# Uniqueness of the Positive Solution of $\Delta u + f(u) = 0$ in an Annulus *

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**Abstract**

We give here an extension of the recent result of Kwong (which in turn extended earlier results of Coffman and McLeod and Serrin) on the uniqueness of the positive radial solution of a semilinear elliptic equation. When reduced to the special case considered by Kwong, our proof is shorter.

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**Proposed Running Head.** Uniqueness of Ground State

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# 1 Introduction

Let $R^n$ $(n > 2)$ denote the usual $n$-dimensional Euclidean space. Given $0 \le a < b \le \infty$, we consider the "annular" domain

$$\Omega = \begin{cases} \{a < |x| < b\} & \text{if } a > 0, \\ \{|x| < b\} & \text{if } a = 0\,. \end{cases} \tag{1.1}$$

When $a > 0$, $\Omega$ is the usual annulus or the exterior of the ball of radius $a$, depending on whether $b$ is finite or not. When $a = 0$, $\Omega$ is either the ball of radius $b$, or (when $b = \infty$) the entire $R^n$. We are concerned here with the uniqueness of positive radial solutions of the problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{when } |x| = a > 0 \\ u(x) = 0 & \text{when } |x| = b, \end{cases} \tag{I}$$

where $f(u)$ belongs to a certain class of nonlinear functions. When $a = 0$, the first boundary condition is automatically fulfilled. When $b = \infty$, the second boundary condition in (I) is interpreted to be $u(x) \to 0$ as $|x| \to \infty$.

This problem arises in many applications. Please consult [8,9] for more information.

Existence of positive solutions for various classes of $f(u)$ has been obtained by many authors, including Berestycki and Lions [1], and Berestycki, Lions, and Peletier [2]. The uniqueness problem is, however, more difficult. In general, a positive solution of (I) need not be radially symmetric. The interest in radial solutions is sparked by the well-known results of Gidas, Ni, and Nirenberg in [4,5]. When $a = 0$ and $b < \infty$, they showed that any positive solution of (I) is necessarily radially symmetric. When $b = \infty$, the same conclusion is true under some additional restrictions on $f(u)$. The result, however, may not hold if either $a > 0$ or $b = \infty$ and $f(u)$ is, for instance, not Lipschitz.

When we confine ourselves to radial solutions, problem (I) is reduced to the following equivalent problem which involves an ordinary differential equation:

$$\begin{cases} u'' + \dfrac{m}{r}u'(r) + f(u) = 0 & a < r < b \\ u'(a) = 0 & \text{if } a > 0 \\ u(b) = 0, & \end{cases} \tag{II}$$

where $r$ denotes the radial variable, and $m = n - 1 > 0$. Since the case with $m \leq 1$ has been completely settled by McLeod and Serrin [10], we assume throughout this paper that $m > 1$.

The study of uniqueness can be traced back to Coffman [3] who obtained an affirmative answer for the case $n = 3$, and $f(u) = u^3 - u$. Peletier and Serrin [13] succeeded in showing uniqueness for functions satisfying a starlike condition. Their result has been further extended by Kaper and Kwong [7]. Other uniqueness results have been obtained by Ni [11] and Ni and Nussbaum [12]. McLeod and Serrin [10] continued the study of Coffman and extended his result to functions with a certain convexity property. Nevertheless, there is still a gap between the class of functions that admits existence and those covered by their results. Recently Kwong narrowed the gap by proving uniqueness when $f(u) = u^p - u$, for all $p \in (1, \dfrac{n+2}{n-2})$. The proof is very complicated and hard to adapt to include more general functions.

In this paper, we give a shorter proof of Kwong's result. Furthermore, the technique works for a much wider class of nonlinear functions including $f(u) = u^p - \sum c_i u^{q_i}$, with $q_i < p$. The approach is a combination of those of Zhang [14] and Kwong [9]. After the present work was completed, we learned that McLeod has also obtained similar results using related techniques. In most aspects our results are more general. On the other hand, we are indebted to Professor McLeod for a lemma (Lemma 1 below, with $a = 0$) that allows us to relax one of the original conditions we imposed on $f(u)$.

We assume that $f$ satisfies the following conditions:

**[F1]** $f \in C^1[0, \infty)$.

**[F2]** There exists a constant $\theta > 0$ such that $f(u) < 0$ for $u < \theta$ and that $f(u) > 0$ for $u > \theta$. To cover the case $b = \infty$, we require in addition that $f(0) = 0$, and $f'(u) \leq 0$ in a neighborhood of $u = 0$.

**[F3]** Let $\phi$ be the point defined by $\int_0^\phi f(\sigma)\,d\sigma = 0$. In $[\phi, \infty)$, the function $G(u) = \dfrac{uf'(u)}{f(u)}$ is nonincreasing and converges to a finite limit $\lambda \geq 1$ as $u \to \infty$. In $[\theta, \phi)$, $G(u) \geq G(\phi)$, and in $[0, \theta)$, $G(u) \leq \lambda$, but it need not be monotone in either interval.

The additional condition in **[F2]** is not needed if we are interested only in boundary value problems in a bounded $\Omega$. If $f(0) \neq 0$, there cannot be a solution of the boundary value problem with $b = \infty$. Even when $f(0) = 0$, an additional hypothesis such as $f'(0) < 0$ is needed to guarantee existence. Since we are concerned only with the uniqueness question, we do not impose these conditions; we confine ourselves to the case where a solution in $[a, \infty)$ is assumed to exist. We impose the monotonicity of $f(u)$, or equivalently $f'(u) \leq 0$, in a neighborhood of 0 to ensure a desirable decay rate for the solutions. We believe that this condition can be relaxed with a refined argument.

Apart from the improvement obtained by introducing $\phi$, condition **[F3]** is equivalent to the I-condition first used by McLeod and Serrin [10]. They also used a less general condition that is sometimes more convenient to verify. In Section 4 we shall discuss the relations between these various conditions and give some examples.

Our main result can be stated as follows:

**Theorem 1** *Suppose that* **[F1]** – **[F3]** *hold. Then problem* (I) *has at most one positive radial solution, and problem* (II) *has at most one positive solution.*

Using the symmetry results of Gidas, Ni, and Nirenberg, we can state the following corollary:

**Corollary** *Suppose that* **[F1]** – **[F3]** *hold. Furthermore assume* $a = 0$ *and* $f'(0) < 0$. *Then problem* (I) *has at most one positive solution.*

Theorem 1 is proved in Section 2, assuming the validity of two key assertions, Lemmas 15 and 16 (which we prove in Section 3). The justification for postponing the proof of these lemmas is that they are the by-product of the investigation of a different boundary value problem which is of interest by itself. Also, Lemma 15 implies Theorem 2, which — roughly speaking — states that no two monotonically decreasing solutions of (II) that intersect below $u = \phi$ can cross the $r$-axis at the same point.

In Section 4, we discuss examples of $f(u)$ that satisfy the condition **[F3].**

There have been much new activities since the paper was completed. Here is a last minute update. Chen and Lin [16] used a different method involving the Pohozaev identity to establish the desired oscillatory behavior of the first variational equation, thereby obtaining uniqueness results. Although there is a large intersection between their criterion and ours, neither one contains the other. Yanagida [19] used the Pohozaev identity in a clever way to obtain the uniqueness of the ground state of the Matukuma equation, $\Delta u + \dfrac{u^p}{1+|x|^2}$. His technique was modified by Kwong and Li [18] to treat problem (II) with an additional first-order term, namely, $u'' + mu'/r + f(u) + g(r)u = 0$, and with general boundary conditions. For the case $g(x) = -1$, the result, besides providing a much shorter proof of the theorem in [9], now covers the case of all homogeneous boundary conditions. For the case $g(x) = 1$, and $f(u) = u^p$ with $p$ subcritical, the result resolves an open question first raised by Brezis and Nirenberg [15] in 1983 on the uniqueness of the positive solutions of

$$\Delta u + u^{p^*} + \lambda u = 0 \tag{1.2}$$

on the unit ball, with a critical exponent $p^*$. In [18], we also extended the technique we used here to obtain uniqueness for the Matukuma equation in a finite interval, a situation in which Yanagida's method no longer works. Some related uniqueness results for Emden-Fowler equations were established in [17] using a method of change of variables and differential inequalities — entirely different from the usual Coffman approach.

## 2 Proof of the Main Theorem

As in all previous work, we use the Kolodner-Coffman method [8]. Let $u(r, \alpha)$ be the solution to the initial value problem

$$\begin{cases} u'' + \dfrac{m}{r}u'(r) + f(u) = 0 & r > a,\ m > 1 \\ u(a) = \alpha, \quad u'(a) = 0. \end{cases} \tag{2.1}$$

Denote the first zero of $u(r, \alpha)$, if there is one, by $b(\alpha)$. It is well known that $u(r)$ is strictly decreasing in the interval $(a, b(\alpha))$. By **[F1]**, $u(r, \alpha)$ cannot be tangential to the $r$-axis at $b(\alpha)$. Hence $b(\alpha)$ is a continuous function of $\alpha$. We also study the variational equation

$$\begin{cases} w'' + \dfrac{m}{r}w' + f'(u)w = 0, & r > a, \\ w(a) = 1, \quad w'(a) = 0, \end{cases} \tag{2.2}$$

which is satisfied by the function

$$w(r, \alpha) = \frac{\partial u(r, \alpha)}{\partial \alpha}. \tag{2.3}$$

For simplicity, we frequently write $w(r)$ for $w(r, \alpha)$ whenever it is clear from the context what value of $\alpha$ has been chosen.

Following Coffman, uniqueness for positive solutions on bounded intervals holds if we can show that the function $w(r)$ changes sign exactly once in $[a, b(\alpha)]$, so that $w(b(\alpha)) < 0$. More work is needed, however, to handle unbounded intervals.

We point out that an approximating procedure could be used to relax the smoothness requirements on $f(u)$, but we shall not pursue this further in the present paper. Suppose that $f(u)$ is the $C^1$ limit of a sequence of nonlinear functions $f_n(u)$, namely, that $f_n(u) \to f(u)$ and $f_n'(u) \to f'(u)$ uniformly on any compact interval. If we can prove that for each $n$, the corresponding function $w_n(r, \alpha)$ changes sign exactly once in $[a, b(\alpha)]$, then by taking the limit, we conclude that $w(r, \alpha)$ also changes sign exactly once in the interior of the interval. But we have only $w(b(\alpha)) \leq 0$. We thus "almost" have uniqueness for the limiting reaction function $f(u)$. If we can exclude the possibility $w(b(\alpha)) = 0$, for instance, using Lemma 8, we can then restore uniqueness.

The first step in our proof of Theorem 1 is to ensure that $w(r)$ has at least one zero in $[a, b(\alpha))$. This is true in general if $f(u)$ is superlinear (namely, if $f(u)/u$ is an increasing function, or equivalently if $\lambda = 1$ in **[F3]**). Our first lemma shows that this is still true if $\lambda > 1$.

**Lemma 1** *The function $w(r)$ must vanish somewhere in $[a, b(\alpha))$.*

**Proof.** As remarked above, we can assume that $\lambda > 1$. Take $\beta = 2/(\lambda - 1)$. The function $v(r) = (r - a)u'(r) + \beta u(r)$ satisfies the differential equation

$$\begin{aligned} L[v] &= v'' + \frac{m}{r}v' + f'(u)v \\ &= -\frac{amu'(r)}{r^2} + \beta\left(uf'(u) - \lambda f(u)\right) \geq 0. \end{aligned} \tag{2.4}$$

The first term in the second line is nonnegative because $u'(r) < 0$. The second term is nonnegative by **[F3].** On the other hand, equation (2.2) can be rewritten as $L[w] = 0$. At $r = a$, $w'(a) = v'(a) = 0$ and $v(a) = u(a) > 0$. At the right endpoint $v(b(\alpha)) = (b(\alpha) - a)u'(b(\alpha)) < 0$. Thus $v(r)$ vanishes somewhere in the interval; let its first zero be $\rho$. In $[a, \rho]$, $vL[v] \geq 0$. By the Sturm comparison theorem, $v(r)$ oscillates more slowly than $w(r)$, implying that $w(r)$ must vanish in $[a, \rho]$. ∎

The remaining effort is devoted to showing that $w(r)$ cannot have a second zero at or before $b(\alpha)$, and to dealing with the case $b(\alpha) = \infty$.

In general, $u(r, \alpha)$ may not have a zero in $(a, \infty)$. Following [8], we partition the set of solutions into the following:

$$N = \{\alpha \in (0, \infty) : b(\alpha) \text{ exists }\}, \tag{2.5}$$

$$G = \{\alpha \in (0, \infty) - N : u(r, \alpha) \to 0 \text{ as } r \to \infty\} \tag{2.6}$$

$$P = (0, \infty) - N - G. \tag{2.7}$$

We sometimes abuse the notation by saying that $u(r, \alpha) \in N$ $(G, P)$, instead of $\alpha \in N$ $(G, P)$.

The following properties of $N$ and $P$ are easy to establish.

**Lemma 2** *Let $\alpha \in P$ and $\alpha > \theta$. Then $\lim_{r \to \infty} u(r, \alpha) = \theta$. As a consequence, $u(r, \alpha)$ has an absolute minimum, say at $r_0 > a$.*

**Proof.** Equation (2.1) is the familiar classical differential equation describing the motion of a particle under the influence of a potential (given by the indefinite integral of $f(u)$) and a frictional force. The position $u = \theta$ is the only place where the potential attains its minimum. It is the only stable equilibrium point, to which all solutions, except the ground states, tend. ∎

**Lemma 3** *The sets $N$ and $P$ are open subsets of $(0, \infty)$, and $(0, \theta) \subset P$.*

**Proof.** That $N$ is open follows from continuity of the solutions on $\alpha$. Let $u(r, \overline{\alpha}) \in P$. Then it has a positive absolute minimum at $r_0$. The derivative $u'(r, \overline{\alpha})$ therefore changes sign, as $r$ passes through $r_0$. For all $\alpha$ close to $\overline{\alpha}$, $u'(r, \alpha)$ must also change sign somewhere close to $r_0$. In other words, $u(r, \alpha)$ has a local minimum near $r_0$, and $u$ has not vanished yet. From the shape of the potential and the dissipativeness of the system, it is easy to see that $u(r, \alpha)$ cannot fall below this local minimum for larger $r$. Hence, $u(r, \alpha) \in P$. ∎

The behavior of the solutions of (2.2), as $r \to \infty$, plays an important role in the proof of Theorem 1 for an unbounded domain. Let us fix $u(r, \alpha) \in G$ and consider (2.2) as a linear differential equation with the known coefficient $f'(u(r, \alpha))$. The set of all its solutions forms a two-dimensional linear space.

**Lemma 4** *Let $\alpha \in G$. There is one (up to a constant multiple) solution $W(r)$ of (2.2) that decays asymptotically, more precisely,*

$$W(r) = O(r^{1-m}), \qquad as \qquad r \to \infty. \tag{2.8}$$

*All other (linearly independent) solutions satisfy*

$$W(r) \to K \neq 0, \qquad as\ r \to \infty. \tag{2.9}$$

*The limit $K$ may be $\pm\infty$.*

**Proof.** This is very likely a known result, but we are not able to locate a complete reference. The decaying solution has been named the recessive solution or the principal solution; see Hartman [6]. The conclusions are consequences of the fact that $f'(u)$ is eventually negative. Since we are concerned only with asymptotic behavior, we may assume in the proof that $f'(u) < 0$ for all $u$. The existence of the principal solution and the behavior of the other solutions can be found in Corollary 6.4 of [6]. It remains to prove that $W(r) = O(r^{1-m})$ for the principal solution. This solution can be obtained by a shooting procedure. Let $W(r,\kappa)$ be the solution of (2.2) with initial conditions

$$w(a) = 1, \quad w'(a) = \kappa. \tag{2.10}$$

For $\kappa$ very negative, $W(r,\kappa)$ crosses the $r$-axis. As $\kappa$ increases, the solution increases and the zero moves towards $r = \infty$. When $\kappa = 0$, we have an increasing solution. Hence there is a maximum value of $\kappa$ for which the solution just misses the $r$-axis. This gives the principal solution; it is the supremum of all the solutions with smaller $\kappa$. Fix one such $\kappa$, and let the first zero of $W(r,\kappa)$ be $\rho$. A simple comparison principle shows that $W(r,\kappa)$ in $[a,\rho]$ is dominated by the solution of the boundary value problem

$$Z'' + \frac{m}{r}Z' = 0, \tag{2.11}$$

$$Z(a) = 1, \quad Z(\rho) = 0. \tag{2.12}$$

By either an explicit solution of (2.11) or a simple application of Sturm's comparison theorem, we see that $Z(r) \leq (a/r)^{m-1}$. Hence $W(r,\kappa) \leq (a/r)^{m-1}$. It follows that the same inequality holds for the limiting principal solution, and the required decaying behavior follows. ∎

We need a simple property of functions that satisfy (2.9).

**Lemma 5** *Suppose $W(r) < 0$ for large $r$ and satisfies* (2.9). *Then*

$$\liminf_{r\to\infty} \frac{rW'(r)}{|W(r)|} \leq 0. \tag{2.13}$$

**Proof.** Suppose (2.13) does not hold. Then for some $r_0$ and some $\epsilon > 0$,

$$\frac{W'(r)}{|W(r)|} \geq \frac{\epsilon}{r}, \quad \text{for all } r > r_0. \tag{2.14}$$

Integrating over $[r_0, r]$ gives

$$|W(r)| \leq \left(\frac{r_0}{r}\right)^{\epsilon} |W(r_0)|. \tag{2.15}$$

The right-hand side tends to zero as $r \to \infty$, contradicting (2.9). ∎

By Lemma 3, $G$ is a closed set. Let

$$\alpha^* = \min G. \tag{2.16}$$

As a first step towards our goal, we show that with $\alpha^*$ chosen as the initial height, the function $w(r, \alpha^*)$ has the expected property.

**Lemma 6** *$w(r, \alpha^*)$ has exactly one zero in $(a, \infty)$.*

**Proof.** Suppose the contrary is true, namely, that $w(r, \alpha^*)$ has at least two zeros. This implies that if $\alpha$ is sufficiently close to $\alpha^*$, then $u(r, \alpha)$ and $u(r, \alpha^*)$ intersect at least twice. Denote the first two intersection points by $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$. These points remain distinct and vary continuously as $\alpha$ changes, at least in a small neighborhood of $\alpha^*$. Let us pick our $\alpha$ in $(0, \alpha^*)$ and decrease it gradually from $\alpha^*$. There must be a limiting point $\overline{\alpha} \geq \theta$ at which the second intersection point $\sigma_2(\overline{\alpha})$ disappears into $\infty$. This is because with the initial height $\theta$, the solution $u(r, \theta) \equiv \theta$ is a constant and intersects $u(r, \alpha^*)$ only once.

Suppose first that $\overline{\alpha} > \theta$. The choice of $\alpha^*$ dictates that $(0, \alpha^*) \subset P$. By Lemma 2, for each $\alpha \in [\overline{\alpha}, \alpha^*)$, $u(r, \alpha)$ has an absolute minimum at some point $r_0(\alpha)$. This point depends continuously on $\alpha$. As $\alpha$ decreases towards $\overline{\alpha}$, $r_0(\alpha) \to r_0(\overline{\alpha})$ and $u(r_0(\alpha), \alpha) \to u(r_0(\overline{\alpha}), \overline{\alpha})$. Hence, for $\alpha$ sufficiently close to $\overline{\alpha}$, we have $u(r_0(\alpha), \alpha) > \frac{1}{2} u(r_0(\overline{\alpha}), \overline{\alpha})$. Since $r_0(\alpha)$ is an absolute minimum for $u(r, \alpha)$, we have

$$u(r, \alpha) > \frac{1}{2} u(r_0(\overline{\alpha}), \overline{\alpha}), \quad \text{for all } r > a. \tag{2.17}$$

On the other hand, as $\alpha$ decreases towards $\overline{\alpha}$, $\sigma_2(\alpha)$, the second point of intersection of $u(r, \alpha)$, with $u(r, \alpha^*)$, diverges to $\infty$. At this point,

$$u(\sigma_2(\alpha), \alpha) = u(\sigma_2(\alpha), \alpha^*). \tag{2.18}$$

Note that the right-hand side tends to zero, while, by (2.17), the left-hand side is bounded away from zero. This is a contradiction.

We therefore must have $\overline{\alpha} = \theta$. For the solution $u(r, \theta)$, the point $r_0$ is not defined. Nevertheless, we can still obtain a positive lower bound for

$u(r,\alpha)$ for all $\alpha$ sufficiently close to $\theta$. This can be done by using the well-known energy technique. The potential energy $\int_0^u f(\sigma)\,d\sigma$ is strictly lower at and around $u=\theta$ than at $u=0$. We omit the easy details. A contradiction can be obtained just as before. ∎

The next step is the most crucial one in the whole proof. We shall show that for $\alpha \in N$, $w(r)$ cannot have its second zero exactly at $b(\alpha)$. Knowing this, we can deduce that if for some $\alpha_1 \in N$, the corresponding $w(r,\alpha_1)$ has only one zero in $[a, b(\alpha_1)]$, then for all $\alpha > \alpha_1$, $w(r,\alpha)$ has only one zero in $[a, b(\alpha)]$; as a consequence, $[\alpha, \infty) \subset N$. The method of proof was first used by Zhang in [14]. It is only in this step that we need to invoke condition **[F3]**. We begin by giving this condition an equivalent formulation.

**Lemma 7** *If we assume that conditions* **[F1]** *and* **[F2]** *hold, condition* **[F3]** *is equivalent to the following condition:*

*For each $v > \phi$, there exist a constant $\gamma > 1$ such that*

$$\gamma f(u) - uf'(u) \begin{cases} \leq 0 & \text{for all } u < v \\ \geq 0 & \text{for all } u > v \end{cases}. \tag{2.19}$$

**Proof.** In $[0,\theta)$, $f(u) < 0$. The first inequality in (2.19) over the interval $[0,\theta)$ is thus equivalent to

$$G(u) \leq \gamma \text{ for } u \in (0,\theta). \tag{2.20}$$

In $[\theta,\infty)$, $f(u) > 0$. The remaining part of the first inequality in (2.19) is equivalent to

$$G(u) \geq \gamma \text{ for } u \in (\theta, v). \tag{2.21}$$

The second inequality in (2.19) is equivalent to

$$G(u) \leq \gamma \text{ for } u \in (v,\infty). \tag{2.22}$$

It follows from (2.21) and (2.22), by continuity, that $G(v) = \gamma$. The conclusion of the lemma is now obvious. ∎

The next lemma is so important that we give two different proofs.

**Lemma 8** *Suppose $\alpha \in N$. The endpoint $b(\alpha)$ cannot be the second zero of the function $w(r)$.*

**Proof.** By Lemma 1, $w(r)$ has one zero $\tau < b(\alpha)$. Suppose first that $u(\tau) \leq \phi$. Then by Lemma 15 of Section 3, $w(r)$ cannot have a second zero in $[\tau, b(\alpha)]$, and we are done.

Next we assume

$$u(\tau) > \phi. \tag{2.23}$$

Suppose that $b(\alpha)$ is the second zero of $w(r)$. Then

$$w(r) > 0 \quad \text{ for } r \in (a, \tau), \tag{2.24}$$

and

$$w(r) < 0 \quad \text{ for } r \in (\tau, b(\alpha)). \tag{2.25}$$

It is straightforward to verify the following identities, by using the differential equations (2.1) and (2.2) satisfied by $u$ and $w$, respectively:

$$\left[r^m(u'w - uw')\right]' + r^m w\left[f(u) - uf'(u)\right] = 0. \tag{2.26}$$

$$\left[r^m((ru)''w - (ru)'w')\right]' + r^m w\left[3f(u) - uf'(u)\right] = 0. \tag{2.27}$$

By integrating these, we obtain

$$\int_a^b [f(u) - uf'(u)]w(r)r^m\,dr = 0, \tag{2.28}$$

and

$$\int_a^b [3f(u) - uf'(u)]w(r)r^m\,dr = r^m w(ru)''\Big|_{r=a} + b^{m+1}w'(b)u'(b). \tag{2.29}$$

If $a = 0$, the first term on the right-hand side of (2.29) vanishes because of the factor $r^m$. If $a > 0$, this term expands to $a^n u''(a) < 0$, since $w(a) = 1$ and $u'(a) = 0$. The last term in (2.29) is nonnegative since $w'(b) \geq 0$ and $u'(b) \leq 0$. Hence the integral in (2.29) is nonnegative. By taking suitable linear combinations of (2.28) and (2.29), we have

$$\int_a^b [\gamma f(u) - uf'(u)]w(r)r^m\,dr \leq 0, \tag{2.30}$$

for all $\gamma > 1$.

By (2.23), $u(r) \geq u(\tau) > \phi$ for $r \in [a, \tau]$ and $u(r) \leq u(\tau)$ for $r \in [\tau, b(\alpha)]$. Using the alternative form of **[F3]** given in Lemma 7, we see that there exists a constant $\gamma > 1$, such that $\gamma f(u) - uf'(u)$ is nonnegative in $(a, \tau)$

and nonpositive in $(\tau, b(\alpha))$. Furthermore, the function cannot be identically zero in $(a, \tau)$. These facts obviously contradict inequality (2.30). ∎

Although the proof is short, its dependence on the seemingly fortuitous identities (2.28) and (2.29) hides some insight. We give here another proof using Sturm's comparison theorem. It is this alternative method that enables us to obtain uniqueness results for the generalized Matukuma equation, $\Delta u + q(|x|)u^p = 0$, reported in the forthcoming paper [18].

**Alternative Proof.** Let

$$v(r) = ru'(r) + \beta u(r), \tag{2.31}$$

where $\beta > 0$ is a constant to be chosen later. Then

$$L[v] = \beta\left(uf'(u) - \gamma f(u)\right), \tag{2.32}$$

where $L$ is the operator defined in (2.4) and $\gamma = (\beta+2)/\beta$. We use Lemma 7 to choose $\gamma$ (and hence $\beta$) so that

$$L[v] \le 0 \quad \text{in } (a, \tau) \quad \text{and} \quad L[v] \ge 0 \quad \text{in } (\tau, b(\alpha)). \tag{2.33}$$

Suppose as before that $\tau$ and $b(\alpha)$ are the two zeros of $w(r)$. At $r = a$, $v(a) > 0$ and $v'(a) \le w'(a) = 0$. By the Sturm comparison theorem, $v(r)$ oscillates faster than $w(r)$ in the right neighborhood of $a$ before $v(r)$ changes sign. Hence $v$ has a first zero $\sigma < \tau$. After passing $\sigma$, $v(r)$ becomes negative, and so it begins to oscillate more slowly than $w(r)$. Thus $v(r)$ cannot have a second zero before $\tau$. After passing $\tau$, $L[v]$ changes sign, and this reverses the comparison condition. Now $v(r)$ begins to oscillate faster than $w(r)$. Furthermore, since $w(\tau) = 0$, $v(r)$ has a head start at $r = \tau$. Hence, $v(r)$ must have a zero $\varsigma$ before the next zero of $w(r)$, namely, $b(\alpha)$. After passing $\varsigma$, $v(r)$ changes sign, and so it switches to oscillate slower than $w(r)$. It follows that $v(r)$ cannot change sign for a third time before $b(\alpha)$. This is a contradiction since $v(b(\alpha)) = bu'(b(\alpha)) < 0$. ∎

When $\alpha \in G$, a similar assertion holds.

**Lemma 9** *Suppose that the last condition in* **[F2]** *is satisfied. Let $\alpha \in G$. Let $w(r)$ have only one zero in $[a, \infty)$. Then $w(r) \to K \neq 0$ as $r \to \infty$.*

**Proof.** The first proof of Lemma 8 works, after replacing $b$ by $\infty$ and using Lemma 16 instead of 15, only if we can control the boundary terms of the various integrals. This is indeed true if we know that $f'(0) < 0$, which implies that $u$ decays exponentially. The general case can also be handled by using estimates of the form (2.8). Instead, let us establish the lemma by modifying the alternative proof. The last sentence in the proof no longer leads to a simple contradiction, because we do not know (at least not yet) that $v(r) < 0$ for large $r$. Suppose $v(r) > 0$ for large $r$. We have already shown that in $(\varsigma, \infty)$, $v(r)$ oscillates less than $w(r)$. By Lemma 4, $w(r)$ is the only principal solution; all others that are eventually positive and oscillate more slowly than $w(r)$; in particular $v(r)$, approach a positive limit (or $\infty$). It follows that $v(r) = ru'(r) + \beta u(r) \to K_1 > 0$. Since $u(r) \to 0$, $ru'(r) \to K_1 > 0$. This is a contradiction. ∎

**Lemma 10** *Suppose that the last condition in* **[F2]** *is satisfied. For $\epsilon > 0$ small enough, $\alpha^* + \epsilon \in N$.*

**Proof.** As before, let $\tau$ be the only zero of $w(r) = w(r, \alpha^*)$ in $[a, \infty)$. By Lemma 9, $w(r)$ cannot be the principal solution. By Lemma 5, there exists an arbitrarily large point $\pi$ such that

$$\frac{w'(\pi)}{|w(\pi)|} < \frac{m-1}{\pi}. \tag{2.34}$$

If we chose $\pi > \tau$, we also have $w(\pi) < 0$. Now let us choose one such large $\pi$ that in $(\pi, \infty)$, $u(r)$ is within the neighborhood of 0 in which $f'(u) < 0$.

Denote

$$v(r) = \frac{u(r, \alpha^* + \epsilon) - u(r, \alpha^*)}{\epsilon}. \tag{2.35}$$

Then $v(r)$ satisfies

$$v'' + \frac{m}{r} v' + f'(\xi(r)) v = 0, \tag{2.36}$$

and

$$v(a) = 1, \quad v'(a) = 0, \tag{2.37}$$

where $\xi(r)$ is some number between $u(r, \alpha^*)$ and $u(r, \alpha^* + \epsilon)$. As $\epsilon \to 0$, $f'(\xi(r)) \to f'(u(r))$ uniformly in any compact interval, in particular in $[a, \pi]$.

Thus $v(r) \to w(r)$ uniformly in $[a, \pi]$. Therefore, when $\epsilon$ is small enough,

$$v(\pi) < 0, \quad \text{and} \quad \frac{v'(\pi)}{|v(\pi)|} < \frac{m-1}{\pi}. \tag{2.38}$$

The first inequality implies that $u(\pi, \alpha^* + \epsilon) < u(\pi, \alpha^*)$. Now suppose $u(r, \alpha^* + \epsilon)$ does not belong to $N$. Then either it remains below $u(r, \alpha^*)$ for all $r > \pi$ or the two solutions intersect at some future point $\sigma$. In other words, either $v(r) < 0$ for all $r > \pi$, or $v(\sigma) = 0$.

We claim that the latter case is impossible. In $(\pi, \sigma)$, the coefficient of the last term in (2.36) is negative. We can compare $v(r)$ with the solution of the equation

$$V'' + \frac{m}{r} V' = 0, \tag{2.39}$$

having initial conditions

$$V(\pi) = v(\pi) < 0, \quad \frac{V'(\pi)}{|V(\pi)|} = \frac{v'(\pi)}{|v(\pi)|} < \frac{m-1}{\pi}. \tag{2.40}$$

Since $v(r)$ oscillates more slowly than $V(r)$, $v(r) \leq V(r)$, for $r \in [\pi, \sigma]$, as long as $V(r)$ remains negative. Direct computation shows that $V(r)$ remains negative and indeed increases to a negative limit as $r \to \infty$. This contradicts the assumption that $v(\sigma) = 0$.

In the remaining case, the coefficient of the last term in (2.36) is negative in $(\pi, \infty)$, implying that $v(r)$ again oscillates more slowly than $V(r)$ in $(\pi, \infty)$. So $\lim_{r\to\infty} v(r) \leq \lim_{r\to\infty} V(r) < 0$. We still have a contradiction because, by definition (2.35), $v(r) \to 0$ as $r \to \infty$. ∎

**Lemma 11** *For $\epsilon > 0$ small enough, $w(r, \alpha^* + \epsilon)$ has exactly one zero in $[a, b(\alpha^* + \epsilon)]$.*

**Proof.** As $\epsilon \to 0$, $w(r, \alpha^* + \epsilon)$ and $w'(r, \alpha^* + \epsilon)$ converge uniformly to $w(r, \alpha)$ and $w'(r, \alpha)$, respectively, in $[a, \pi]$. In the interval $[\pi, b(\alpha^* + \epsilon)]$, $u(r, \alpha^* + \epsilon)$ remains below $u(r, \alpha)$, so that the coefficient of the third term in (2.2) (for $w(r, \alpha^* + \epsilon)$) is negative. The proof that $w(r, \alpha^* + \epsilon)$ does not vanish in this interval is identical to that used in Lemma 10 to show that $v(r)$ cannot have vanished in $[\pi, \infty)$. We omit the details. ∎

We can now wrap up the proof of Theorem 1. We first assume that the hypotheses of Lemmas 9 and 11 hold. By the definition of $\alpha^*$, we know that $(0, \alpha^*) \subset P$. By Lemmas 10 and 11, we see that for some $\epsilon > 0$, $(\alpha^*, \alpha^* + \epsilon) \subset N$, and that $w(r, \alpha)$ has exactly one zero in $[a, b(\alpha)]$. Thus $b(\alpha)$ is a strictly decreasing function of $\alpha$. Using a continuity argument, we see that as $\alpha$ increases, the number of zeros of $w(r, \alpha)$ in $[a, b(\alpha)]$ cannot increase, unless at some moment $w(b(\alpha))$ becomes the second zero, a possibility excluded by Lemma 8. Hence $w(r, \alpha)$ changes sign exactly once for all $\alpha > \alpha^*$. As a consequence, $b(\alpha)$ remains strictly decreasing for all $\alpha > \alpha^*$, and $(\alpha^*, \infty) \subset N$. Uniqueness for (II) on a bounded interval holds. Furthermore, $\alpha^*$ is the only initial height that yields a ground state solution in $[a, \infty)$.

Now suppose that the last condition in **[F2]** is not satisfied. Then Lemmas 9 and 11 may not be valid, and we cannot conclude that $\alpha^*$ (if it exists at all) gives the only ground state. However, we can still prove that if $\alpha \in N$, then $w(r, \alpha)$ has only one zero in $[a, b(\alpha)]$, and so uniqueness for the boundary value problem on a bounded interval is still valid. Suppose that $w(r, \alpha)$ has more than one zero in $[a, b(\alpha)]$. By Lemma 8, its second zero $\sigma$ must be in $(a, b(\alpha))$. We can then modify $f(u)$ in the interval $[0, \sigma]$ so that the new function satisfies the last condition of **[F2]**. Then the new $w(r)$ function cannot have a second zero in $(a, b(\alpha))$; in particular, it cannot be zero at $\sigma$. This is a contradiction because the two $w(r)$ coincide in $[a, \sigma]$.

From what we have proved, we can describe the structure of the set of solutions. If the last condition in **[F2]** is satisfied, then $N = (\alpha^*, \infty)$, $G = \{\alpha^*\}$, and $P = (0, \alpha^*)$. In general, there is a point $\alpha^{**} = \max G$ such that $N = (\alpha^{**}, \infty)$. There could be more than one member of $G$ in $(0, \alpha^{**})$. We conjecture, however, that the contrary prevails.

## 3 A Boundary Value Problem

In this section, we assume only that the nonlinear function $f(u)$ satisfies the following:

**[F4]** $f \in C^1[0,\infty)$,
and there exists a point $\phi$ such that

$$\int_0^{\phi} f(\sigma)\, d\sigma = 0, \tag{3.1}$$

whereas

$$\int_0^{u} f(\sigma)\, d\sigma \leq 0, \quad \text{for all } u < \phi. \tag{3.2}$$

We also allow the middle term of our differential equation to have a more general coefficient. The interval $[a,b]$ will be bounded except in Theorem 2 and Lemma 16.

We consider a new boundary value problem

$$\begin{cases} u'' + g(r)u'(r) + f(u) = 0, \quad u > 0, \qquad a < r < b \\ u(a) = \alpha, \quad u'(a) < 0 \\ u(b) = 0. \end{cases} \tag{III}$$

We assume that

$$\alpha \leq \phi \tag{3.3}$$

and that $g(r) : [a,b] \to (0,\infty)$ is a continuous nonincreasing function. The continuity requirement on $g(r)$ implies that we must take $a > 0$ for the differential equation considered in Section 2.

Our goal is to show that (III) has a unique positive solution. As a bonus, we furnish two of the links needed in Section 2. Note that we require the solution to be initially decreasing. Otherwise, it is possible that there exists another solution that starts with a positive slope at $r = a$, increases beyond $u = \phi$, and then comes back to vanish at $r = b$.

We first outline our method. The natural approach is to apply the Kolodner-Coffman method on solutions shooting from $r = a$. This method requires knowledge of the oscillatory behavior of the $w(r)$ function, which seems to be difficult to obtain directly. We study a related shooting process, this time from the $r$-axis, backward towards $r = a$. The techniques of Peletier and Serrin [13] and Kaper and Kwong [7] can be used to show that no two solutions of this backward shooting can intersect below $u = \phi$. This fact implies the nonoscillatory nature of the $w(r)$ function associated with this second type of shooting. As the two $w(r)$ functions satisfy the same linear differential equation, we can deduce the desired oscillatory property of the first $w(r)$ function from that of the second one.

We shall omit some details, especially those that follow from simple energy arguments. We first observe that all solutions to (III) must be monotonically decreasing in $[a, b]$.

Existence of a solution to (III) is most easily established by shooting. We let $u(r, \gamma)$ be the solution to the same differential equation with the fixed initial height $u(a) = \alpha$ and the initial slope $u'(a) = \gamma$. For $\gamma$ sufficiently negative, $u(r, \gamma)$ must cross the $r$-axis at a point very close to $a$. For $\gamma = 0$, the solution will not cross the $r$-axis, because it does not have enough energy. Thus, as $\gamma$ gradually increases from $-\infty$, the first zero of $u(r, \gamma)$ must at some point disappear into $\infty$. Hence, by continuity, there exists some suitable $\gamma$ for which this first zero occurs at the given endpoint $b$.

To establish uniqueness, we shall show that the function

$$w(r, \gamma) = \frac{\partial u(r, \gamma)}{\partial \gamma} \tag{3.4}$$

does not have a zero in $[a, b(\gamma)]$, where $b(\gamma)$ is defined as the first zero of $u(r, \gamma)$. In particular, $w(b(\gamma)) > 0$. Coffman's argument can then be modified to show that $b(\gamma)$ is a strictly increasing function of $\gamma$, and uniqueness follows.

As usual, the function $w(r, \gamma)$ satisfies the first variational equation

$$w'' + g(r)w' + f'(u)w = 0 \tag{3.5}$$

and the initial conditions

$$w(a) = 0, \quad w'(a) = 1. \tag{3.6}$$

We look at a related terminal value problem. Let $\lambda = u'(b)$, the terminal slope of the solution that solves (III). Let $U(r, \beta)$ be the solution of the differential equation in (III) with the terminal values

$$U(\beta, \beta) = 0, \quad U'(\beta, \beta) = \lambda. \tag{3.7}$$

Here $U'$ denotes the derivative with respect to the first argument $r$. We assume that $\beta$ varies within a neighborhood of the endpoint $b$, so small that $U(r, \beta) < \beta + \epsilon$, for all $r \in [a, \beta]$, where $\epsilon > 0$ is some sufficiently small number. Using an energy argument, we see that $U(r, \beta)$ must be strictly decreasing in $[a, \beta]$.

A monotone separation property was first established for $g(r) = m/r$ in [13], and then for general $g(r)$ in [7], for ground states (solutions that decay to zero at $\infty$). The same ideas in fact work to give a similar result on bounded intervals, which we state as Lemma 12 below. The proof is even simpler because $r = \infty$ is not involved. We omit the details. Let $\beta < b$. At this moment, we do not know whether the two solutions $U(r, \beta)$ and $U(r, b)$ intersect in $[a, b]$ or not. Let $[\sigma, b] \subset [a, b]$ be the maximal subinterval in which they do not intersect.

**Lemma 12** *Suppose $U(r, \beta)$ and $U(r, b)$, $\beta < b$, do not intersect in $[\sigma, b]$. For all pairs of points $\sigma \leq \rho < r$ at which the two solutions have the same height, i.e., $U(\rho, \beta) = U(r, b)$,*

$$U'(\rho, \beta) \leq U'(r, b) < 0. \tag{3.8}$$

The next step, as in [13] and [7], is to use the monotone separation lemma to deduce that the two solutions actually cannot intersect below $u = \phi$. As the proof works without change here, it is again omitted. Since $U(r, b) \leq \phi$ in $[a, b]$, the two solutions cannot intersect at all in $[a, b]$.

**Lemma 13** *For $\beta < b$, $U(r, \beta) < U(r, b)$ in $[a, b]$.*

Define

$$W(r, \beta) = \frac{\partial U(r, \beta)}{\partial \beta}. \tag{3.9}$$

Then $W(r) = W(r, b)$ satisfies the same differential equation (2.2) as $w(r)$ does. By differentiating the first identity in (3.7) with respect to $r$, we obtain

$$U'(\beta, \beta) + W(\beta, \beta) = 0. \tag{3.10}$$

In particular, letting $\beta = b$, we obtain

$$W(b) = -\lambda > 0. \tag{3.11}$$

**Lemma 14** *The function $W(r)$ cannot have a zero in $(a, b)$.*

**Proof.** The proof is standard. Suppose $\tau$ is the last zero of $W(r)$ in $(a, b)$, if there is one. Then at some point $r_0 < \tau$, $W(r_0) < 0$. This implies that for $\beta < b$ and sufficiently close to $b$, $U(r_0, b) - U(r_0, \beta) < 0$. This contradicts Lemma 13. ∎

**Lemma 15** *The function $w(r)$ cannot have a zero in $(a, b)$.*

**Proof.** Recall that $w(r)$ has a zero at $r = a$. Since $w(r)$ and $W(r)$ are solutions of the same second-order linear ordinary differential equation, Sturm's separation theorem applies. If $w(r)$ has a second zero in $(a, b)$, $W(r)$ must also have one. This contradicts Lemma 14. ∎

We have thus completed the proof of the following theorem in the case $b < \infty$. The case $b = \infty$ is already included in the results in [7] and [13].

**Theorem 2** *Suppose that $f(u)$ satisfies* **[F4]** *and $g(r)$ is a positive nonincreasing continuous function on the interval $[a, b)$, $(b \leq \infty)$. The boundary value problem* (III) *has a unique positive solution.*

The proof of Lemma 9 in Section 3 requires a version of Lemma 15 for $b = \infty$. The technique used to establish Lemma 15 cannot be extended, because, for $b = \infty$, it is not clear how to define the corresponding terminal value shooting problem. Instead we use a limiting argument.

**Lemma 16** *Suppose $u(r)$ is a solution of the differential equation over $[a, \infty)$ in* (III), *such that $u(a) < \phi$ and $\lim_{r\to\infty} u(r) = 0$. Then the function $w(r, \gamma)$, as defined in* (3.4), *cannot be a principal solution of* (3.5).

**Proof.** We extend $u(r)$ backward a little from $r = a$ to $r = a - \epsilon$. If $\epsilon > 0$ is chosen small enough, we can make sure that $u(a - \epsilon) < \phi$, and $u'(a - \epsilon) < 0$. Define the function $\overline{w}(r)$ in the same way we define $w(r)$, but using $a - \epsilon$ as the shooting point instead. Then $\overline{w}(r)$ satisfies the same differential equation (3.5). Suppose that the lemma is false and $w(r)$ is a principal solution, with a zero at $r = a$. By Sturm's separation theorem, $\overline{w}(r)$ must have a zero $\rho \in (a, \infty)$. Next we modify $f(u)$ in $(0, u(\rho))$ so that the new function $\overline{f}(u)$ vanishes near 0. If we keep the modification within a sufficiently small neighborhood of 0, we can make sure that $u(a - \epsilon)$ remains below $\phi$ of this new function. In $[a - \epsilon, \rho]$, $u(r)$ is still the solution of the modified differential equation. When we continue $u(r)$ beyond $\rho$, we obtain an extended solution that vanishes at a finite zero (provided that the modification of $f(u)$ has been suitably chosen). This contradicts Lemma 15.

Let us now show how $\overline{f}(u)$ can be constructed. If we just redefine $f(u)$ to be identically zero in a small interval $(0, \delta)$, then the corresponding new $u(r)$ will have a finite zero. This can be seen by comparing the new solution with the original one in $[\sigma, \infty)$, where $\sigma$ is the point at which $u(\sigma) = \delta$. Denote the finite zero of the new solution by $b_0$. Then $u(b_0 + 1) < 0$. However, the new $f$ has a jump at $\delta$ and so is not $C^1$. To obtain $\overline{f}(u)$, we must smooth the jump by changing the function in $(\delta, 2\delta)$, keeping the modification so small that $u(b_0 + 1)$ remains strictly negative. ∎

## 4 Examples

The most difficult part in **[F3]** to verify is usually the monotonicity of $G(u)$. We say that $f(u)$ has the property **[G]** in an interval $I$ if

**[G]** In $I$, $f(u) > 0$, and $G(u) = uf'(u)/f(u)$ is nonincreasing.

New examples can be generated from known ones by the following lemma.

**Lemma 17** *If $f(u)$ satisfies* **[G]***, then $f(cu)$, $f^c(u)$, and $u^d f(u)$ also satisfy* **[G]***, for any $c > 1, d > 0$. If $f(u)$ and $g(u)$ satisfy* **[G]***, so does the product $f(u)g(u)$.*

A stronger condition was used in [10]. It is essentially the same as that stated in the next lemma. In [10], it is required only that some power of $u$ exists with the desired properties (2.21) and (2.22). Our lemma goes one step further by specifying the optimal choice of such a power.

**Lemma 18** *Let $\lambda = \lim_{r\to\infty} G(u)$ be as in* **[F3]***. If*

$$\frac{f(u)}{u^\lambda} \quad \textit{is nondecreasing} \tag{4.1}$$

*and*

$$u\left(\frac{f(u)}{u^\lambda}\right)' \quad \textit{is nonincreasing,} \tag{4.2}$$

*then* **[F3]** *holds.*

**Proof.** The function in (4.2) expands to

$$\frac{uf'(u) - \lambda f(u)}{u^\lambda}. \tag{4.3}$$

Note that for $u > \theta$ (so that $f(u) > 0$),

$$\frac{uf'(u)}{f(u)} = \frac{uf'(u) - \lambda f(u)}{u^\lambda}.\frac{u^\lambda}{f(u)} + \lambda. \tag{4.4}$$

By (4.1) and (4.2) the two fractions on the right-hand side are both nonincreasing. Since $\lambda$ is a constant, the entire expression is nonincreasing in $(\theta, \infty)$. The remaining inequalities in **[F3]** are easy to verify. ∎

From Lemma 18, it is easy to see that the function

$$f(u) = u^p - \sum cu^q, \quad 1 \leq q < p,\, c > 0, \tag{4.5}$$

satisfies **[F3]**. A simple example that does not come under this form is

$$f(u) = u^4 - u^3 + cu^2 - u, \quad 0 < c < 3/2. \tag{4.6}$$

It should be noted that the condition in Lemma 18 does not imply **[F3]**. Examples that satisfy **[G]** can be easily generated by solving the differential equation

$$\frac{f'}{f} = \frac{G(u)}{u}, \tag{4.7}$$

for any nonincreasing $G(u)$. We obtain

$$f(u) = \exp\left(\int \frac{G(u)}{u}!u\right). \tag{4.8}$$

In particular, if $G(u)$ is not strictly decreasing, then the conditions of Lemma 18 cannot be satisfied.

The following criterion is also useful.

**Lemma 19** *Let $f(u) = u^p - F(u)$. Suppose that $f(u) > 0$ for $u$ beyond some $u_0 > 0$. If*

$$u^2F''(u) + u(1-2p)F'(u) + p^2F(u) \geq 0, \quad u > u_0, \tag{4.9}$$

*then $f(u)$ satisfies* **[G]** *in $[u_0, \infty)$.*

**Proof.** Direct computation of $G'(u)$ gives a fraction whose numerator is

$$-P(u)u^p + \left(u^2F(u)F''(u) + uF(u)F'(u) - u^2F'^2(u)\right), \tag{4.10}$$

where $P(u)$ is the expression in (4.9). Since $u^p > F(u)$ for $u > u_0$ and the coefficient of the first term is negative, the expression in (4.10) is less than

$$-P(u)F(u) + \left(u^2F(u)F''(u) + uF(u)F'(u) - u^2F'^2(u)\right)$$

$$= -\left(uF'(u) - pF(u)\right)^2 < 0. \tag{4.11}$$

Hence $G(u)$ is nonincreasing in $[u_0, \infty)$. ∎

It is easy to see that $F(u) = u^q$ satisfies (4.9). The sum of any set of functions satisfying (4.9) also satisfies (4.9). The example (4.5) can be generated this way.

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