 1997.
[28] D. E. Stewart and J.C. Trinkle. An implicit time-stepping scheme for rigid body dynamics with inelastic collisions and coulomb friction. International J. Numer. Methods Engineering, 39(15):281287, 1996.
[29] David E. Stewart. Rigid-body dynamics with friction and impact. SIAM Review, 42(1):3-39, 2000.
[30] David E. Stewart and Jeffrey C. Trinkle. An implicit time-stepping scheme for rigid-body dynamics with inelastic collisions and coulomb friction. International Journal for Numerical Methods in Engineering, 39:2673-2691, 1996.
[31] D.E. Stewart. Rigid-body dynamics with friction and impact. SIAM Rev., 42(1):3-39 (electronic), 2000.
[32] J. A. Tzitzouris. Numerical resolution of frictional multi-rigid-body systems via fully implicit timestepping and nonlinear complementarity. PhD thesis, The Johns Hopkins University, 2001.

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[^1]:    ${ }^{1}$ Formulation in terms of accelerations and forces is the most natural approach, since the Newton-Euler equations are typically expressed in these quantities.

[^2]:    The submitted manuscript has been created by the University of Chicago as Operator of Argonne National Laboratory ("Argonne") under Contract No. W-31-109-ENG-38 with the U.S. Department of Energy. The U.S. Government retains for itself, and others acting on its behalf, a paid-up, nonexclusive, irrevocable worldwide license in said article to reproduce, prepare derivative works, distribute copies to the public, and perform publicly and display publicly, by or on behalf of the Government.

