
#### Abstract

We establish new uniqueness results for boundary value problems of the superlinear Emden-Fowler type, $u^{\prime \prime}(t)+F(t, u)=0, \quad u(t)>0, \quad t \in$ $(a, b)$, with either a Dirichlet or Neumann condition at each endpoint. The first result extends a known criterion to nonlinear terms that may change sign. The proof uses the theory of differential inequalities, after changing the independent variable to the quantity $u$. The second result deals with nonlinear functions of the form $\sum c t^{\gamma} u^{p}, c, \gamma>0$. The proof uses part of a method due to Coffman and employs as an independent variable the quantity $t^{a} u$ for some $a>0$. We also look at a special case $F(t, u)=t^{4}\left(u^{5}+u\right)$, not covered by the previous two results. We show that some of the earlier ideas still apply after we work through several rather technical estimates, with the help of the symbolic manipulation software MAPLE.


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Proposed Running Head. Uniqueness for Emden-Fowler BVP

## 1 Introduction

We are interested in boundary value problems for second-order nonlinear ordinary differential equations of the form

$$
\begin{equation*}
u^{\prime \prime}(t)+F(t, u)=0, \quad u(t)>0, \quad t \in(a, b) \tag{1.1}
\end{equation*}
$$

where $-\infty<a<b<\infty$, subject to either a Dirichlet or a Neumann condition at each of the endpoints $a$ and $b$. More precisely, we impose one of the three sets of boundary conditions:

$$
\begin{gather*}
u(a)=u(b)=0,  \tag{BC1}\\
u^{\prime}(a)=u(b)=0, \tag{BC2}
\end{gather*}
$$

and

$$
\begin{equation*}
u(a)=u^{\prime}(b)=0 \tag{BC3}
\end{equation*}
$$

The third type can be reduced to (BC2) by using a reflection. We shall not consider the case of two Neumann conditions. We shall refer to the boundary value problems consisting of (1.1) and one of the above boundary conditions as (BVP1), (BVP2), and (BVP3), respectively.

The question studied in this paper is the uniqueness of the (positive) solution, assuming its existence. Recently, there has been considerable interest in this problem in connection with the study of radially symmetric ground state solutions of a nonlinear reaction-diffusion equation either in an annulus or outside a ball. Symmetry reduces the partial differential equation in question to an ordinary differential equation of the Lane-Emden type. A well-known change of variable then transforms it into the Emden-Fowler form. The resulting equation usually involves a singularity at the origin $t=0$. An additional singularity arises when the domain is unbounded; in this case, $b=\infty$ and the boundary condition at this "endpoint" is replaced by $\lim _{t \rightarrow \infty} u(t)=0$.

The classical Emden equation is the following special case of (1.1):

$$
\begin{equation*}
u^{\prime \prime}(t)+\frac{n-1}{t} u^{\prime}(t)+u^{p}(t)=0 \tag{1.2}
\end{equation*}
$$

where $n$ is the dimension of the Euclidean space in which the reaction-diffusion equation holds. If solutions that change sign are also considered, the nonlinear term in (1.2) must be replaced by $|u(t)|^{p-1} u(t)$. The classical Emden-Fowler equation is obtained from (1.2) and takes the form

$$
\begin{equation*}
u^{\prime \prime}(t)+t^{\gamma} u^{p}(t)=0, \quad \gamma \in(-\infty, \infty), p>1 \tag{1.3}
\end{equation*}
$$

Nehari's work [12] on the equation (1.1) has much influence in later work by Coffman [3] [4], Moroney [11], Wong [17], and others. Wong's excellent survey [16]
contains known results for these equations up to 1975. Much development has occurred since then. The paper of Brezis and Nirenberg [2] led to a surge of activity on equations with nonlinear terms of the form $u^{p}+u^{q}$, for various values of $p$ and $q$.

In this paper, we look at the uniqueness problem, assuming existence of the pertinent solution. In general, it is hard to establish uniqueness, even in the regular case $b<\infty$. We are not concerned here with "global" uniqueness, as is the case for most classical results such as those given in the monograph by Bernfeld and Lakshmikantham [1], but with uniqueness within the class of positive solutions. Many of the equations we study admit the trivial solution as well as infinitely many other solutions that change sign.

The nonlinear function $F:[a, b] \times[0, \infty) \rightarrow R$ is said to be superlinear (sublinear) if for each fixed $t \in[a, b]$,
$\frac{F(t, u)}{u}$ is nondecreasing (nonincreasing) in $u \in(0, \infty)$ but not a
constant in any interval $\left(0, u_{0}\right)$.

The nonconstancy requirement is used to exclude functions that behave like a linear function for small $u$. Prototypes of $F(t, u)$ are $q(t) u^{p}$ with $q(t)>0$. It is superlinear when $p>1$ and sublinear when $p<1$. We do not assume that $F(t, u)$ is positive, as in some previous work. Thus $F(t, u)=q(t)\left(u^{p}-u\right)$ is, by our definition, superlinear for $p>1$ and $q(t)>0$.

For sublinear equations, uniqueness theorems have been obtained by Picard and Urysohn; see [7] for a recent treatment. For equations that behaves sublinearly for large values of $u$, there is the result of Peletier and Serrin [15], later improved by Kaper and Kwong [5].

The corresponding theory for superlinear equations is richer because uniqueness no longer holds unless some rather technical restrictions are imposed on the coefficients. The most tantalizing fact is that numerical experiments usually show positive results under much less stringent conditions than required by analytic proofs. In the rest of this paper, we are concerned only with superlinear equations. We mention that the results in [9] apply to some equations that are neither superlinear nor sublinear.

Moroney [11] first showed that (BVP2) has a unique solution if $F(t, u)$ is positive and nonincreasing in $t$ for fixed $u$. Coffman [4] gave a different proof and complemented the result by showing that the same is true if $F(t, u)$ is positive and nondecreasing in $t$ but $(b-t)^{2} F(t, u)$ is nonincreasing in $t$. Kwong [7] showed that the monotonicity of $F(t, u)$ in $t$ is not needed. In Section 2 we improve this result further to allow $F(t, u)$ to change sign. Coffman [3] also
established uniqueness for (BVP1) with the classical Emden-Fowler equations (1.3). Ni [13] studied equation (1.1) in the form

$$
\begin{equation*}
u^{\prime \prime}(t)+\frac{n-1}{t} u^{\prime}(t)+f(u)=0 \tag{1.4}
\end{equation*}
$$

and extended Coffman's results in many directions. The effort was continued by Ni and Nussbaum [14]. Among the many interesting results they obtained are a uniqueness theorem for functions of the form $f(u)=u^{p}+\epsilon u^{q}$ with $p, q \leq$ $\frac{n}{n-2}$, and a nonuniqueness theorem for the same functions with $q<\frac{n+2}{n-2}<$ $p$. Kwong [7] noticed that the method first used by Coffman for establishing uniqueness results can be simplified by using Sturm comparison arguments.

For the singular or noncompact case, Coffman [4] obtained uniqueness for (BVP2) for the equation

$$
\begin{equation*}
u^{\prime \prime}(t)+\frac{2}{t} u^{\prime}(t)+u^{3}(t)-u=0, \quad t \in(a, \infty), \quad a \geq 0 \tag{1.5}
\end{equation*}
$$

McLeod and Serrin [10] extended the result to a wide range of $n$ and $p$. Kwong [6] completed the study by confirming uniqueness for all admissible solution of $n$ and $p$. The results were further extended by Kwong and Zhang [9] recently.

In this paper we explore another avenue for obtaining uniqueness results. Let us consider (BVP2). Suppose that $u_{1}(t)$ and $u_{2}(t)$ are two solutions of the boundary value problem. It is well known that they must intersect at least once in $[a, b]$. Uniqueness follows if we can show that they intersect exactly once. The main idea is to rewrite the original equation using $u(t)$ or some related function as the independent variable (with an appropriate new dependent variable) and then use comparison techniques of differential inequalities to show that beyond the first point of intersection of $u_{1}(t)$ and $u_{2}(t)$, some function of $u_{1}^{\prime}(t)$ remains always larger than the corresponding function of $u_{2}^{\prime}(t)$. This implies that the two solutions cannot intersect again. In Section 2 we illustrate this approach by extending one of Coffman's theorems as mentioned above. Some interesting corollaries are given. We also show how new criteria may be obtained from old ones by using the method of change of variables. In Section 3 we give some further application of our method, dealing with reaction terms of the form $F(t, u)=\sum t^{\gamma} u^{p}$. In Section 4, we study (BVP1) of one particular equation $u^{\prime \prime}+t^{4}\left(u^{5}+u\right)=0$, which is covered neither by known results nor by those in earlier sections. We show how an elaboration of our method still leads to a positive result. An obvious conjecture is the validity of a more general result.

The proofs of the theorems in Sections 3 and 4 require very technical computation done with the help of the symbolic manipulation software MAPLE.

## 2 Some General Uniqueness Results

We first give a uniqueness criterion for (BVP2) which extends earlier results of Coffman [4] and Kwong [7].

The condition we impose on $F(t, u)$ is the following:
[F1] The function $F(t, u)$ is Lipschitz continuous in $u$ for fixed $t$ and is superlinear. There exists a positive concave function $\phi:(a, b) \rightarrow(0, \infty)$ such that $\phi^{2} F(t, u)$ is nonincreasing in $t$ for each fixed $u$.

In the Coffman-Kwong theorem, $\phi(t)$ is $(b-t)$, but there is an additional positivity assumption on $F(t, u)$, under which [F1] does not offer any more than the above particular choice of $\phi(t)$. The situation, however, is different if $F(t, u)$ is allowed to change sign. Simple examples of functions that satisfy [F1] are given by $\frac{f(u)}{\phi^{2}(t)}$, with any choice of $\phi(t)$, such as $\frac{u^{3}-u}{b+1-t}$. Note that this function satisfies [F1] with $\phi(t)=\sqrt{b+1-t}$ but not with $\phi(t)=b-t$.

Lipschitz continuity on $F(u, t)$ is assumed to ensure uniqueness for initial value problems. The requirement can be slightly relaxed either by refining the arguments or by using a limiting process.

Theorem 1 Suppose that $F(t, u)$ satisfies [F1]. Then (BVP2) has a unique solution, and it is strictly decreasing in $[a, b]$.

Proof. The monotonicity of $u(t)$ in $(a, b)$ is obvious if $F(t, u)$ is nonnegative. Once we know monotonicity, strict monotonicity is a consequence of uniqueness of initial value problems. Indeed, if there is a point $c \in(a, b)$ at which $u^{\prime}(c)=0$, it must also be a point of inflection. Then $u^{\prime \prime}(c)=-F(c, u(c))=0$; but now we have the contradiction that $u(t) \equiv u(c)$ is the only solution that satisfies the initial conditions at $t=c$. Let us first prove the uniqueness of $u(t)$ assuming this monotonicity property.

Suppose that there are two distinct solutions $u_{1}(t)$ and $u_{2}(t)$ of the boundary value problem. Without loss of generality we may assume that $u_{1}(a)>u_{2}(a)$. It is well known that the two solutions must intersect at least once in $(a, b)$. Indeed, if this is not true, then $u_{1}(t) \geq u_{2}(t)$ in $(a, b)$, implying by superlinearity that $F\left(t, u_{1}(t)\right) / u_{1}(t) \geq F\left(t, u_{2}(t)\right) / u_{2}(t)$. We deduce from the Sturm comparison theorem that $u_{1}(t)$ oscillates faster than $u_{2}(t)$ in $(a, b)$, but this is obviously false.

Let $c \in(a, b)$ be the first point of intersection of $u_{1}(t)$ and $u_{2}(t)$. Then $u_{1}(c)=u_{2}(c)=\beta$ and $u_{1}^{\prime}(c)<u_{2}^{\prime}(c)<0$. Let $d \leq b$ be the next point of
intersection, which exists since the two solutions intersect again at $b$. Denote $u_{1}(d)=u_{2}(d)=\gamma$. Now we consider the solution curves as functions of their abscissa $u \in(\gamma, \beta)$. Define $R_{i}(u)=\phi(t) u_{i}^{\prime}(t), i=1,2$, as a function of $u$ instead of $t$. We have

$$
\begin{equation*}
R_{2}(\gamma)<R_{1}(\gamma) \tag{2.1}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{2}(\beta)>R_{1}(\beta) \tag{2.2}
\end{equation*}
$$

By continuity, there exists a point $\alpha \in(\gamma, \beta)$ at which

$$
\begin{equation*}
R_{2}(\alpha)=R_{1}(\alpha) \tag{2.3}
\end{equation*}
$$

In the interval $[\alpha, \beta]$, the functions $R_{i}(u)$ satisfy the differential equations

$$
\begin{aligned}
\frac{d R_{i}(u)}{d u} & =\left.\frac{d R_{i}}{d t}\right|_{t=t_{i}(u)} /\left.\frac{d u_{i}}{d t}\right|_{t=t_{i}(u)} \\
& =\left.\frac{\phi(t) u_{i}^{\prime \prime}(t)+\phi^{\prime}(t) u_{i}^{\prime}(t)}{u_{i}^{\prime}(t)}\right|_{t=t_{i}(u)} \\
& =\phi^{\prime}\left(t_{i}(u)\right)-\frac{\phi\left(t_{i}(u)\right) F\left(t_{i}(u), u\right)}{R_{i}(u)}
\end{aligned}
$$

where $t=t_{i}(u)$ denotes the inverse function corresponding to $u=u_{i}(t)$. It is easy to see from a simple picture that $t_{1}(u) \leq t_{2}(u)$ for $u \in(\alpha, \beta)$. It follows from the monotonicity of $F(t, u)$ that the function $R_{2}(u)$ satisfies in $[\alpha, \beta]$ the differential inequality

$$
\begin{equation*}
\frac{d R_{2}(u)}{d u} \leq \phi^{\prime}\left(t_{1}(u)\right)-\frac{\left.\phi^{( } t_{1}(u)\right) F\left(t_{1}(u), u\right)}{R_{2}(u)} \tag{2.4}
\end{equation*}
$$

Given the initial comparison condition (2.3), we can thereby conclude, using the theory of differential inequalities, that $R_{2}(u) \leq R_{1}(u)$, a contradiction to (2.2).

Let us complete the proof of the theorem by showing that a solution to the boundary value problem must be nonincreasing. Let $c$ be the last local maximum of the solution $u(t)$ and denote $\beta=u(c)$. Then in $[c, b], u(t)$ is nonincreasing. Let $d$ be the last local minimum before $c$, and denote $\alpha=u(d)$. Then in $[d, c], u(t)$ is nondecreasing. Now switch to using $u \in[\alpha, \beta]$ as the independent variable as before. Define $R_{1}(u)=-\phi\left(t_{1}(u)\right) u^{\prime}\left(t_{1}(u)\right)$ and $R_{2}(u)=$ $\phi\left(t_{2}(u)\right) u^{\prime}\left(t_{2}(u)\right)$, where $t_{1}(u)$ is the inverse function of $u(t)$ in $[d, c]$ and $t_{2}(u)$ the inverse of $u(t)$ in $[c, b]$. Obviously $t_{2}(u) \geq t_{1}(u)$. The functions $R_{i}(u)$ satisfy the same differential equation and differential inequality, respectively. From the strict initial comparison condition $R_{2}(\alpha)<R_{1}(\alpha)=0$ follows the contradiction $R_{2}(\beta)<R_{1}(\beta)$.

In the proof of Theorem 1, we have actually shown the uniqueness of a nonincreasing solution to (1.1) having prescribed boundary values. We state this more explicitly below.

Theorem 2 Suppose that $[\mathbf{F} 1]$ is satisfied. Let $\mu_{1}>\mu_{2}>0$ be two given real values. There is at most one nonincreasing solution of (1.1) on $[a, b]$ satisfying the boundary conditions

$$
\begin{equation*}
u(a)=\mu_{1}, \quad u(b)=\mu_{2} \tag{2.5}
\end{equation*}
$$

Furthermore, let $w_{1}(t)$ and $w_{2}(t)$ be solutions of (1.1) on $[a, b]$ satisfying

$$
\begin{equation*}
w_{1}(a)=u(a)=w_{2}(a) \tag{2.6}
\end{equation*}
$$

and

$$
\begin{equation*}
w_{1}^{\prime}(a)<u(a)<w_{2}^{\prime}(a) \leq 0 \tag{2.7}
\end{equation*}
$$

Let $[a, c]$ be the largest interval in which $w_{1}(t) \geq \mu_{2}$. Then $w_{1}(t)$ is nonincreasing in $[a, c]$,

$$
\begin{equation*}
w_{1}(t)<u(t) \text { for all } t \in(a, c] \tag{2.8}
\end{equation*}
$$

and

$$
\begin{equation*}
w_{2}(t)>u(t) \text { for all } t \in(a, b] \tag{2.9}
\end{equation*}
$$

Proof. The proof of the first part of the theorem is precisely that part in the proof of Theorem 1 where we showed that $u_{1}(t)$ and $u_{2}(t)$ cannot have two intersection points $c$ and $d$.

The comparison assertion between $w_{1}(t)$ and $u(t)$ in $[a, c]$ is a consequence of the first part of the theorem once we know that $w_{1}(t)$ is nonincreasing in [a,c]. Indeed, if $w_{1}(\tau)=u(\tau)$ for some $\tau \in[a, c]$, we arrive at the contradiction that (1.1) has two distinct monotone solutions satisfying the same boundary conditions at $a$ and $\tau$. So let us suppose that $w_{1}(t)$ is not nonincreasing in $[a, c]$. Then there is a first point $d \in[a, c]$ at which $w_{1}^{\prime}(d)=0$. In $[a, d]$, $w_{1}(t)$ is nonincreasing. It therefore cannot intersect $u(t)$ in $[a, d]$, again by the first part of the theorem. Denote $\alpha=w_{1}(d)$. Within the horizontal strip $\alpha \leq u \leq \mu_{1}$, the graph of $w_{1}(t)$ lies to the left of that of $u(t)$. We can rewrite (1.1) using $u$ in $\left[\alpha, \mu_{1}\right]$ as the independent variable, and $R_{1}(u)=\phi(t) u^{\prime}(t)$ and $R_{2}(u)=\phi(t) w_{1}^{\prime}(t)$ as we did in the proof of Theorem 1. The inequalities

$$
\begin{equation*}
R_{1}(\alpha)<R_{2}(\alpha)=0 \quad \text { and } \quad R_{1}\left(\mu_{1}\right)>R_{2}\left(\mu_{2}\right) \tag{2.10}
\end{equation*}
$$

yield a contradiction just as before.
Finally, suppose that the comparison conclusion between $w_{2}(t)$ and $u(t)$ is false. Let $\tau$ be the first point of intersection of the two solution curves. Just as
in the proof of Theorem 1, we can show that $w_{2}(t)$ must be monotone in the interval $[a, \tau]$. The first part of the theorem now leads to a contradiction.

Note that Theorem 2 does not guarantee the existence of a solution satisfying the given boundary conditions, nor does it rule out the possibility of a second solution that is nonmonotone. Also, Theorem 2 asserts only a comparison between $w_{1}(t)$ and $u(t)$ up to the point $c$. Beyond $c, w_{1}(t)$ at first dips under the height $\mu_{2}$, but at a later point, $w_{1}(t)$ may bounce back to overtake $u(t)$. On the other hand, $w_{2}(t)$ lies above $u(t)$ in the entire interval $[a, b]$.

An analog of Theorem 2 concerning solutions having the same terminal values at the right endpoint $b$ can easily be obtained as a corollary. It will be used later in the study of (BVP1).

Corollary 1 Suppose that $[\mathbf{F} 1]$ is satisfied and $\mu_{1}>\mu_{2}>0$ are two given real values. Let $u(t)$ be the unique solution of (1.1) on $[a, b]$ satisfying (2.5), as asserted in Theorem 2. Let $v_{1}(t)$ and $v_{2}(t)$ be the solutions of (1.1) on $[a, b]$ satisfying

$$
\begin{equation*}
v_{1}(b)=u(b)=v_{2}(b) \tag{2.11}
\end{equation*}
$$

and

$$
\begin{equation*}
0>v_{1}^{\prime}(b)>u^{\prime}(b)>v_{2}^{\prime}(b) \tag{2.12}
\end{equation*}
$$

Then

$$
\begin{equation*}
v_{1}(t)<u(t) \text { for all } t \in[a, b) \tag{2.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\max \left\{v_{2}(t): t \in[a, b]\right\}>\mu_{2} \tag{2.14}
\end{equation*}
$$

Let us now turn our attention to (BVP1). We need a complementary condition on $F(t, u)$ :
[F2] There exists a positive concave function $\psi:(a, b) \rightarrow(0, \infty)$ such that $\psi^{2} F(t, u)$ is nondecreasing in $t$ for each fixed $u$.

Theorem 3 Suppose that $F(t, u)$ satisfies $[\mathbf{F 1}]$ and $[\mathbf{F 1}]$. Then (BVP1) has a unique solution, and it has a unique maximum in $[a, b]$.

Proof. The fact that any solution of (BVP1) can only have one maximum in $[a, b]$ is a consequence of the monotonicity of any solution of (BVP2).

Suppose that there are two distinct solutions $u_{1}(t)$ and $u_{2}(t)$. By superlinearity, they must intersect at least once in $(a, b)$. Between two consecutive points
of intersection (including $a$ and $b$ ), one of the solutions cannot be monotone (because of Theorem 2), and it must therefore attain its unique maximum between the two points. It follows that the two solutions cannot have more than one other intersection point besides $a$ and $b$. Whichg we may assume that $u_{1}^{\prime}(b)>u_{2}^{\prime}(b)$. It follows that at the other endpoint $a, u_{1}^{\prime}(a)<u_{2}^{\prime}(a)$. Let $c$ be the point at which $u_{1}(t)$ attains its maximum. By Corollary 1, $\max \left\{u_{2}(t): t \in[c, b]\right\}>u_{1}(c)$. Hence, $\max \left\{u_{2}(t): t \in[a, b]\right\}>\max \left\{u_{1}(t): t \in[a, b]\right\}$. Let $c^{\prime}$ be the point at which $u_{2}(t)$ attains its maximum. By applying Corollary 1 to a reflection of the interval $\left[a, c^{t}\right]$, we obtain exactly the opposite inequality between the two maxima. This contradiction completes the proof of the theorem.

Examining the above proof more closely reveals that the full strength of [F1] and [F2] is not actually needed. If we now have an a priori estimate of where the maximum of a solution of (BVP1) is located, say within a subinterval $\left[c^{\prime}, c\right] \subset[a, b]$, then all we need is the monotonicity of $\phi^{2} F(t, u)$ and $\psi^{2} F(t, u)$ in, respectively, $\left[c^{\prime}, b\right]$ and $[a, c]$. We give below an application of this observation. In the rest of this section, we consider only nonnegative $C^{1}$ functions $F(t, u)$ for the sake of simplicity. With due care, most of the results can be extended to the more general situation.

Theorem 4 Suppose that $F(t, u)$ is $C^{1}$, superlinear, and nonnegative. If for every fixed $u, F(t, u)$ is nondecreasing in $t$ and satisfies the inequality

$$
\begin{equation*}
\frac{F_{t}(t, u)}{F(t, u)} \leq \frac{2}{t} \tag{2.15}
\end{equation*}
$$

where $F_{t}$ denotes the partial derivative of $F$ with respect to $t$, then (BVP1) has a unique solution for all $0<a<b$.

Proof. A simple comparison argument using the monotonicity of $F(t, u)$ with respect to $t$ shows that a solution of (BVP1) must be skewed towards the endpoint $b$. Hence, the maximum of the solution must be attained within the interval $[(a+b) / 2, b] \subset[b / 2, b]$.

Next, it is easy to verify that (2.15) implies that $(b-t)^{2} F(t, u)$ cannot have a critical point in $(b / 2, b]$. In other words, $(b-t)^{2} F(t, u)$ is nonincreasing in $[b / 2, b]$, and so the proof of Theorem 3 works to give uniqueness.

Corollary 2 Suppose that $F(t, u)=q(t) f(u)$ is $C^{1}$, superlinear, and nonnegative. If $q(t)$ is nondecreasing and $2 t q^{\prime}(t) \leq q(t)$, then (BVP1) has a unique solution for all $0<a<b$.

The hypothesis on $q(t)$ in the lemma is satisfied in particular if $q(t)$ is a positive linear combination of nonnegative powers of $t$ up to the square or, more generally, if

$$
\begin{equation*}
q(t)=\int_{0}^{2} t^{\gamma} d \mu(\gamma) \tag{2.16}
\end{equation*}
$$

for some nonnegative Borel measure $d \mu(\gamma)$. As a simple example, we can take $d \mu(\gamma)=1$ to obtain $q(t)=\left(t^{2}-1\right) / \ln (t)$.

Note that uniqueness is valid for a similar class of coefficients,

$$
\begin{equation*}
q(t)=\int_{0}^{2} \frac{d \mu(\gamma)}{t^{\gamma}} \tag{2.17}
\end{equation*}
$$

this time because the hypotheses of Theorem 3 are satisfied.
In the rest of this section as well as in the next, many of the criteria are established for nonlinear functions of the form

$$
\begin{equation*}
F(t, u)=\sum c_{i} t^{\gamma_{i}} u^{p_{i}} \tag{2.18}
\end{equation*}
$$

where $c_{i}>0$, with given conditions relating $p_{i}$ and $\gamma_{i}$. Often the results can be extended to the case in which the finite sum is replaced with an integral containing a nonnegative Borel measure. This can be done by either modifying the proof or by using the following limiting argument. Let $S$ denote the set of pairs $(p, \gamma)$ that satisfy the condition imposed on $\left(p_{i}, \gamma_{i}\right)$ by the criterion, and let $d \mu$ be a nonnegative Borel measure defined on $S$. In general, $S$ is a twodimensional subset of the plane. Sometimes the result involves a fixed value of $p$ or $\gamma$; then the set $S$ is usually an interval. The nonlinear function generated by $d \mu$ over $S$ is defined to be

$$
\begin{equation*}
F(t, u)=\mu\left(t^{\gamma} u^{p} ; S\right)=\int_{S} t^{\gamma} u^{p} d \mu \tag{2.19}
\end{equation*}
$$

The function $F(t, u)$ can be approximated by a sequence of functions each in the form of a finite sum. Let $b_{i}(\beta)$ denote the first zero of the solution with the initial conditions $u(a)=0, u^{\prime}(a)=\beta$ for the $i^{\text {th }}$ function in this sequence. By continuity, the sequence $b_{i}$ converges uniformly to the corresponding first-zero function for the limiting nonlinear function $F(t, u)$. By the uniqueness results on boundary value problems for nonlinear terms in the form a finite sum, each $b_{i}(\beta)$ is a strictly decreasing function of $\beta$. Hence $b(\beta)$ is a decreasing (not necessarily strict) function. Strict monotonicity can be established by observing that the possibility of having an interval of values of $\beta$ for which $b(\beta)$ is a constant is ruled out by superlinearity.

In Section 4 we show how the idea used in the proof of Theorem 4 can be further exploited. In the meantime, with the help of some well-known transformations, we can increase the versatility of our criteria. We mention only a couple of examples.

Theorem 5 Suppose that $F(t, u)$ is $C^{1}$, superlinear, and nonnegative and that for each fixed $v$ either

$$
\begin{equation*}
t F(t, t v) \text { is nonincreasing in } t \tag{2.20}
\end{equation*}
$$

and

$$
\begin{equation*}
t^{3} F(t, t v) \text { is nondecreasing in } t \tag{2.21}
\end{equation*}
$$

or

$$
\begin{equation*}
t^{3} F(t, t v) \text { is nonincreasing in } t \tag{2.22}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial}{\partial t}\left[t^{3} F(t, t v)\right] \leq 2 t^{2} F(t, t v) \tag{2.23}
\end{equation*}
$$

Then (BVP1) has a unique solution for all $0<a<b$.

Proof. The change of variables

$$
\begin{equation*}
s=\frac{1}{t} \quad \text { and } \quad u=\frac{v}{s} \tag{2.24}
\end{equation*}
$$

transforms (1.1) into

$$
\begin{equation*}
\frac{d^{2} v}{d s^{2}}+\frac{1}{s^{3}} F\left(\frac{1}{s}, \frac{v}{s}\right)=0 \tag{2.25}
\end{equation*}
$$

Theorem 3 and Corollary 2 can now be applied to the two sets of conditions, respectively.

Corollary 3 Suppose that

$$
\begin{equation*}
F(t, u)=q(t) u^{p} \tag{2.26}
\end{equation*}
$$

where $q(t)$ satisfies either

$$
\begin{equation*}
t^{p+1} q(t) \text { is nonincreasing, } \tag{2.27}
\end{equation*}
$$

and

$$
\begin{equation*}
t^{p+3} q(t) \text { is nondecreasing } \tag{2.28}
\end{equation*}
$$

or

$$
\begin{equation*}
t^{p+3} q(t) \text { is nonincreasing } \tag{2.29}
\end{equation*}
$$

and

$$
\begin{equation*}
-t q^{\prime}(t) \leq(p+5) q(t) \tag{2.30}
\end{equation*}
$$

Then (BVP1) has a unique solution for all $0 \leq a<b$. In particular, the hypotheses on $q(t)$ are satisfied if

$$
\begin{equation*}
q(t)=\mu\left(t^{-\gamma},[p+1, p+3]\right) \tag{2.31}
\end{equation*}
$$

or

$$
\begin{equation*}
q(t)=\mu\left(t^{-\gamma},[p+3, p+5]\right) \tag{2.32}
\end{equation*}
$$

for some nonnegative Borel measure $d \mu(\gamma)$.

Corollary 4 Suppose that

$$
\begin{equation*}
F(t, u)=\sum c_{i} \frac{u^{p_{i}}}{t^{\gamma_{i}}} \tag{2.33}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{i}>0, \quad \text { and } \quad p_{i}+1 \leq \gamma_{i} \leq p_{i}+3 . \tag{2.34}
\end{equation*}
$$

Then (BVP1) has a unique solution for all $0 \leq a<b$.

Ni [13] showed that for (BVP1) and (BVP2) (1.4) has a unique solution if $f(u)>0, n>3$, and

$$
\begin{equation*}
\frac{n}{n-2} f(u) \geq u f^{\prime}(u) \tag{2.35}
\end{equation*}
$$

An alternative proof was given in [7]. We now give a third proof and an extension. The change of variables $u(t)=v(t) / t^{n-2}$ and $s=t^{n-2}$ transforms (1.4) to

$$
\begin{equation*}
v^{\prime \prime}(s)+\frac{1}{(n-2)^{2}} s^{k-2} f\left(\frac{v}{s}\right) \tag{2.36}
\end{equation*}
$$

where $k=\frac{n}{n-2}$. For this equation, [F1] is satisfied with $\phi(t)=1$, from superlinearity. With the choice of $\psi(t)=t$, we see that $[\mathbf{F 2}]$ becomes

$$
\begin{equation*}
s^{k} f\left(\frac{v}{s}\right) \quad \text { is increasing in } s . \tag{2.37}
\end{equation*}
$$

On differentiating this expression with respect to $s$ and then substituting $v / s=$ $u$, we see that this condition is equivalent to Ni's criterion.

It is now easy to see how the method can be extended to cover timedependent reaction terms.

Theorem 6 Suppose that $f(t, u)$ is $C^{1}$, superlinear, and nonnegative and that for each fixed $v$,

$$
\begin{equation*}
s^{k-2} f\left(s^{k / n}, \frac{v}{s}\right) \quad \text { is nonincreasing in } s \tag{2.38}
\end{equation*}
$$

and

$$
\begin{equation*}
s^{k} f\left(s^{k / n}, \frac{v}{s}\right) \quad \text { is nondecreasing in } s \tag{2.39}
\end{equation*}
$$

where $k=\frac{n}{n-2}$ (more generally, whenever the function in (2.38) satisfies any uniqueness criterion applicable to $F(t, u)$ of (1.1)). Then (BVP1) for the equation

$$
\begin{equation*}
u^{\prime \prime}(t)+\frac{n-1}{t} u^{\prime}(t)+f(t, u)=0 \tag{2.40}
\end{equation*}
$$

has a unique solution.

When $f(t, u)$ is a sum of products of powers of $t$ and $u$, we have the following result.

Corollary 5 Let $n>2, c_{i}>0, \gamma_{i}$ and $p_{i}(i=1, \cdots, N)$ be given constants. There exists a unique solution to (BVP1) for

$$
\begin{equation*}
u^{\prime \prime}(t)+\frac{n-1}{t} u^{\prime}(t)+\sum_{i=1}^{N} t^{\gamma_{i}} u^{p_{i}}=0 \tag{2.41}
\end{equation*}
$$

if either, for each i,

$$
\begin{equation*}
\max \left\{1, \frac{n+\gamma_{i}}{n-2}-2\right\} \leq p_{i} \leq \frac{n+\gamma_{i}}{n-2} \tag{2.42}
\end{equation*}
$$

or, for each i,

$$
\begin{equation*}
\max \left\{1, \frac{n+\gamma_{i}}{n-2}\right\} \leq p_{i} \leq \frac{n+\gamma_{i}}{n-2}+2 \tag{2.43}
\end{equation*}
$$

Proof. The assumption that $p_{i} \geq 1$ is needed to ensure that the equation is superlinear. The second set of conditions implies that the transformed equation satisfies the hypotheses of Corollary 2.

Specializations to cases where all the $\gamma_{i}$ or $p_{i}$ are equal can be obtained as before.

## $3 \quad F(t, u)=\sum c t^{\gamma} u^{p}$

The techniques we use in this section are most suited to handle functions of the form $F(t, u)=\sum c_{i} t^{\gamma_{i}} u_{i}^{p}$, with $\gamma_{i}>0$. Extensions to more general functions appear plausible. In order not to obscure the main ideas with too much technical detail, we first illustrate how the method works on a special case,

$$
\begin{equation*}
u^{\prime \prime}(t)+t^{\gamma}\left(u^{5}(t)+u^{4}(t)\right)=0 \tag{3.1}
\end{equation*}
$$

Using results from Section 2, we already know that in this case, (BVP1) has a unique solution if $-2 \leq \gamma \leq 2$. We wish to extend this range. For functions with only one term, namely $F(t, u)=t^{\gamma} u^{p}$, it is known that (BVP1) has a unique solution for any value of $\gamma$ (see [3]). There is, of course, no reason to believe that such a nice property is preserved when the function is perturbed by a different power of $u$. We show that at least when the perturbing power is close enough to $p$, and when $\gamma>0$, uniqueness prevails. The argument used is a combination of Coffman's method and a new choice of the independent variable.

The solution $u(t)$ for (BVP1) is a member of the family of solutions $u(t ; \beta)$ of an initial value problem for (3.1), in which the initial values at $t=a$ are assigned as follows:

$$
\begin{equation*}
u(a ; \beta)=0, \quad u^{\prime}(a ; \beta)=\beta \tag{3.2}
\end{equation*}
$$

When we omit the parameter $\beta$ in the notation $u(t ; \beta)$, we are referring to the original solution of our (BVP1).

According to Coffman's method (see Kwong [7] for a survey), uniqueness for (BVP1) follows if we can show that

$$
\begin{equation*}
w(b)=\left.\frac{\partial u(t ; \beta)}{\partial \beta}\right|_{t=b}<0 \tag{3.3}
\end{equation*}
$$

The function $w(t)$ satisfies the variational equation, obtained by differentiating (3.1) with respect to $\beta$,

$$
\begin{equation*}
w^{\prime \prime}(t)+t^{\gamma}\left(5 u^{4}(t)+4 u^{3}(t)\right) w(t)=0 \tag{3.4}
\end{equation*}
$$

and the initial condition

$$
\begin{equation*}
w(a)=0 \tag{3.5}
\end{equation*}
$$

The assertion (3.3) is usually derived from a study of the oscillatory behavior of $w(t)$ as a solution of the second-order "linear" differential equation (3.4); indeed, if we can prove that $w(t)$ has exactly one zero in ( $a, b]$, then (3.3) follows.

The function $z(t)=(2+\gamma) u(t) / 4+t u^{\prime}(t)$ satisfies the differential inequality

$$
\begin{equation*}
z^{\prime \prime}(t)+t^{\gamma}\left(5 u^{4}(t)+4 u^{3}(t)\right) z(t)=-(2+\gamma) t^{\gamma} u^{4}(t) / 4 \leq 0 \tag{3.6}
\end{equation*}
$$

Near $a, z(t)$ is positive. Let $c>a$ be the first zero of $z(t)$. According to Sturm's comparison theorem, the function $w(t)$ oscillates more slowly than the function $z(t)$ in $[a, c]$. Thus the first zero of $w(t)$ must be larger than $c$. In a right neighborhood of $c, z(t)$ is negative. At this point, we do not know whether $z(t)$ will remain negative in the rest of the interval, $[c, b]$. Let $d>c$ be either the next zero of $z(t)$ or $b$ if $z(t)$ does not have another zero.

Consider an auxiliary function $W(t)$, which is the solution of (3.4) satisfying, at $t=c$,

$$
\begin{equation*}
W(c)=0, W^{\prime}(c)=-1 \tag{3.7}
\end{equation*}
$$

Even though $z(t)$ satisfies the same differential inequalities in $[c, d]$ as it does in [a,c], the sign of $z(t)$ in these intervals determines the speed of its oscillation relative to that of the solution of the corresponding differential equation. In the interval $[c, d], W(t)$ oscillates faster than $z(t)$. Thus, if $d$ is a zero of $z(t)$, $W(t)$ must have a zero within $[c, d]$, and $W(t)$ must take negative values at some points in $[c, d]$. Let us assume we can prove that this is impossible. Then, as a consequence, $d$ cannot be a zero of $z(t), d=b$, and $W(t)$ remains nonnegative in the entire interval $[c, b]$. Since $w(t)$ and $W(t)$ satisfy the same second-order linear differential equation, their zeros must separate each other, by Sturm's separation theorem. We already know that $w(t)$ has a zero in $(c, b]$. It cannot have another zero, lest $W(t)$ have one between these two zeros; (3.3) is now proved.

It remains to establish our claim that $W(t)$ cannot be negative in $[c, d]$. Suppose the contrary; so $W(\tau)<0$ for some $\tau \in[c, d]$. Just as $w(t)$ can be interpreted as the rate of change of $u(t ; \beta)$ with respect to the initial slope $\beta$, $W(t)$ can be interpreted as the rate of change of $u(t ; \eta)$ with respect to the slope $\eta$ at the point $c$, where $u(t ; \eta)$ now stands for the family of solutions of the initial value problem

$$
\begin{equation*}
u(c ; \eta)=u(c), \quad u^{\prime}(c ; \eta)=\eta \tag{3.8}
\end{equation*}
$$

The inequality $W(\tau)<0$ means that there exists a solution of (3.1) that intersects $u(t)$ at $c$, having a slightly larger slope than $u(t)$ at $c$ (so it stays above $u(t)$ in a right neighborhood of $c$ ), but is below $u(t)$ at $t=\tau$. By continuity, the two solutions must intersect again before $t=\tau$. Since $u(t)<0$ in $(a, b]$, we can assume, by choosing the slope of $u(t ; \eta)$ at $c$ sufficiently close to that of $u(t)$, that the second solution is also decreasing in $[c, \tau]$. We now show that this leads to a contradiction.

Let $\alpha=(2+\gamma) / 4$. Since $\left(t^{\alpha} u(t)\right)^{\prime}=t^{\alpha-1} z(t)<0$ in $(c, d), t^{\alpha} u(t)$ is decreasing in the interval. We can rewrite the equation in $[c, d]$ using $v=t^{\alpha} u(t)$ as the new independent variable and $R_{1}(v)=u^{\prime}(t)$ and $R_{2}(v)=u^{\prime}(t ; \eta)$ as the dependent variables. As before, we obtain the differential equation for $R_{i}(v)$, using the relation $d R_{i} / d v=\left(d R_{i} / d t\right) /(d v / d t)$. In the following, we list the
equation and suppress the subscript $i=1,2$.

$$
\begin{equation*}
\frac{d R}{d v}=-\frac{t^{\gamma}\left(u^{5}+u^{4}\right)}{t^{\alpha} u^{\prime}+\alpha t^{\alpha-1} u} . \tag{3.9}
\end{equation*}
$$

After substituting $u=v / t^{\alpha}$ and $u^{\prime}=R$, we have the equation

$$
\begin{equation*}
\frac{d R}{d v}=G(v, R ; t):=-\frac{4\left(v^{5}+t^{\alpha} v^{4}\right)}{t^{\alpha+1}\left(2 v+\gamma v+4 t^{\alpha+1} R\right)} \tag{3.10}
\end{equation*}
$$

If we can show that $G(v, R ; t)$, the righthand side of (3.10), is nonincreasing in $t$ for fixed $v$ and $R$, then as in the proof of Theorem 1, this implies that the two solutions $u(t)$ and $u(t, \eta)$ cannot intersect beyond the point $t=c$, giving us the desired contradiction. The monotonicity $G(v, R ; t)$ is, however, not obvious. The following argument, though conceptually simple, involves a fair amount of computation, which we carried out with the help of the symbolic manipulation software MAPLE. We shall give more details on the computer aspect of our work in the Appendix.

Differentiating $G(v, R ; t)$ with respect to $t$ gives a fraction, the numerator of which, aside from factors of powers of $v$ and $t$, is

$$
\begin{equation*}
\left(12+8 \gamma+\gamma^{2}\right) v^{2}+(8+4 \gamma) v t^{\alpha}+(48+8 \gamma) v R t^{1+\alpha}+(40+4 \gamma) R t^{1+2 \alpha} \tag{3.11}
\end{equation*}
$$

The denominator of the fraction is the square of the denominator of the original fraction and so is positive. It thus remains to show that the expression (3.11) is nonpositive. This fact is not readily obvious; although the last two terms are nonpositive (recall that $R=u^{\prime} \leq 0$ ), the first two are not. Since $v(t)$ is decreasing in $[c, \tau]$, we have $v^{\prime}(t) \leq 0$, implying that

$$
\begin{equation*}
t^{\alpha}\left(2 u+\gamma u+4 t u^{\prime}\right) \leq 0 \tag{3.12}
\end{equation*}
$$

Let us denote this function by $-P(t)$, with $P(t) \geq 0$, and replace the variables $u$ and $u^{\prime}$ by the new variables $v$ and $R$. We obtain

$$
\begin{equation*}
-P(t)=2 v+\gamma v+4 t^{1+\alpha} R \leq 0 \tag{3.13}
\end{equation*}
$$

Solving this inequality for $R$ and then substituting the answer into (3.11), we have

$$
\begin{equation*}
-\left(12+8 \gamma+\gamma^{2}\right) v^{2}-(12+2 \gamma) P v-\left(12+8 \gamma+\gamma^{2}\right) v t^{\alpha}-(10+\gamma) P t^{\alpha} \tag{3.14}
\end{equation*}
$$

which is now obviously nonpositive.
One may wonder why we had the confidence to carry on the complicated computation, as there was very little indication of possible success at the point when (3.10) was derived. In fact, our initial investigation used chosen values
of $\gamma$, thus rendering the computation less formidable. Besides, the availability of a symbolic algebra software makes it easy to carry out experimentations; a simple program mechanizes the process. After a number of successful trials with specific values of $\gamma$, we took a general $\gamma$. Then we went on to the general case summarized by the following theorem.

Theorem 7 Suppose that

$$
\begin{equation*}
F(t, u)=\sum t^{\gamma_{i}} u^{p_{i}} \tag{3.15}
\end{equation*}
$$

with

$$
\begin{equation*}
\gamma_{i} \geq-2, \quad \text { and } \quad p_{i} \geq 1 \tag{3.16}
\end{equation*}
$$

Denote

$$
\begin{equation*}
\alpha=\min _{i}\left\{\frac{\gamma_{i}+2}{p_{i}-1}\right\} . \tag{3.17}
\end{equation*}
$$

If for all i,

$$
\begin{equation*}
p_{i} \geq \frac{\gamma_{i}}{\alpha}-1 \tag{3.18}
\end{equation*}
$$

then (BVP1) has a unique solution for all $0 \leq a<b$.

Proof. The basic steps in the proof of the general case are identical to those of the special case discussed above. The only difference is that the computation involved is many times more complicated. With the help of MAPLE, however, this is not a major obstacle. Only the final simplified result of each computational step will be presented in this proof. The MAPLE program used to produce the result will be discussed in the Appendix.

The function $w(t)$, as defined in (3.3), satisfies the differential equation

$$
\begin{equation*}
L[w(t)]=w^{\prime \prime}(t)+\left[\sum p_{i} t^{\gamma_{i}} u^{p_{i}-1}(t)\right] w(t)=0 \tag{3.19}
\end{equation*}
$$

The quantity $\alpha$ is chosen as in (3.17) so that the function $z(t)=\alpha u(t)+t u^{\prime}(t)$ satisfies the differential inequality

$$
\begin{equation*}
L[z] \leq 0 . \tag{3.20}
\end{equation*}
$$

The same arguments using the auxiliary function $W(t)$ carry over without change. Using $R=u^{\prime}(t)$ and $v=t^{\alpha} u$ as dependent and independent variables, we obtain a differential equation of the form (3.10). All that remains is to show that the righthand side of (3.10), namely, $G(v, R ; t)$, is nonincreasing in $t$ for fixed $v$ and $R$. Note that each term in the expression for $F(t, u)$ gives rise to
a term of an identical form in the numerator of $G(v, R ; t)$. In the subsequent computation, each of these can be handled independently of the others; and since each term has the same form, we have to keep track of only one typical term.

As before, each term is differentiated with respect to $t$, and the numerator of the resulting fraction is treated with a substitution by using $R$ from an inequality of the form (3.13). The final expression, analogous to (3.14), can be separated into two parts; the first one contains the factor $P$, and the other does not. After we discard factors that are powers of $t$ and $v$ (the actual exponents depend on the term), what remains of the first part is the same for all terms and has the form

$$
\begin{equation*}
-2(\gamma+2)(p+\gamma+1) \tag{3.21}
\end{equation*}
$$

where $\gamma$ and $p$ pertain to the particular term for which $\alpha$ is defined. The other part, again after we discard powers of $t$ and $v$, assumes the form

$$
\begin{equation*}
-\left(\gamma p_{i}+\gamma-\gamma_{i} p+\gamma_{i}+2 p_{i}+2\right) \tag{3.22}
\end{equation*}
$$

Since the first part is obviously nonpositive, uniqueness follows when the expression in (3.22) is nonpositive. It is easy to see that this is equivalent to requiring that (3.18) holds.

A few special cases of the theorem are given below as corollaries. As shown in Section 2, uniqueness criteria of the form in this paper can be stated more generally for functions induced by Borel measures. We recall the notation introduced in (2.19).

Corollary 6 Let $p \geq 1$ and $d \mu(\rho)$ be a nonnegative Borel measure on $[p, p+2]$. If

$$
\begin{equation*}
f(u)=\mu\left(u^{\rho},[p, p+2]\right) \tag{3.23}
\end{equation*}
$$

then (BVP1) for the differential equation

$$
\begin{equation*}
u^{\prime \prime}(t)+t^{\gamma} f(u(t))=0, \quad 0 \leq a<t<b \tag{3.24}
\end{equation*}
$$

has a unique solution for all $\gamma \geq-2$, or $\gamma \leq-(p+3)$.

Proof. Let us first look at the case when $\gamma>0$. We apply Theorem 7 with all the $\gamma_{i}$ in (3.15) equal to $\gamma$ and $p \leq p_{i} \leq p+2$. In this case, the number $\alpha$ defined in (3.17) is $\gamma /(p+1)$. The inequality (3.18) can be verified easily, and so the conclusion of Theorem 7 holds.

In the second case, we apply Theorem 7 first to the nonlinear function $F(t, u)$ in which $\gamma_{i}=\gamma-p_{i}-3$ and $p \leq p_{i} \leq p+2$. The change of variable argument used to prove Theorem 5 (see (2.25)) then gives the desired conclusion.

This is an extension of the well-known fact that uniqueness holds for $F(t, u)=$ $t^{\gamma} u^{p}$, for all $\gamma$ and $p>1$. Note, however, that Corollary 6 leaves a gap in the admissible values of $\gamma$. It is interesting to find out what really happens if $-(p+3)<\gamma<-2$.

In case the function $f(u)$ is generated by a wider range of powers of $u$, Theorem 7 no longer gives uniqueness for all large $\gamma$. Indeed, if $q>p+2$ and $f(u)=\mu\left(u^{\rho},[p, q]\right)$, then uniqueness holds for (3.24) if

$$
\begin{equation*}
-2 \leq \gamma \leq \frac{2(p+2)}{q-p-2} \tag{3.25}
\end{equation*}
$$

On the other hand, the change of variable argument fails to provide any result for negative values of $\gamma$.

By fixing the exponent of $u$ we obtain the following corollary.

Corollary 7 Let $F(t, u)=q(t) u^{p}$. Let $\delta>0$ be any positive number. If

$$
\begin{equation*}
q(t)=\mu\left(t^{\gamma},\left[\delta, \frac{(p+1)(\delta+2)}{p-1}\right]\right) \tag{3.26}
\end{equation*}
$$

or

$$
\begin{equation*}
q(t)=\mu\left(t^{-\gamma},\left[\delta+p+3, \frac{(p+1)(\delta+2)}{p-1}+p+3\right]\right) \tag{3.27}
\end{equation*}
$$

then (BVP1) has a unique solution.

## 4 A Special Case $F(t, u)=t^{4}\left(u^{5}+u\right)$

Ample numerical evidence indicates that established uniqueness criteria, such as those derived in previous sections, are not very sharp. This is typical of nonlinear analysis.

In this section we consider uniqueness for (BVP1) of an equation that is not covered by any of the criteria known so far. The method used is an elaboration of ideas used earlier in Section 2. The technicalities are so involved that we believe it is easier to present the details only for a special case. Obviously the method has wider applications.

We wish to show that for any $0<a<b$, (BVP1) of

$$
\begin{equation*}
u^{\prime \prime}+t^{4}\left(u^{5}+u\right)=0 \quad \text { on }(a, b) \tag{4.1}
\end{equation*}
$$

has only one solution.
Suppose there are two solutions, $u_{1}(t)$ and $u_{2}(t)$, with $u_{1}^{\prime}(a)>u_{2}^{\prime}(a)$. They must intersect at least once before $t=b$. Let the first intersection be at $t=c$. As usual, we can easily show that $w(t)=u_{1}(t)-u_{2}(t)$ oscillates more slowly than $z(t)=3 u_{1}(t) / 2+t u_{1}^{\prime}(t)$, by comparing the second-order "linear" equations that they satisfy. Hence, the point $c$ must be larger than the first zero of $z(t)$, namely, the first point $\tau$ at which $\tau u_{1}^{\prime}(\tau) / u_{1}(\tau)=-3 / 2$. From the differential equation satisfied by $z(t)$, we see that $z(t)$ cannot be tangent to the $t$-axis. Hence the point $\tau$ depends continuously on the initial conditions as well as on the left endpoint $a$.

Suppose now that we can assert

$$
\begin{equation*}
\tau>\frac{2 b}{3} \tag{4.2}
\end{equation*}
$$

Then it is easy to see that the function $(b-t)^{2} t^{4}$ is nonincreasing in the interval $[\tau, b]$, which contains $[c, b]$. In other words, condition $[\mathbf{F} \mathbf{1}]$ is satisfied with $\phi(t)=(b-t)$. The existence of two distinct nonincreasing solutions $u_{1}(t)$ and $u_{2}(t)$ on $[c, b]$ would have contradicted Theorem 2.

Hence it remains to establish inequality (4.2) to verify uniqueness. Since we no longer need to mention the solution $u_{2}(t)$, we will drop the subscript when referring to $u_{1}(t)$. Alternatively, we may view (4.2) as a fact to be established for any solution that vanishes at $a$ and $b$.

Note that the solutions of (4.1) oscillates faster in $(a, b)$ than those of the linear equation

$$
\begin{equation*}
U^{\prime \prime}+t^{4} U=0 \tag{4.3}
\end{equation*}
$$

Hence $u(t) \leq U(t)$ in $(a, b)$ if $u(a)=U(a)=0$ and $u^{\prime}(a)=U^{\prime}(a)>0$. This implies in particular that $b<\infty$.

We suppose now that (4.2) is not satisfied. Let us pull the left endpoint $a$ back towards the origin, while keeping the initial slope fixed. We know that both $\tau$ and $b$ will change continuously. We have one of two possibilities. Either there is a point $a \geq 0$ at which (4.2) is barely violated, for which case we have $\tau=2 b / 3$, or when $a=0$, we still have $\tau<2 b / 3$.

In the latter case, let us start with the solution at $a=0$ and deform it by decreasing the initial slope $u^{\prime}(0)$. If the initial slope is sufficiently small, then $u^{\prime}(t)$ remains small in $(0, b) ; u^{\prime}(t)$ is smaller than $U^{\prime}(t)$. The nonlinear term $u^{5}(t)$ in (4.1) is relatively small in comparison with the linear term. It follows that $U(t)$ gives a very good approximation for $u(t)$. It can be verified easily, for example numerically, that for $U(t)$ the inequality corresponding to (4.2) holds. Thus (4.2) holds for $u(t)$ if $u^{\prime}(0)$ is sufficiently small. From our assumption that (4.2) is not satisfied originally, we must be able to get a value at which (4.2) is barely violated, so that

$$
\begin{equation*}
\tau=2 b / 3 \tag{4.4}
\end{equation*}
$$

Hence in all cases, we have a suitable solution in $(a, b)$ for some $a \geq 0$, such that (4.4) holds. In the rest of the proof we concentrate on this particular solution and claim that a contradiction ensues.

Before starting the complicated computation, we scale the solution both horizontally and vertically. Scaling is done for convenience rather than necessity. The horizontal scaling is chosen so as to transform $\tau$ into 1 and $b$ into $3 / 2$. A constant factor is then introduced into the nonlinear term. The vertical scaling is used to retain a unit coefficient for the term $t^{4} u^{5}$. By abusing the notation, we denote the scaled solution again by $u(t)$. The differential equation it satisfies is now

$$
\begin{equation*}
u^{\prime \prime}(t)+t^{4} u^{5}(t)+\lambda t^{4} u(t)=0, \quad \text { on } \quad(a, 3 / 2) \tag{4.5}
\end{equation*}
$$

where $\lambda>0$ is some number determined by the scaling. By the definition of the point $\tau$,

$$
\begin{equation*}
\frac{u^{\prime}(1)}{u(1)}=-\frac{3}{2} \tag{4.6}
\end{equation*}
$$

Let us denote

$$
\begin{equation*}
u(1)=\kappa \tag{4.7}
\end{equation*}
$$

Then

$$
\begin{equation*}
u^{\prime}(1)=-\frac{3 \kappa}{2} \tag{4.8}
\end{equation*}
$$

We first derive an upper bound on $\kappa$ By integrating (4.5) twice, first over $[1, s], s<3 / 2$ and then over $[1,3 / 2]$, we obtain the identity

$$
\begin{equation*}
u(1)=\frac{u^{\prime}(1)}{2}+\int_{1}^{3 / 2}(3 / 2-t) t^{4}\left(u^{5}(t)+\lambda u(t)\right) d t \tag{4.9}
\end{equation*}
$$

Making use of (4.7) and (4.8), we have

$$
\begin{equation*}
\kappa=4 \int_{1}^{3 / 2}(3 / 2-t) t^{4}\left(u^{5}(t)+\lambda u(t)\right) d t \tag{4.10}
\end{equation*}
$$

Concavity implies that in the interval $[1,3 / 2], u(t) \geq(3-2 t) \kappa$. Substituting this inequality into (4.10) and making a change of variable in the integral, we obtain

$$
\begin{equation*}
1 \geq 4\left(\kappa^{4}+\lambda\right) \int_{0}^{1 / 2}(1 / 2-s)(1+s)^{4} d t \tag{4.11}
\end{equation*}
$$

With the help of MAPLE, this simplifies to

$$
\begin{equation*}
\kappa^{4} \leq \frac{12320}{2281}-\frac{6886}{2281} \lambda \tag{4.12}
\end{equation*}
$$

Let $\sigma \in(0,1)$ be the point at which $u(t)$ achieves it maximum. Our next step is to obtain an upper bound for $\sigma$. Let $\alpha=1-\sigma$. We need a lower bound on $\alpha$. Integrating (4.5) over $[\sigma, 1]$, we get

$$
\begin{equation*}
\int_{\sigma}^{1} t^{4}\left(u^{5}(t)+\lambda u(t)\right) d t=\frac{3 \kappa}{2} \tag{4.13}
\end{equation*}
$$

Because of concavity, $u(t)$ in $[\sigma, 1]$ lies entirely under its tangent line at the point $t=1$. The highest point of the tangent line in the interval is directly above $t=\sigma$. This point is in fact higher than the maximum point on $u(t)$ and is therefore an upper bound for $u(t), t \in([a, 3 / 2]$. In other words,

$$
\begin{equation*}
u(t) \leq\left(1+\frac{3 \alpha}{2}\right) \kappa, \quad t \in[a, 3 / 2] . \tag{4.14}
\end{equation*}
$$

Substituting this into (4.13) gives the inequality

$$
\begin{equation*}
\frac{1}{5}\left[1-(1-\alpha)^{5}\right]\left(1+\frac{3 \alpha}{2}\right)\left[\left(1+\frac{3 \alpha}{2}\right)^{4} \kappa^{4}+\lambda\right] \geq \frac{3}{2} \tag{4.15}
\end{equation*}
$$

Using (4.12) in estimating the expression in the second pair of square brackets (and ignoring the negative term involving $\lambda$ ), we have

$$
\begin{equation*}
\frac{1}{5}\left[1-(1-\alpha)^{5}\right]\left(1+\frac{3 \alpha}{2}\right)^{5}\left[\frac{12320}{2281}\right]-\frac{3}{2} \geq 0 \tag{4.16}
\end{equation*}
$$

The lefthand side is a polynomial in $\alpha$. MAPLE has a command to find exact upper and lower bounds for all the real roots of a given polynomial, with specified accuracy. For the above polynomial, MAPLE found two real roots, a negative one and a positive one larger than 18475/131072. We repeat the fact that the bound is exact, not just a numerical approximation. Inequality (4.16) therefore implies that

$$
\begin{equation*}
\alpha>\frac{18475}{131072} \tag{4.17}
\end{equation*}
$$

By integrating twice the differential equation over $[a, \sigma]$, we get the inequality

$$
\begin{equation*}
\int_{a}^{\sigma}(t-a) t^{4}\left(u^{4}(\sigma)+\lambda\right) d t \geq 1 \tag{4.18}
\end{equation*}
$$

This is a special case of a generalization of the well-known Lyapunov inequality for disfocality in oscillation theory; see [8]. Inequalities (4.14) and (4.18) yield

$$
\begin{equation*}
\left[\left(1+\frac{3 \alpha}{2}\right)^{4} \kappa^{4}+\lambda\right] \int_{a}^{\sigma}(t-a) t^{4} d t \geq 1 \tag{4.19}
\end{equation*}
$$

Replacing $a$ by 0 will increase only the lefthand side; so we have

$$
\begin{equation*}
\frac{(1-\alpha)^{6}}{6}\left[\left(1+\frac{3 \alpha}{2}\right)^{4} \kappa^{4}+\lambda\right] \geq 1 \tag{4.20}
\end{equation*}
$$

Moving the term involving $\lambda$ to the righthand side and using inequality (4.17), we have

$$
\begin{equation*}
(1-\alpha)^{6}\left(1+\frac{3 \alpha}{2}\right)^{4} \kappa^{4} \geq 6-\left(\frac{112597}{131072}\right)^{6} \lambda \tag{4.21}
\end{equation*}
$$

The coefficient of the lefthand side is an increasing function of $\alpha$. It therefore is not larger than that obtained by substituting the righthand side of (4.17) for $\alpha$. Dividing (4.21) by this coefficient, we obtain the inequality

$$
\begin{equation*}
\kappa^{4}>6.93-0.465 \lambda \tag{4.22}
\end{equation*}
$$

which contradicts (4.12). The proof is now complete.

It is interesting to see how far this proof can be extended to cover more general powers. We conclude by posing the challenge:

Challenge: Classify completely according to uniqueness or nonuniqueness of (BVP1) or (BVP2) on all intervals [a,b], $0<a<b$, for equations of the form

$$
\begin{equation*}
u^{\prime \prime}(t)+t^{\gamma} u^{p}(t)+t^{\delta} u^{q}(t)=0, \quad \text { on }(a, b) \tag{4.23}
\end{equation*}
$$

in terms of the values of $\gamma, \delta \in(-\infty, \infty)$ and $p, q \geq 1$.

Results in this paper have provided some partial answers.

## APPENDIX. The MAPLE Program

The program that leads to the results in Section 3 is contained in a file called $u Q$ and has the following lines of MAPLE instructions. Line numbers are added for reference and are not part of the file.

```
##############################################
f(u):=t^h*u^q;
# u1 is du/dt and u2 is the second derivative
#
u2 := -f(u);
# df(z) is the chain rule
df := proc (z) diff(z,t)+diff(z,u)*u1+diff(z,u1)*u2 end;
#
dr := proc (RR,vv) dR:=df(RR)/df(vv);
    dR:=expand(subs(u1=solve(R=RR,u1),u=solve(v=vv,u),dR));
    normal(dR); end;
el := proc (ex,eq,x) subs(x=solve(eq,x),ex); end;
#
RR:=u1; a:=(g+2)/(p-1); vv:=t`a*u;
#
rDR:=dr(RR,vv);
DrDR:=normal(diff(",t));
nDrDR:=numer(");
eq:=numer(df(vv))=-P;
subs(u=solve(vv=v,u),");
nDrDR:=normal(expand(el("'"',el(",R=RR,u1),R)));
#
k:=op(")[1];
K0:=coeff(",P,0);
K1:=coeff(k,P,1);
K0:=factor(KO);
K1:=factor(K1);
```

\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#\#

Line 1 defines the nonlinear term, and can be changed if a different equation is studied. As pointed out in Section 3, we have to keep track of only one typical term in $F(t, u)$. In this case it is $\mathrm{t}^{\mathrm{h}} \mathrm{q}$. We have used h and q instead of $\gamma_{i}$ and $p_{i}$ for convenience. Lines that start with \# are comment or separator lines. Line 4 is simply the differential equation $u^{\prime \prime}+f(u)=0$, being solved for $u^{\prime \prime}$.

Line 6 defines a procedure with the name df that takes an argument $\mathbf{z}$. In the definition, $\boldsymbol{z}$ is a dummy variable, so that an actual invocation of the procedure should be $d f$ (expression in $t$ and $u$ ). It uses the chain rule and the differential equation to find the total derivative of the expression with respect to $t$. Lines 8 to 10 define the procedure that gives the righthand side of (3.10). It takes two arguments RR and vv that correspond to $R$ and $v$, respectively. The last part of line 8 simply defines the local variable dR as $\frac{d \mathrm{RR}}{d t} / \frac{d \mathrm{vv}}{d t}$. What line 9 does is to first eliminate in $d R$ the variables $u 1$ and $u$ in favor of $R$ and $v$, and then multiply everything out. The normal command in line 10 is used to combine all the pieces obtained in line 9 into one single fraction in the lowest possible reduced form. Line 11 defines the procedure of eliminating from a given expression (ex) a given variable ( x ) according to a given equation (eq that contains x ).

Then comes the actual computation. Line 13 inputs our choices of $\mathrm{RR}=$ $R=u^{\prime}=\mathrm{u} 1, \mathrm{a}=\alpha=(\gamma+2) /(p-1)=(\mathrm{g}+2) /(\mathrm{p}-1)$, and $\mathrm{vv}=v=$ $t^{\alpha} u=\mathrm{t} \mathbf{a} * \mathrm{u}$. Line 15 invokes dr to find the righthand side of (3.10). Line 16 finds the derivative of the answer from the previous line and simplifies it; the symbol " is a convenient abbreviation for the previous answer. Line 17 takes the numerator of the derivative. Line 18 defines the equation given by (3.13). Lines 19 and 20 carry out the task of solving the equation for $R$ and substituting into the numerator of the derivative to obtain (3.14). The answer from line 20 is an expression consisting of a long expression multiplied by a power of $v$ and a power of $t$. The operator op in line 22 extracts the long expression. The next two lines separate the expression into two parts, K0 that contains no $P$ and K1 that does. The last two lines factor these two parts.

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