# THE INFLUENCE OF NONLOCAL NONLINEARITIES ON THE LONG-TIME BEHAVIOR OF SOLUTIONS OF BURGERS' EQUATION 

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#### Abstract

We study the long-time behavior of solutions of Burgers' equation with nonlocal nonlinearities $u_{t}=u_{x x}+\varepsilon u u_{x}+\frac{1}{2}\left(a\|u(\cdot, t)\|^{p-1}+b\right) u$, $0<x<1, a, \varepsilon \in, b>0, p>1$, subject to $u(0, t)=u(1, t)=0$. A stability-instability analysis is given in some detail, and some finitetime blow-up results are given.


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## 1 Introduction

In this paper we consider the following initial-boundary value problem:

$$
\begin{cases}u_{t}=u_{x x}+\varepsilon u u_{x}+\frac{1}{2}\left(a\|u(\cdot, t)\|^{p-1}+b\right) u, & 0<x<1,  \tag{P}\\ u(0, t)=u(1, t)=0, & t>0, \\ u(x, 0)=u_{0}(x), & 0, \leq x \leq 1 .\end{cases}
$$

Here $a, b, \varepsilon$, and $p$ are given constants, with $\varepsilon \geq 0$ (without loss of generality), $b>0$, and $p>1$; and $u_{0}(\cdot)$ is a continuous function with $u_{0}(0)=u_{0}(1)=0$. Moreover, we take

$$
\|u(\cdot, t)\|=\left(\int_{0}^{1}|u(x, t)|^{q} d x\right)^{1 / q}
$$

In this paper, we require that $u_{0} \geq 0$. Then by the maximum principle, $u(\cdot, t) \geq 0$ for all $t$ in the existence interval.

Our interest in (P) is twofold. First, (P) is closely related to a onedimensional turbulence model proposed by Burgers ([2], [3]) and studied by Horgan and Olmstead [8]. (See also Drazin and Reid [5].) The major difference between ( P ) and the earlier model is the use of the $L^{q}$ norm in (P) rather than the $L^{2}$ norm. A number of authors ([1], [6], [8], [12]) have investigated nonlocal problems as models for local problems; they also restricted their attention to the case $q=2$. To the best of our knowledge, no one has considered problems in which a convective term ( $\varepsilon u u_{x}$ ) is present. Yet, convective terms have a remarkable effect on the dynamical behavior of solutions of equations. For example, consider

$$
\left\{\begin{array}{lll}
u_{t}=u_{x x}+\varepsilon u u_{x}+\frac{1}{2} b u, & 0<x<1, & t>0 \\
u(0, t)=u(1, t)=0, & & t>0 \\
u(x, 0)=u_{0}(x) \geq 0, & 0 \leq x \leq 1 . &
\end{array}\right.
$$

When $\varepsilon=0$ and $b>2 \pi^{2}$, this problem possesses the exponentially growing solution $e^{\left(b / 2-\pi^{2}\right) t} \sin \pi x$, whereas for $\varepsilon>0$, all solutions are bounded for any $b$ (since a supersolution of the form $M(1-\alpha x)$ with large $M$ and
$\alpha \in(0,1)$ exists). We shall see that this phenomenon persists, although to a somewhat less pronounced effect, for ( P ).

The second reason for our interest in ( P ) is that (with $b=0$ ) it is closely related to the same initial-boundary value problem for the equation

$$
\begin{equation*}
u_{t}=u_{x x}+\varepsilon u u_{x}+a|u|^{p-1} u, \tag{1.1}
\end{equation*}
$$

where the nonlocal nonlinearity is replaced by the more standard local term. This problem was studied extensively in [4] and [11], where the stabilizing effect of convective terms was noted. The study of (P) was taken up with the objective of obtaining analogous results for a closely related problem. It turns out that the results we derived for ( P ) are more complete than those for (1.1) obtained in [4] and [11].

In [14], Straughan et al. consider, from a computational point of view, the same initial-boundary value problem for

$$
L u=u_{t}+2 u u_{x}-R^{-1} u_{x x},
$$

where

$$
L u= \begin{cases}u-R u\|u\|^{2}, & \text { or } \\ u, & \text { or } \\ u+R u\|u\| . & \end{cases}
$$

Here $\|\cdot\|$ denotes the $L^{2}$ norm, and $R>0$ can be thought of as the Reynolds number. Under appropriate scaling, this problem is included in ( P ) when $q=2$. In particular, we are able to verify theoretically for ( P ) all except one of the numerical observations made in [14].

The plan of our paper is as follows. In Section 2, we state the results in the absence of convection. In Section 3, we characterize the set of stationary solutions of $(\mathrm{P})$ when $\varepsilon>0$. In Section 4, we discuss the long-time behavior of solutions of $(\mathrm{P})$, including stability, asymptotic stability, global existence, and nonexistence. The necessary local existence theorems and comparison theorems are discussed in Appendix A. These are standard but we could not find any reference to such results for nonlocal problems of the type considered here. In Appendix B, we gave the proof of a technical inequality, namely, that of Lemma 3.2.

## 2 Discussion for the Case $\varepsilon=0$

In this section, we consider the following problem:

$$
\left\{\begin{array}{lll}
u_{t}=u_{x x}+\frac{1}{2}\left(a\|u(\cdot, t)\|^{p-1}+b\right) u, & 0<x<1, & t>0  \tag{0}\\
u(0, t)=u(1, t)=0, & & t>0 \\
u(x, 0)=u_{0}(x), & 0 \leq x \leq 1
\end{array}\right.
$$

Because ( $\mathrm{P}^{0}$ ) is an explicitly resolvable problem, we can easily examine the questions of stability, global existence, and nonexistence of nonnegative solutions.

First, for the stationary solutions of $\left(\mathrm{P}^{0}\right)$, we need to solve

$$
\left\{\begin{array}{l}
v^{\prime \prime}+\frac{1}{2}\left(a\|v\|^{p-1}+b\right) v=0, \quad 0<x<1  \tag{0}\\
v(0)=v(1)=0
\end{array}\right.
$$

The nonnegative solutions of ( $\mathrm{S}^{0}$ ) must be in the form $v(x)=c_{1} \varphi_{1}(x)=$ $c_{1} \sin \pi x$. Substituting such $v$ in the equation, we find

$$
\begin{equation*}
-\pi^{2}+\frac{1}{2}\left(a c_{1}^{p-1}\left\|\varphi_{1}\right\|^{p-1}+b\right)=0 \tag{2.1}
\end{equation*}
$$

Thus, for $a>0$, there is no positive solution when $b \geq 2 \pi^{2}$ and one positive solution when $b<2 \pi^{2}$. For $a<0$, there exists a unique positive solution when $b>2 \pi^{2}$ but none when $b \leq 2 \pi^{2}$.

Next, for $\left(\mathrm{P}^{0}\right)$, we seek solutions of the form

$$
\begin{equation*}
u(x, t)=\sum_{n=1}^{\infty} a_{n}(t) \varphi_{n}(x) \tag{2.2}
\end{equation*}
$$

where $\varphi_{n}(x)=\sin n \pi x$. Let $f(t)=\|u(\cdot, t)\|^{p-1}=\left(\int_{0}^{1}\left|\sum_{n=1}^{\infty} a_{n} \varphi_{n}\right|^{q} d x\right)^{\frac{p-1}{q}}$.
Through a straightforward computation, we have

$$
\begin{equation*}
f(t)=(h(t))^{p-1} e^{\frac{a(p-1)}{2} \int_{0}^{t} f(\eta) d \eta} \tag{2.3}
\end{equation*}
$$

where

$$
h(t)=e^{\frac{b}{2} t}\left(\int_{0}^{1}\left|\sum_{n=1}^{\infty} a_{n}(0) e^{-n^{2} \pi^{2} t} \varphi_{n}(x)\right|^{q} d x\right)^{\frac{1}{q}}
$$

From (2.3), after a quadrature, we can rewrite $f(t)$ as follows:

$$
\begin{equation*}
f(t)=(h(t))^{p-1} /\left[1-\frac{a(p-1)}{2} \int_{0}^{t}(h(\eta))^{p-1} d \eta\right] . \tag{2.4}
\end{equation*}
$$

Let

$$
\begin{equation*}
H(t)=\int_{0}^{t} h^{p-1}(\eta) d \eta \tag{2.5}
\end{equation*}
$$

If $b<2 \pi^{2}$, it follows that $H(t) \rightarrow H_{\infty}<\infty$ for some constant $H_{\infty}$ which is proportional to $\left|a_{N}(0)\right|$ where $N$ is the smallest integer such that $a_{n} \not \equiv 0$. Thus, if $a(p-1) H_{\infty}<2$, which will be the case if $a_{N}(0)$ is sufficiently small, then $f(t) \rightarrow 0$ as $t \rightarrow \infty$, and the trivial solution is stable in $L^{q}$. On the other hand, if $a(p-1) H_{\infty}>2$, then clearly $f(t) \rightarrow \infty$ in finite time, so some solutions of $\left(\mathrm{P}^{0}\right)$ blow up in finite time. Finally, if $a(p-1) H_{\infty}=2$, then $f(t)$ can have a finite, nonzero limit as $t \rightarrow \infty$. This situation occurs, for example, when $u$ is a stationary solution of $\left(\mathrm{P}^{0}\right)$.

To show that $v(x)$ is unstable, we look for solutions of $\left(\mathrm{P}^{0}\right)$ of the form

$$
u(x, t)=a_{1}(t) \varphi_{1}(x)=\alpha(t) v(x),
$$

where $\alpha(t)=a_{1}(t) / c_{1}$ and $\alpha(0) \neq 1$. Then, using ( $S^{0}$ ), we see that

$$
\alpha^{\prime}(t)=c^{2}\left(\alpha^{p-1}(t)-1\right) \alpha(t)
$$

where $c^{2}=\frac{1}{2} a\|v\|_{q}^{p-1}$. If $\alpha(0)<1$, then $\alpha^{\prime}(t)<0$ for small $t>0$ and consequently for all $t$ (since $p>1$ ). Therefore $\alpha(t)<\alpha(0)$ and $\alpha^{\prime}(t)<-c^{2}\left(1-\alpha^{p-1}(0)\right) \alpha(t)$, so $\alpha(t) \rightarrow 0$ as $t \rightarrow \infty$. If $\alpha(0)>1$, it is easy to see that $\alpha^{\prime}>0$ for all $t$ for which $\alpha(t)$ exists. It follows that $\alpha(t)$ blows up in finite time. Thus, for any $r>1$,

$$
\liminf _{t \rightarrow \infty}\|u(\cdot, t)-v\|_{r}>0
$$

and consequently $v$ is not stable (in any norm).
If $b \geq 2 \pi^{2}$, then zero is unstable. To see this, we note that if $a_{1}(0) \neq 0$, then, in view of the asymptotic behavior of $h(t)$ near $+\infty, H(t)$ grows either exponentially or linearly so that it passes $2 / a(p-1)$ in finite time. Consequently, whenever $\left(u_{0}, \varphi_{1}\right) \neq 0$, solutions of ( $\mathrm{P}^{0}$ ) blow up in finite time.

For the case $a<0$, we again have subcases. If $b<2 \pi^{2}$, zero is the only nonnegative stationary solution. From the form of $h(t)$ and the fact that $a<0$, we see that $h(t) \rightarrow 0$ as $t \rightarrow \infty$, and consequently $\|u(\cdot, t)\| \rightarrow 0$ also.

If $b=2 \pi^{2}$, then $h(t) \rightarrow\left|a_{1}(0)\right|\left\|\varphi_{1}\right\|$. If $\left|a_{1}(0)\right|=0$, then from (2.4), since $a<0$, we have $f(t) \rightarrow 0$ as $t \rightarrow \infty$. If $\left|a_{1}(0)\right| \neq 0$, then $H(t)$ behaves like $\left(\left|a_{1}(0)\right|\left\|\varphi_{1}\right\|\right)^{p-1} t$, so $f(t) \rightarrow 0$ in this case too. Therefore zero is globally asymptotically stable in this case also.

If $b>2 \pi^{2}$, we claim that zero is unstable and that the positive stationary solution, $v(x)$, is stable. For the first of these, let $u(x, t)=\varepsilon(t) v(x)$. Then, as before,

$$
\varepsilon^{\prime}(t)=c^{2} \varepsilon(t)\left(1-\varepsilon^{p-1}(t)\right)
$$

where $c^{2}=|a|\|v\|^{p-1}$. Thus, if $0<\varepsilon(0)<1, \varepsilon^{\prime}>0$ for small $t$, and hence for all $t$ for which $\varepsilon(0) \leq \varepsilon(t) \leq 1$. If follows that if $(0, T)$ is the largest interval on which $\varepsilon^{\prime}(t)>0$, then $\varepsilon(t) \rightarrow 1^{-}$as $t \nearrow T$ and $T=+\infty$. This establishes the instability of the null solution.

To show that $v(x)$ is stable for $a<0$ and $b>2 \pi^{2}$, we write

$$
\begin{equation*}
u(x, t)=a_{1}(t) \varphi_{1}(x)+\sum_{n=2}^{\infty} a_{n}(t) \varphi_{n}(x), \tag{2.6}
\end{equation*}
$$

where

$$
a_{1}(0)=(1+\delta) c_{1} \quad \text { and } \quad|\delta| \ll 1, \quad \delta \neq 0
$$

and where $v(x)=c_{1} \varphi_{1}(x)$ solves $\left(\mathrm{S}^{0}\right)$. Then a tedious, but routine, computation yields

$$
\begin{equation*}
a_{n}(t)=a_{n}(0) e^{\left(-n^{2} \pi^{2}+\frac{1}{2} b\right) t}\left[1-\frac{a(p-1)}{2} H(t)\right]^{-\frac{1}{p-1}} \tag{2.7}
\end{equation*}
$$

and

$$
\begin{equation*}
u(x, t)-v(x)=\left(a_{1}(t) / c_{1}-1\right) v(x)+\sum_{n=2}^{\infty} a_{n}(t) \varphi_{n}(x) \tag{2.8}
\end{equation*}
$$

or

$$
\begin{align*}
u(x, t)-v(x) & =\left[(1+\delta) e^{\left(\frac{1}{2} b-\pi^{2}\right) t}\left[1-\frac{a(p-1)}{2} H(t)\right]^{-\frac{1}{p-1}}-1\right] v(x) \\
& +e^{\left(\frac{1}{2} b-\pi\right) t}\left[1-\frac{a(p-1)}{2} H(t)\right]^{-\frac{1}{p-1}} \sum_{n=2}^{\infty} a_{n}(0) e^{-\left(n^{2}-1\right) \pi^{2} t} \varphi_{n}(x) . \tag{2.9}
\end{align*}
$$

Since $a_{1}(0) \neq 0$, we see that $h(t) \sim(1+\delta)\|v\| e^{\left(\frac{1}{2} b-\pi^{2}\right) t}$ as $t \rightarrow \infty$. This, together with (2.1), shows us that

$$
\begin{aligned}
{\left[1-\frac{1}{2} a(p-1) H(t)\right]^{\frac{1}{p-1}} } & \sim\left(\frac{-a\|v\|^{p-1}}{b-2 \pi^{2}}\right)^{\frac{1}{p-1}}(1+\delta) e^{\left(\frac{b}{2}-\pi^{2}\right) t} \\
& \sim(1+\delta) e^{\left(\frac{b}{2}-\pi^{2}\right) t}
\end{aligned}
$$

as $t \rightarrow+\infty$. Using these estimates in (2.9) and the triangle inequality, we see that

$$
\begin{aligned}
\|u(\cdot, t)-v\|_{q} & \leq \varepsilon_{1}(t)\|v\|_{q} \\
+ & e^{-\pi^{2} t}\left(1-\varepsilon_{2}(t)\right)\left[\int_{0}^{1}\left|\sum_{n=2}^{\infty} a_{n}(0) e^{-\left(n^{2}-2\right) \pi^{2} t} \varphi_{n}(x)\right|^{q} d x\right]^{1 / q}
\end{aligned}
$$

where $\varepsilon_{1}(t), \varepsilon_{2}(t) \rightarrow 0$ as $t \rightarrow \infty$. Consequently $\|u(\cdot, t)-v\|_{q} \rightarrow 0$ as $t \rightarrow+\infty$.

## 3 The Stationary Solutions When $\varepsilon>0$

If we write $u=\sigma v$ with $\sigma>0$ such that $\sigma^{p-1}|a|=1$, and if we replace $\varepsilon$ by $\sigma \varepsilon$, we can reduce ( P ) to an equivalent problem for $v$ with $a=1$ or $a=-1$. For this reason, we seek all positive solutions of

$$
\left\{\begin{array}{l}
w_{x x}+\varepsilon w w_{x}+\frac{1}{2}\left(\delta\|w\|^{p-1}+b\right) w=0, \quad 0<x<1  \tag{S}\\
w(0)=w(1)=0
\end{array}\right.
$$

where $\delta=1$ or $\delta=-1$. We let

$$
\|w\|=\|w\|_{q}=\left(\int_{0}^{1} w^{q} d x\right)^{1 / q}
$$

and set

$$
\begin{equation*}
y=\int_{0}^{x} w(\eta) d \eta, \quad Y=\int_{0}^{1} w(\eta) d \eta \tag{3.1}
\end{equation*}
$$

Under this change of variable, we have

$$
\frac{d}{d x}=w \frac{d}{d y}
$$

Thus, in lieu of (S), with $v(y)=w^{2}(x(y))$, we find

$$
\begin{gather*}
v_{y y}+\varepsilon v_{y}+\delta Y_{q}^{p-1}+b=0  \tag{3.2a}\\
v(0)=v(Y)=0 \tag{3.2b}
\end{gather*}
$$

where $Y_{q}=\|w\|_{q}$ and $Y_{1}=Y$. The positive solutions of (3.2a,b) are given by

$$
\begin{equation*}
v(y, Y)=\varepsilon^{-1} Y\left(\delta Y_{q}^{p-1}+b\right) J(\varepsilon y, \varepsilon Y) \tag{3.3a}
\end{equation*}
$$

where

$$
\begin{equation*}
J(y, z)=\frac{1-e^{-y}}{1-e^{-z}}-\frac{y}{z} \tag{3.3b}
\end{equation*}
$$

It follows that $\delta Y_{q}^{p-1}+b>0$, since $v=w^{2}$.
The values of $Y, Y_{q}$ are then determined by solving the nonlinear system

$$
\begin{equation*}
1=\int_{0}^{1} d x=\int_{0}^{Y} \frac{d y}{\sqrt{v(y)}} \tag{3.4a}
\end{equation*}
$$

$$
\begin{equation*}
Y_{q}^{q}=\int_{0}^{1} w^{q}(x) d x=\int_{0}^{Y} v^{\frac{1}{2}(q-1)}(y) d y . \tag{3.4b}
\end{equation*}
$$

We define, for $s \geq 0, z>0$,

$$
H_{s}(z)=\int_{0}^{1}[J(\sigma z, z)]^{\frac{1}{2}(s-1)} d \sigma .
$$

A routine calculation shows that with

$$
\begin{align*}
& Z_{q}=\varepsilon Y_{q},  \tag{3.5}\\
& Z_{1}=Z=\varepsilon Y,
\end{align*}
$$

we have from (3.4a,b), for $1 \leq q<\infty$,

$$
\begin{equation*}
Z_{q} \equiv Z_{q}(Z)=Z\left[H_{q}(Z) H_{0}^{q-1}(Z)\right]^{1 / q} \quad(Z \geq 0) \tag{3.6}
\end{equation*}
$$

while

$$
\begin{equation*}
b=Z H_{0}^{2}(Z)-\delta \varepsilon^{-(p-1)} Z_{q}^{p-1}(Z)=b(Z) \tag{3.7}
\end{equation*}
$$

When $q=+\infty$,

$$
Z_{q}(Z)=Z^{-1 / 2}\left[\frac{e^{-Z}-(1-Z)}{1-e^{-Z}}+\ln \left(\frac{1-e^{-Z}}{Z}\right)\right]^{1 / 2} H_{0}(Z)
$$

Useful asymptotic formulas are, for $1 \leq q<\infty$,

$$
\frac{Z_{q}(Z)}{Z}= \begin{cases}2 /(1+q)^{1 / q}, & Z \rightarrow \infty \\ {\left[\pi^{q-1} \Gamma^{2}\left(\frac{1}{2}(1+q)\right) / \Gamma(1+q)\right]^{1 / q},} & Z \rightarrow 0^{+}\end{cases}
$$

and, for $q=\infty$,

$$
\frac{Z_{\infty}(Z)}{Z}= \begin{cases}2, & Z \rightarrow \infty \\ \pi / 2, & Z \rightarrow 0^{+}\end{cases}
$$

We have plotted $Z_{q}(Z)$ and $Z_{q}(Z) / Z$ against $Z$ for various $q$ in Figure 1 and Figure 2, respectively.

For fixed $b$, the cardinality of the set of stationary solutions turns out to be the same as the cardinality of the set of solutions of (3.7). To see this, we need some lemmas.

Lemma 3.1. For any $q \geq 1, p>1$, all the positive solutions of ( S ) are concave.

Proof. The proof follows immediately from (3.3a,b). We have

$$
w_{x x}=w \frac{d}{d y}\left(w \frac{d w}{d y}\right)=\frac{1}{2} \sqrt{v(y)} v_{y y}(y) .
$$

Clearly, from (3.3a,b) $v_{y y}<0$.
This establishes the concavity of the steady states found in [14] numerically.

Lemma 3.2. The function $h(Z):=Z H_{0}^{2}(Z)(Z>0)$ is strictly increasing.
The proof of this lemma is long and is therefore included in Appendix B.

Lemma 3.3. The function $h_{q}(Z):=Z^{q} H_{q}(Z) H_{0}^{q-1}(Z)$ is strictly increasing for $q \geq 1$.

Proof. We can write

$$
h_{q}(Z)=Z^{\frac{1}{2}(q+1)} H_{q}(Z)(h(Z))^{\frac{1}{2}(q-1)} .
$$

Therefore, by Lemma 3.2, it suffices to show that $H_{q}(Z)$ is strictly increasing. This in turn will hold if

$$
\frac{d}{d z} J(\sigma z, z)=\left[\sigma e^{-\sigma z}\left(1-e^{-z}\right)-e^{-z}\left(1-e^{-\sigma z}\right)\right] /\left(1-e^{-z}\right)^{2} \geq 0 .
$$

If we write $x=e^{-z}$, then $x \in(0,1]$, and

$$
G(\sigma, x) \equiv \frac{d}{d z} J(\sigma z, z)=\left[\sigma x^{\sigma}-\frac{x\left(1-x^{\sigma}\right)}{1-x}\right] /(1-x) .
$$

In view of the convexity of $1-x^{\sigma}$, we find

$$
\sigma x^{\sigma-1} \geq \frac{1-x^{\sigma}}{1-x} .
$$

Hence $G(\sigma, x) \geq 0$, and the lemma is true.

Lemma 3.4. Let $w_{1}$, $w_{2}$ be two nonnegative stationary solutions of ( P ) with $\left\|w_{1}\right\|_{1} \leq\left\|w_{2}\right\|_{1}$. Then either $w_{1}<w_{2}$ on $(0,1)$ or $w_{1} \equiv w_{2}$.

Proof. Any such solution can be written in the form

$$
\begin{aligned}
\varepsilon^{2} v\left(z, \frac{Z}{\varepsilon}\right) & =Z\left(\delta \varepsilon^{-(p-1)} Z_{q}^{p-1}(Z)+b\right) J(z, Z) \\
& =Z h(Z) J(z, Z)
\end{aligned}
$$

where $z=\varepsilon y, Z=\varepsilon\|w\|_{1}$ and $v(y)=w^{2}(x)$, in view of (3.7).
A straightforward computation shows that

$$
\begin{aligned}
\varepsilon^{2} \frac{\partial v}{\partial Z}\left(z, \frac{Z}{\varepsilon}\right) & =\varepsilon v, 2\left(z, \frac{Z}{\varepsilon}\right) \\
& =h^{\prime}(Z) Z J(z, Z)+h(Z) \frac{\partial}{\partial Z}(Z J(z, Z)) \\
& =h^{\prime}(Z) Z J(z, Z)+\frac{h(Z)\left(1-e^{-z}\right)\left(1-e^{-Z}-Z e^{-Z}\right)}{\left(1-e^{-Z}\right)^{2}} \\
& >0 .
\end{aligned}
$$

Thus, if $Y_{i}=\left\|w_{i}\right\|_{1}$, we have $Y_{1} \leq Y_{2}$ and

$$
v\left(y, Y_{1}\right) \leq v\left(y, Y_{2}\right)
$$

We need to show that if $w_{1} \not \equiv w_{2}$, then $w_{1}<w_{2}$ on $(0,1)$. Note that $w_{i}^{2}(x)=v\left(y_{i}, Y_{i}\right)$, and $\frac{d y_{i}}{d x}=w_{i}(x)$ for $i=1,2$. Thus, for any $x \in(0,1)$,

$$
x=\int_{0}^{y_{1}(x)}\left(v\left(\sigma, Y_{1}\right)\right)^{-1 / 2} d \sigma=\int_{0}^{y_{2}(x)}\left(v\left(\sigma, Y_{2}\right)\right)^{-1 / 2} d \sigma,
$$

where

$$
y_{i}(x)=\int_{0}^{x} w_{i}(\eta) d \eta
$$

We see that if $Y_{1}=Y_{2}$, then $y_{1}(x) \equiv y_{2}(x)$, so that $w_{1}(x) \equiv w_{2}(x)$. Hence, we may assume $Y_{1}<Y_{2}$.

Since $v\left(\sigma, Y_{1}\right)<v\left(\sigma, Y_{2}\right)$, if $\sigma>0$, we must have $y_{1}(x)<y_{2}(x)$ on $(0,1]$. Let $\bar{x}$ denote the unique $x$ in $(0,1)$ where $w_{2}^{\prime}(\bar{x})=0$ (Lemma 3.1). Then for
$x \in(0, \bar{x}), y_{2}(x)<y_{2}(\bar{x})$. Now

$$
\begin{aligned}
& v\left(y_{2}(x), Y_{2}\right)-v\left(y_{1}(x), Y_{1}\right) \\
& \quad=v\left(y_{2}(x), Y_{2}\right)-v\left(y_{1}(x), Y_{2}\right)+v\left(y_{1}(x), Y_{2}\right)-v\left(y_{1}(x), Y_{1}\right) .
\end{aligned}
$$

Since $v\left(\cdot, Y_{2}\right)$ is increasing on $\left(0, y_{2}(\bar{x})\right)$ and $v, 2>0$, the differences on the right are positive on $(0, \bar{x}]$. Thus $w_{1}<w_{2}$ on $(0, \bar{x}]$.

A similar argument using the change of variable

$$
y=\int_{x}^{1} w(\eta) d \eta
$$

yields (3.2a) with $\varepsilon$ replaced by $-\varepsilon$ and, in place of (3.3a),

$$
\left.v(y, Y)=\varepsilon^{-1}\left(\delta Y_{q}^{p-1}+b\right)\left[y-\frac{Y\left(e^{\varepsilon y}-1\right)}{\left(e^{\varepsilon Y}\right.}-1\right)\right] .
$$

Again, one finds that $v_{, 2}>0$ and ultimately that $w_{1}<w_{2}$ if $1-x<1-\bar{x}$ or $x>\bar{x}$.

These lemmas allow us to conclude the following theorem.

Theorem 3.5. Let $w(x)$ be a nonnegative solution of (S). Then, if $Z=$ $\varepsilon\|w\|_{1}, Z$ solves (3.7). Conversely, if $Z>0$ is a solution of (3.7), then $w(x)=(v(y))^{1 / 2}$ solves $(\mathrm{S})$ with $d x / d y=(v(y, Y))^{-1 / 2}, Y=Z / \varepsilon$, and $v$ is given by (3.3a,b).

We next count the solutions of (3.7) for fixed $\varepsilon>0$. Let $Z(b)$ be a branch of solutions of (2.7). By the implicit function theorem, as long as

$$
Q(Z):=\frac{\partial}{\partial Z}\left[Z H_{0}^{2}(Z)-\delta \varepsilon^{-(p-1)} Z_{q}^{p-1}(Z)\right] \neq 0
$$

on this branch, $Z$ will be a $C^{1}$ function of $b$ and

$$
Z^{\prime}(b) Q(Z(b))=1
$$

For fixed $b$, the number of solutions of (3.7) is then the same as the number of sign changes of $Q(Z)$ on $(0, \infty)$. However, the sign analysis of $Q$ is
complicated when $\delta>0$, since, in principle, a function of bounded variation (the difference of two increasing functions) can have infinitely many zeros.

The case $\delta<0$ is easy to treat. The function $b(Z)$ given by (3.7) is strictly increasing for $Z \geq 0$ and has the range $\left[2 \pi^{2}, \infty\right)$. Therefore, for each $b>2 \pi^{2}$, there is exactly one positive solution of (S), and there is no nontrivial solution for $b \leq 2 \pi^{2}$.

We observe that $b(Z)$ has the following asymptotic properties:

$$
b(Z)=\left\{\begin{array}{lll}
4 Z-\delta \varepsilon^{-(p-1)} \varepsilon_{\infty}^{p-1} Z^{p-1} & Z \rightarrow \infty, & 1 \leq q \leq \infty, \\
2 \pi^{2}-\delta \varepsilon^{-(p-1)} \varepsilon_{0}^{p-1} Z^{p-1} & Z \rightarrow 0^{+}, & 1 \leq q \leq \infty,
\end{array}\right.
$$

where

$$
\varepsilon_{\infty}= \begin{cases}2(1+q)^{-1 / q} & 1 \leq q<\infty \\ 2 & q=\infty\end{cases}
$$

and

$$
\varepsilon_{0}= \begin{cases}{\left[\pi^{q-1} \Gamma^{2}\left(\frac{1}{2}(1+q)\right) / \Gamma(1+q)\right]^{1 / q}} & 1 \leq q<\infty \\ \frac{\pi}{2} & q=\infty\end{cases}
$$

These asymptotic formulas, together with the numerical results of Figures $4-11$, allow us to assert the following for $\delta>0$ :
( $\mathrm{N}-\mathbf{1}$ ) If $1<p<2$, there is $b(p, q, \varepsilon)<2 \pi^{2}$ such that
(a) if $b<b(p, q, \varepsilon)$, there are no positive stationary solutions;
(b) if $b=b(p, q, \varepsilon)$ or $b>2 \pi^{2}$, there is exactly one solution; and
(c) if $b(p, q, \varepsilon)<b<2 \pi^{2}$, there are exactly two solutions.
( $\mathrm{N}-2$ ) If $p=2$, we have the following:
(a) If $\varepsilon<\frac{1}{4} \varepsilon_{\infty}$, there is exactly one positive stationary solution for each $b<2 \pi^{2}$, none for $b \geq 2 \pi^{2}$.
(b) If $\varepsilon=\frac{1}{4} \varepsilon_{\infty}$, there is $b\left(\varepsilon_{\infty}\right)<2 \pi^{2}$ such that
(i) if $b \leq b\left(\varepsilon_{\infty}\right)$ or $b>2 \pi^{2}$, there are no positive stationary solutions; and
(ii) if $b\left(\varepsilon_{\infty}\right)<b<2 \pi^{2}$, there is exactly one solution.
(c) If $\varepsilon>\frac{1}{4} \varepsilon_{\infty}$, there is $b\left(\varepsilon_{\infty}\right)<2 \pi^{2}$ such that
(i) if $b<b\left(\varepsilon_{\infty}\right)$, there are no positive stationary solutions;
(ii) if $b=b\left(\varepsilon_{\infty}\right)$ or $b>2 \pi^{2}$, there is exactly one solution; and
(iii) if $b\left(\varepsilon_{\infty}\right)<b<2 \pi^{2}$, there are exactly two solutions.
$(\mathrm{N}-\mathbf{3})$ If $p>2$, there is a critical number $\varepsilon_{1}(p, q)$ such that
(a) if $\varepsilon \leq \varepsilon_{1}$, we have the following:
(i) If $b \geq 2 \pi^{2}$, there are no positive stationary solutions.
(ii) If $b<2 \pi^{2}$, there is exactly one solution.
(b) if $\varepsilon>\varepsilon_{1}$, there is $b(p, q, \varepsilon)>2 \pi^{2}$ such that
(i) if $b>b(p, q, \varepsilon)$, there are no positive stationary solutions;
(ii) if $b=b(p, q, \varepsilon)$ or $b<2 \pi^{2}$, there is exactly one solution; and (iii) if $2 \pi^{2}<b<b(p, q, \varepsilon)$, there are exactly two solutions.

A word of explanation about the figures is in order. In Figures 4-6, we have set $q=2$ and chosen $p=1.5,2$, and 3 , respectively, plotting the solution set of (2.7) for various $\varepsilon$. The assertions above are based on these figures, the asymptotic formulas following (3.7), and (H). In Figures 7-11, we have fixed $\varepsilon$ and $p$ and plotted the solution sets as functions of $Q$.

## 4 Stability and Global Nonexistence When $\varepsilon>0$

We next consider the following problem equivalent to ( P ):

$$
\left\{\begin{array}{lll}
u_{t}=u_{x x}+\varepsilon u u_{x}+\frac{1}{2}\left(\delta\|u(\cdot, t)\|^{p-1}+b\right) u, & 0<x<1, & 0<t<T, \\
u(0, t)=u(1, t)=0, & 0<t<T, \\
u(x, 0)=u_{0}(x), & 0 \leq x \leq 1 .
\end{array}\right.
$$

For simplicity, we let $D_{T}=(0,1) \times(0, T)$ and $D_{T} \cup \Gamma_{T}=[0,1] \times[0, T)$. Our primary interest is in the stability properties of the steady states and in the asymptotic behavior of solutions of $\left(\mathrm{P}^{\prime}\right)$ for a given initial datum $u_{0}$. To pursue this interest, we first establish a relationship between solutions of ( $\mathrm{P}^{\prime}$ ) and those of ( S ).

Lemma 4.1. If $u$ is a bounded monotone (in time) solution of $\left(\mathrm{P}^{\prime}\right)$, then $u$ tends to a solution of $(\mathrm{S})$ as $t \rightarrow \infty$.

Proof. First, we note that such a solution must be global in time, by the continuation statement in Appendix A. Suppose that $\lim _{t \rightarrow \infty} u(x, t)=\varphi(x)$, and let

$$
\begin{equation*}
F(x, t)=\int_{0}^{1} G(x, y) u(y, t) d y \tag{4.1}
\end{equation*}
$$

where

$$
G(x, y)= \begin{cases}x(1-y), & 0 \leq x \leq y \leq 1 \\ y(1-x), & 0 \leq y \leq x \leq 1\end{cases}
$$

is the Green's function for $-\frac{d^{2}}{d y^{2}}$ with Dirichlet boundary conditions.

Under the assumptions for $u, F$ is bounded in $[0,1] \times[0, \infty)$ and

$$
\begin{align*}
F_{t}(x, t)= & \int_{0}^{1} G(x, y) u_{t}(y, t) d y \\
= & -u(x, t)-\frac{\varepsilon}{2} \int_{0}^{x} u^{2}(y, t) d y+\frac{\varepsilon}{2} x \int_{0}^{1} u^{2}(y, t) d y \\
& +\frac{1}{2}\left(\delta\|u(\cdot, t)\|^{p-1}+b\right) \int_{0}^{1} G(x, y) u(y, t) d y  \tag{4.2}\\
\rightarrow & -\varphi(x)-\frac{\varepsilon}{2} \int_{0}^{x} \varphi^{2}(y) d y+\frac{\varepsilon}{2} x \int_{0}^{1} \varphi^{2}(y) d y \\
& +\frac{1}{2}\left(\delta\|\varphi\|^{p-1}+b\right) \int_{0}^{1} G(x, y) \varphi(y) d y
\end{align*}
$$

as $t \rightarrow \infty$. This limit has a constant sign that depends on whether $u_{t} \geq 0$ or $u_{t} \leq 0$. In actual fact, the limit is zero for $x \in[0,1]$; otherwise $F$ would not have a finite limit as $t \rightarrow \infty$. Therefore

$$
\begin{aligned}
\varphi(x)= & -\frac{\varepsilon}{2} \int_{0}^{x} \varphi^{2}(y) d y+\frac{\varepsilon}{2} x \int_{0}^{1} \varphi^{2}(y) d y \\
& +\frac{1}{2}\left(\delta\|\varphi\|^{p-1}+b\right) \int_{0}^{1} G(x, y) \varphi(y) d y
\end{aligned}
$$

and hence $\varphi$ is a solution of (S).

By means of this lemma, we can obtain a complete result for stability and instability of stationary solutions of $\left(\mathrm{P}^{\prime}\right)$ with $\delta=1$. This time, we treat the solution of ( S ) as a function depending on the parameter $b$ and denote it by $w(x, b)$.

Theorem 4.2. Let $w(x, b)$ be a continuously differentiable positive solution of (S) with $\delta=1$ on some $b$ interval $[\alpha, \beta]$, and let $Z(b)$ be the corresponding solution of (3.7). Then if $Z^{\prime}(b)>0$ on $[\alpha, \beta]$, the solutions are stable, whereas they are unstable if $Z^{\prime}(b)<0$.

Proof. For the case $Z^{\prime}(b)>0$ we first show that $w\left(x, b_{1}\right)<w\left(x, b_{2}\right)$ on $(0,1)$ for $\alpha \leq b_{1}<b_{2} \leq \beta$.

From $Z^{\prime}(b)>0$, it follows that $Z\left(b_{1}\right)<Z\left(b_{2}\right)$ if $\alpha \leq b_{1}<b_{2} \leq \beta$. In view of $(3.7), Z(b)=\varepsilon\|w(\cdot, b)\|_{1}$ and $v(\varepsilon y, Z / \varepsilon, b)=w^{2}(x, b)$. Using the
form of $v(\varepsilon y, Z / \varepsilon, b)$ in Lemma 3.3, we see that

$$
\begin{aligned}
\varepsilon^{2} v\left(\varepsilon y, \frac{Z\left(b_{1}\right)}{\varepsilon}, b_{1}\right) & =\left(\varepsilon^{-(p-1)} Z_{q}^{p-1}\left(Z\left(b_{1}\right)\right)+b_{1}\right) Z\left(b_{1}\right) J\left(\varepsilon y, Z\left(b_{1}\right)\right) \\
& <\left(\varepsilon^{-(p-1)} Z_{q}^{p-1}\left(Z\left(b_{2}\right)\right)+b_{1}\right) Z\left(b_{2}\right) J\left(\varepsilon y, Z\left(b_{2}\right)\right) \\
& <\left(\varepsilon^{-(p-1)} Z_{q}^{p-1}\left(Z\left(b_{2}\right)\right)+b_{2}\right) Z\left(b_{2}\right) J\left(\varepsilon y, Z\left(b_{2}\right)\right) \\
& =\varepsilon^{2} v\left(\varepsilon y, \frac{Z\left(b_{2}\right)}{\varepsilon}, b_{2}\right),
\end{aligned}
$$

since $\frac{\partial v}{\partial Z}>0$ and $b_{1}<b_{2}$.
Thus, letting $Y(b)=\|w(\cdot, b)\|_{1}$, we have

$$
v\left(y, Y\left(b_{1}\right), b_{1}\right)<v\left(y, Y\left(b_{2}\right), b_{2}\right) \quad \text { for } \quad \alpha \leq b_{1}<b_{2} \leq \beta
$$

Then following the same reasoning as in Lemma 3.3, we find that

$$
w\left(x, b_{1}\right)<w\left(x, b_{2}\right) \quad \text { on }(0,1) \quad \text { for } \quad \alpha \leq b_{1}<b_{2} \leq \beta .
$$

Let $u\left(x, t, b_{1}\right)$ be a solution of $\left(\mathrm{P}^{\prime}\right)$ with $u_{0}\left(x, b_{1}\right)=w\left(x, b_{2}\right)$. Then, on $(0,1)$, we have

$$
\begin{aligned}
& u_{0}^{\prime \prime}+\varepsilon u_{0} u_{0}^{\prime}+\frac{1}{2}\left(\left\|u_{0}\right\|^{p-1}+b_{1}\right) u_{0} \\
& \quad=w_{x x}\left(x, b_{2}\right)+\varepsilon w\left(x, b_{2}\right) w_{x}\left(x, b_{2}\right)+\frac{1}{2}\left(\left\|w\left(\cdot, b_{2}\right)\right\|^{p-1}+b_{1}\right) w\left(x, b_{2}\right) \\
& \quad<w_{x x}\left(x, b_{2}\right)+\varepsilon w\left(x, b_{2}\right) w_{x}\left(x, b_{2}\right)+\frac{1}{2}\left(\left\|w\left(\cdot, b_{2}\right)\right\|^{p-1}+b_{2}\right) w\left(x, b_{2}\right) \\
& \quad=0
\end{aligned}
$$

Hence, recalling the Corollary in Appendix A, we have $u_{t}<0$ in $D_{T}$. From the comparison theorem in Appendix A and the monotonicity of $u$, we also have, on $(0,1)$

$$
w\left(x, b_{1}\right)<u\left(x, t, b_{1}\right) \leq w\left(x, b_{2}\right)
$$

By Lemma 4.1, $\varphi\left(x, b_{1}\right)=\lim _{t \rightarrow \infty} u\left(x, t, b_{1}\right)$ exists, and $w\left(x, b_{1}\right) \leq \varphi\left(x, b_{1}\right) \leq$ $w\left(x, b_{2}\right)$. Letting $b_{2} \rightarrow b_{1}^{+}$yields $\varphi\left(x, b_{1}\right) \equiv w\left(x, b_{1}\right)$, which shows that $w\left(x, b_{1}\right)$ is stable from above. We can also prove similarly that $w\left(x, b_{1}\right)$ is stable from below.

If $Z^{\prime}(b)<0$, then $Z\left(b_{2}\right)<Z\left(b_{1}\right)$, and consequently, $\left\|w\left(\cdot, b_{2}\right)\right\|_{1}<$ $\left\|w\left(\cdot, b_{1}\right)\right\|_{1}$ for $\alpha \leq b_{1}<b_{2} \leq \beta$. Thus there is a subinterval $\left[x_{0}, x_{1}\right]$ contained in $[0,1]$ such that $w\left(x, b_{2}\right)<w\left(x, b_{1}\right)$ on $\left[x_{0}, x_{1}\right]$. Let $u\left(x, t, b_{2}\right)$ be a solution of $\left(\mathrm{P}^{\prime}\right)$ with $u_{0}\left(x, b_{2}\right)=w\left(x, b_{1}\right)$. Then, on $(0,1)$, we find that

$$
\begin{aligned}
& u_{0}^{\prime \prime}+\varepsilon u_{0} u_{0}^{\prime}+\frac{1}{2}\left(\left\|u_{0}\right\|^{p-1}+b_{2}\right) u_{0} \\
& \quad=w_{x x}\left(x, b_{1}\right)+\varepsilon w\left(x, b_{1}\right) w_{x}\left(x, b_{1}\right)+\frac{1}{2}\left(\left\|w\left(\cdot, b_{1}\right)\right\|^{p-1}+b_{2}\right) w\left(x, b_{1}\right) \\
& \quad>w_{x x}\left(x, b_{1}\right)+\varepsilon w\left(x, b_{1}\right) w_{x}\left(x, b_{1}\right)+\frac{1}{2}\left(\left\|w\left(\cdot, b_{1}\right)\right\|^{p-1}+b_{1}\right) w\left(x, b_{1}\right) \\
& \quad=0 .
\end{aligned}
$$

Therefore, $u_{t}>0$ in $D_{T}$. Hence, $u\left(x, t, b_{2}\right)$ is increasing in $t$, and it follows that $w\left(x, b_{2}\right)$ is unstable from above. Similarly, $w\left(x, b_{2}\right)$ is unstable from below when $b_{1}>b_{2}$.

Using this theorem combined with the characterization of the stationary solutions in Section 3, we obtain the following stability and instability results for the case $\delta=1$.
(C-1) For $1<p<2$, if $b(p, q, \varepsilon)<b<2 \pi^{2}$, there are two branches of solutions of (S) - one stable, the other unstable; if $b>2 \pi^{2}$, the unique solution is stable.
(C-2) For $p=2$ :
(a) If $\varepsilon<\frac{1}{4} \varepsilon_{\infty}$ and $b<2 \pi^{2}, w(x, b)$ is unstable.
(b) If $\varepsilon=\frac{1}{4} \varepsilon_{\infty}$ and $b\left(\varepsilon_{\infty}\right)<b<2 \pi^{2}, w(x, b)$ is unstable.
(c) If $\varepsilon>\frac{1}{4} \varepsilon_{\infty}$, and
(i) if $b\left(\varepsilon_{\infty}\right)<b<2 \pi^{2}$, there are two branches - one stable, the other unstable; and
(ii) if $b>2 \pi^{2}, w(x, b)$ is stable.
(C-3) For $p>2$ :
(a) If $\varepsilon \leq \varepsilon_{1}$ and $b<2 \pi^{2}, w(x, b)$ is unstable.
(b) If $\varepsilon>\varepsilon_{1}$ and
(i) if $b<2 \pi^{2}, w(x, b)$ is unstable; and
(ii) if $2 \pi^{2}<b<b(p, q, \varepsilon)$, there are two branches - one stable, the other unstable.

In particular, we see from Figures $4-6$ that, as $\varepsilon$ increases, the portion of the bifurcation curve along which $Z^{\prime}(b)>0$ becomes more pronounced. This illustrates the stabilizing effect of the convection term in the dynamical equation.

From Figures $7-11$, we see that, for fixed $\varepsilon$, the choice $q=1$ leads to the "most stability" while the choice $q=+\infty$ leads to the "least stability" in the sense of the preceding paragraph. Increasing $q$ has the effect of decreasing the set $\left\{b \mid Z^{\prime}(b)>0\right\}$.

The case $\delta=-1$, because of the lack of a comparison principle, is not so amenable to analysis. However, bearing in mind numerical evidence (see [14]), we may conjecture that: the positive stationary solution branch (where it exists) is stable.

Next we discuss the asymptotic stabililty of the trivial stationary solution of $\left(\mathrm{P}^{\prime}\right)$. Henry [7] used a linearization method based on semigroup theory to analyze the asymptotic stability. His principle has wide application, but it does not readily extend to the current nonlocal problem. Therefore, we adopt another approach.

$$
\text { Let } \begin{aligned}
& F_{2}(t)= \int_{0}^{1} u^{2} d x \text { to find } \\
& \qquad \begin{aligned}
F_{2}^{\prime}(t) & =2 \int_{0}^{1} u u_{x x} d x+2 \varepsilon \int_{0}^{1} u u_{x} d x+b F_{2}+\delta\|u\|^{p-1} F_{2} \\
& =-2 \int_{0}^{1} u_{x}^{2} d x+\left(b+\delta\|u\|^{p-1}\right) F_{2} \\
& \leq\left(-2 \pi^{2}+b+\delta\|u\|^{p-1}\right) F_{2}
\end{aligned}
\end{aligned}
$$

since $\pi^{2} \int_{0}^{1} u^{2} d x \leq \int_{0}^{1} u_{x}^{2} d x$.
For $\delta=1$, for any $b<2 \pi^{2}$ and sufficiently small initial value $u_{0}$, we have $F_{2}(t) \leq F_{2}(0) e^{-k t}$, which means that $v \equiv 0$ is an asymptotically stable solution of $(\mathrm{S})$ in $L_{2}(0,1)$. Similarly, $v \equiv 0$ is a globally asymptotically stable solution of $(\mathrm{S})$ in $L_{2}(0,1)$ when $b \leq 2 \pi^{2}$ for $\delta=-1$.

Finally, we investigate the global existence and nonexistence of solutions of $\left(\mathrm{P}^{\prime}\right)$. We first give the following theorem for the case $\delta=1$.

Theorem 4.3. For $1<p<2$ and any $\varepsilon>0$, or $p=2$ and $\varepsilon>\frac{1}{2}(1+q)^{-1 / q}$, the solutions of $\left(\mathrm{P}^{\prime}\right)$ are uniformly bounded on $[0,1] \times[0, \infty)$.

Proof. We look for a supersolution $v$ in the form $v(x)=M(1-\sigma x)$ with $M>0$ and $0<\sigma<1$ to be chosen. We shall succeed if $M$ and $\sigma$ satisfy

$$
\begin{equation*}
-\varepsilon \sigma M^{2}(1-\sigma x)+\frac{1}{2}\left[b+M^{p-1}(\alpha(1+q))^{-\frac{p-1}{q}}\right] M(1-\sigma x) \leq 0 \tag{4.3}
\end{equation*}
$$

and

$$
\begin{equation*}
M(1-\sigma) \geq \max _{0 \leq x \leq 1} u_{0}(x) . \tag{4.4}
\end{equation*}
$$

The inequality (4.3) is equivalent to

$$
\begin{equation*}
M^{p-1}\left[\varepsilon \sigma M^{2-p}-\frac{1}{2}(\alpha(1+q))^{-\frac{p-1}{q}}\right] \geq \frac{1}{2} b . \tag{4.5}
\end{equation*}
$$

For $1<p<2$, (4.4) and (4.5) will hold if $M$ is sufficiently large; while if $p=2$, letting $\varepsilon=\frac{1}{2}(1+\gamma)(1+q)^{-1 / q}(\gamma>0) \sum_{f}^{f}$, we have that $\sigma^{\frac{1+q}{q}} \leq$ $(1+\gamma)^{-1}$, i.e. $\sigma \leq(1+\gamma)^{-\frac{q}{1+q}}<1$. For such $\sigma$, we can choose $M$ so large that (4.4) holds.

For the case $\delta=-1$, since the solution of $\left(\mathrm{P}^{\prime}\right)$ is a subsolution of $\left(\mathrm{P}^{\prime}\right)$ with $\delta=1$ and $1<p<2$, all solutions are bounded on $[0,1] \times[0, \infty)$.

Next for $p>2$ or $p=2$ with small $\varepsilon$, we prove that with sufficiently large initial data, problem ( $\mathrm{P}^{\prime}$ ) with $\delta=1$ does not have global solutions. To show this, we employ two different arguments.

Theorem 4.4A. Let $p>2$. Then there exists $c_{0}=c_{0}(\varepsilon, p, b)<\infty$ such that if $u_{0} \geq c_{0} \sin \pi x$, the solution of $\left(\mathrm{P}^{\prime}\right)$ blows up in finite time. The same is true for $p=2$ if $\varepsilon$ is small enough.

Proof. We seek a subsolution $w(x, t)$ in the form $w(x, t)=h(t) \sin \pi x$ with $h(t)$ becoming unbounded in finite time. To obtain this, we need

$$
\begin{equation*}
h^{\prime}(t) \leq-\pi^{2} h(t)+\pi \varepsilon h^{2}(t) \cos \pi x+\frac{1}{2}\left(A_{0} h^{p}(t)+b h(t)\right), \tag{4.6}
\end{equation*}
$$

where $A_{0}=\left[\int_{0}^{1}(\sin \pi x)^{q} d x\right]^{\frac{p-1}{q}}$.
Set

$$
\begin{equation*}
Q(s)=\frac{A_{0}}{2} s^{p-2}-\pi \varepsilon \tag{4.7a}
\end{equation*}
$$

for $b>2 \pi^{2}$, or set

$$
\begin{equation*}
Q(s)=\frac{A_{0}}{2} s^{p-2}-\pi \varepsilon-\left(\pi^{2}-\frac{b}{2}\right) s^{-1} \tag{4.7b}
\end{equation*}
$$

for $b \geq 2 \pi^{2}$.
For $p>2$, if $s_{0}$ is the largest positive root of $Q$, then $Q(s), Q^{\prime}(s)>0$ for $s>s_{0}$.

Choosing $c_{0}=s_{0}$ and letting $h(t)$ be the solution of the following problem

$$
\begin{align*}
& h^{\prime}(t)=Q(h) h^{2}(t), \quad t>0  \tag{4.8}\\
& h(0)=c_{0}
\end{align*}
$$

one easily sees that (3.6) is satisfied and $h(t)$ blows up in finite time.
For $p=2$ and sufficiently small $\varepsilon>0$, the above discussion also holds.

Remark. The result for $p=2$ is in contrast to that in [4], where it is shown that for any $\varepsilon>0$, the solution of a local problem with our nonlocal term replaced by $|u|^{p-1} u$ remains bounded on $[0,1] \times[0, \infty)$ when $p=2$.

For any large initial value, with more restriction on $p$ and $q$, we also have the following theorem.

Theorem 4.4B. Let $p \geq 3$ and $q \geq 2$. Then there exists $c_{1}=c_{1}(\varepsilon, p, b)$ such that if $\int_{0}^{1} u_{0}(x) d x>c_{1}$, the solution of $\left(\mathrm{P}^{\prime}\right)$ blows up in finite time.

Proof. Now we use a variation of the eigenfunction method [9].
Set

$$
\begin{equation*}
J(t)=\int_{0}^{1} u(x, t) \psi(x) d x \tag{4.9}
\end{equation*}
$$

where $\psi(x)=\frac{\pi}{2} \sin \pi x$.
A routine calculation shows that

$$
\begin{align*}
J^{\prime}(t) \geq- & \pi^{2} \int_{0}^{1} u(x, t) \psi(x) d x-\varepsilon \int_{0}^{1} u^{2}(x, t) \psi^{\prime}(x) d x \\
& +\frac{1}{2}\left[\int_{0}^{1} u^{q}(x, t) d x\right]^{\frac{p-1}{q}} \int_{0}^{1} u(x, t) \psi(x) d x  \tag{4.10}\\
& +\frac{b}{2} \int_{0}^{1} u(x, t) \psi(x) d x
\end{align*}
$$

Using Hölder's inequality, we see that

$$
\begin{gather*}
\varepsilon \int_{0}^{1} u^{2}(x, t) \psi^{\prime}(x) d x \leq A_{1}\left[\int_{0}^{1} u^{q}(x, t) d x\right]^{2 / q}  \tag{4.11}\\
\quad \int_{0}^{1} u(x, t) \psi(x) d x \leq A_{2}\left[\int_{0}^{1} u^{q}(x, t) d x\right]^{1 / q} \tag{4.12}
\end{gather*}
$$

with $A_{1}=\varepsilon\left\|\psi^{\prime}\right\|_{L^{\infty}(0,1)}$ and $A_{2}=\pi / 2$.
Let

$$
\begin{equation*}
R(s)=A_{3} s^{(p-1) / q}-A_{1} s^{2 / q} \tag{4.13a}
\end{equation*}
$$

for $b>2 \pi^{2}$, or let

$$
\begin{equation*}
R(s)=A_{3} s^{(p-1) / q}-A_{1} s^{2 / q}-\left(\pi^{2}-\frac{b}{2}\right) s^{1 / q} \tag{4.13b}
\end{equation*}
$$

for $b \leq 2 \pi^{2}$, where $A_{3}=\frac{1}{2} \int_{0}^{1} u_{0}(x) \psi(x) d x$.
If $s_{1}$ is the largest positive root of $R$, we find that $R(s), R^{\prime}(s)>0$ for $s>s_{1}$. Letting $c_{1}=s_{1}$ and $\int_{0}^{1} u_{0}(x) d x>c_{1}$ implies that $\int_{0}^{1} u^{q}(x, t) d x>s_{1}$ for all $t$. Hence,

$$
\begin{equation*}
J^{\prime}(t) \geq R\left(\int_{0}^{1} u^{q}(x, t) d x\right) \geq R\left(\frac{2}{\pi} J^{q}(t)\right) \tag{4.14}
\end{equation*}
$$

Since $q>1, J(t)$ must blow up in finite time.

## Appendix A

Here we establish the comparison principle and local existence of solutions for the following general problem:

$$
\left\{\begin{array}{lll}
u_{t}=u_{x x}+(f(u))_{x}+g(u,\|u\|), & 0<x<1, & 0<t<T,  \tag{G}\\
u(0, t)=u(1, t)=0, & 0<t<T, \\
u(x, 0)=u_{0}(x), & 0 \leq x \leq 1, &
\end{array}\right.
$$

where $\|u\|=\left[\int_{0}^{1}|u|^{q} d x\right]^{1 / q}$.
First, we define the subsolution and supersolution of (G). As in Section 4 , we let $D_{T}=(0,1) \times(0, T)$ and $D_{T} \cup \Gamma_{T}=[0,1] \times[0, T)$.

Definition. A function $u(x, t)$ is called a subsolution of $(\mathrm{G})$ on $D_{T}$ if $u \in C^{2,1}\left(D_{T}\right) \cap C\left(D_{T} \cap \Gamma_{T}\right)$, satisfying

$$
\begin{array}{lll}
u_{t} \leq u_{x x}+(f(u))_{x}+g(u,\|u\|) & 0<x<1, & 0<t<T, \\
u(0, t) \leq 0, u(1, t) \leq 0 & & 0<t<T, \\
u(x, 0) \leq u_{0}(x) & 0 \leq x \leq 1 . &
\end{array}
$$

A supersolution is defined by $\left(\mathrm{G}^{\prime}\right)$ with each " $\leq$ " replaced by " $\geq$ ".

Comparison Theorem. Suppose that $f$ and $g$ are continuously differentiable and that $g, 2 \geq 0$. Let $u$ and $v$ be a nonnegative supersolution and a nonnegative subsolution, respectively, of $(\mathrm{G})$, with $u(x, 0) \geq v(x, 0)$ for $x \in(0,1)$. Then $u \geq v$ in $D_{T} \cup \Gamma_{T}$.

Proof. For every $t \in(0, T)$ and every nonnegative $\varphi(x, t) \in C^{2,1}\left(\bar{D}_{T}\right)$ with $\varphi(0, t)=\varphi(1, t)=0$, the subsolution $v$ satisfies the following integral inequality:

$$
\begin{aligned}
\int_{0}^{1} v(x, t) \varphi(x, t) d x & \leq \int_{0}^{1} v_{0}(x) \varphi(x, 0) d x \\
& +\int_{0}^{t} \int_{0}^{1}\left[v \varphi_{\tau}-\left(v_{x}+f(u)\right) \varphi_{x}+g(v,\|v\|) \varphi\right] d x d \tau
\end{aligned}
$$

The supersolution $u$ satisfies the above with reversed inequality.
We integrate by parts in both the above inequality and that satisfied by $u$ and subtract the two resultant expressions. Then we have

$$
\begin{align*}
\int_{0}^{1} & (v(x, t)-u(x, t)) \varphi(x, t) d x \\
\leq & \int_{0}^{1}(v(x, 0)-u(x, 0)) \varphi(x, 0) d x  \tag{A1}\\
& +\int_{0}^{t} \int_{0}^{1}(v-u)\left(\varphi_{\tau}+\varphi_{x x}-A(x, \tau) \varphi_{x}+B(x, \tau) \varphi\right) d x d \tau \\
& +\int_{0}^{t} \int_{0}^{1} \varphi(x, \tau) \int_{0}^{1} C(s, \tau)(v-u) d s d x d \tau
\end{align*}
$$

where

$$
\begin{aligned}
& A(x, t)=f^{\prime}\left(\theta_{1}(x, t)\right), \\
& B(x, t)=g_{1}\left(\theta_{2}(x, t),\|v\|\right), \\
& C(x, t)=g_{2}\left(u, \theta_{3}(t)\right) \theta_{4}^{\frac{1}{q}-1}(t) \theta_{5}^{q-1}(x, t),
\end{aligned}
$$

with $\theta_{1}, \theta_{2}, \theta_{5}$ between $u$ and $v$, and $\theta_{3}, \theta_{4}$ between $\|u\|$ and $\|v\|$.

Note that by the hypotheses for $f$ and $g, A, B$, and $C$ are bounded on $\bar{D}_{T}$ in the uniform norm. We denote the bound by $M_{0}$.

Now we define two sequence $\left\{A_{n}\right\}$ and $\left\{B_{n}\right\}$ in such a way that
(i) $A_{n}, B_{n} \in C^{\infty}\left(\bar{D}_{T}\right)$,
(ii) $\left|A_{n}\right| \leq M_{0},\left|B_{n}\right| \leq M_{0}$,
(iii) $A_{n} \rightarrow A, B_{n} \rightarrow B$ in $\left(D_{T}\right)$ as $n \rightarrow \infty$,
and we set up a backward problem on $D_{t}$ :

$$
\left\{\begin{array}{lll}
\varphi_{n \tau}+\varphi_{n x x}-A_{n} \varphi_{n x}+B_{n} \varphi=0, & 0<x<1, & 0<\tau<t,  \tag{*}\\
\varphi_{n}(0, \tau)=\varphi_{n}(1, \tau)=0, & & 0<\tau<t, \\
\varphi_{n}(x, t)=\chi(x), & 0 \leq x \leq 1 . &
\end{array}\right.
$$

Here, $\chi(x) \in C_{0}^{\infty}(0,1), 0 \leq \chi \leq 1$.

Recalling standard theory (in [10] for example), we find that $\varphi=\lim _{n \rightarrow \infty} \varphi_{n}$ is a solution of ( $\mathrm{G}^{*}$ ) with $A_{n}, B_{n}$ replaced by $A, B$, and $\varphi \in C^{2,1}\left(\bar{D}_{T}\right)$. The initial and boundary values for $\varphi_{n}$ imply that $\varphi \geq 0$ in $D_{T}$.

Substituting $\varphi$ in (A1) yields

$$
\begin{aligned}
\int_{0}^{1}(v(x, t)-u(x, t)) \chi(x) d x & \leq M_{1} \int_{0}^{1}(v(x, 0)-u(x, 0))^{+} d x \\
& +\int_{0}^{t} \int_{0}^{1} \varphi(x, \tau) \int_{0}^{1} C(s, \tau)(v-u) d s d x d \tau
\end{aligned}
$$

where $M_{1}=\sup _{\bar{D}_{T}}|\varphi|$.
Since this inequality holds for every $\chi$, we can choose a sequence $\left\{\chi_{n}\right\}$ on $(0,1)$ converging to

$$
\chi= \begin{cases}1, & \text { if } v(x, t)-u(x, t)>0, \\ 0, & \text { otherwise } .\end{cases}
$$

Noting that $C \geq 0, v-u=(v-u)^{+}-(v-u)^{-}$, we find that

$$
\begin{aligned}
\int_{0}^{1}(v(x, t)-u(x, t))^{+} d x \leq & M_{1} \int_{0}^{1}(v(x, 0)-u(x, 0))^{+} d x \\
& +M_{0} M_{1} \int_{0}^{t} \int_{0}^{1}(v(x, \tau)-u(x, \tau))^{+} d x d \tau
\end{aligned}
$$

which leads, by Gronwall's inequality, to

$$
\int_{0}^{1}(v(x, t)-u(x, t))^{+} d x \leq M_{1}\left(1+e^{M t}\right) \int_{0}^{1}(v(x, 0)-u(x, 0))^{+} d x .
$$

Thus, the conclusion follows from the condition on initial data.

Corollary. If $u_{0}^{\prime \prime}+f^{\prime}\left(u_{0}\right) u_{0}^{\prime}+g\left(u_{0},\left\|u_{0}\right\|\right) \geq 0(\leq 0)$ on $(0,1)$, then $u_{t}(x, t) \geq 0(\leq 0)$ in $D_{T}$.

Proof. The condition on $u_{0}$ implies that $u_{0}$ is a subsolution (supersolution) of (G). Thus $u(x, t) \geq u_{0}(x)\left(\leq u_{0}(x)\right)$ in $D_{T} \cup \Gamma_{T}$. Let $v(x, t)=$ $u(x, t+h)(h>0)$. Then $v$ is a supersolution (subsolution) of $(\mathrm{G})$, and
therefore $u(x, t+h) \geq u(x, t)(\leq u(x, t))$. Since $h$ is arbitrary, $u$ is increasing (decreasing) in $t$ for fixed $x$, and hence $u_{t} \geq 0(\leq 0)$.

Next we establish the existence of solutions of $(\mathrm{G})$ on $D_{T} \cup \Gamma_{T}$ for sufficiently small $T$ and certain initial values. This time we assume only that $f$ and $g$ are continuously differentiable. We shall also define $f_{M} \equiv \sup _{|u| \leq M}|f(u)|$ and $g_{M} \equiv \sup _{|u| \leq M}|g(u,\|u\|)|$.

Let $G(x, y ; t)$ denote the Green's function for

$$
L u=u_{t}-u_{x x}, \quad 0<x<1, \quad t>0,
$$

with boundary conditions

$$
u(0, t)=u(1, t)=0, \quad t>0,
$$

that is,

$$
G(x, y ; t)=2 \sum_{n=1}^{\infty} \sin (n \pi x) \sin (n \pi y) e^{-n^{2} \pi^{2} t} .
$$

Then

$$
G(x, 0 ; t)=G(x, 1 ; t)=0 .
$$

Also, $u$ is a solution of $(G)$ on $D_{T} \cup \Gamma_{T}$ if and only if for $(x, t) \in D_{T} \cup \Gamma_{T}$

$$
\begin{align*}
u(x, t)= & \int_{0}^{1} G(x, y ; t) u_{0}(y) d y \\
& +\int_{0}^{t} \int_{0}^{1}[G(x, y ; t-\eta) g(u(y, \eta),\|u(\cdot, \eta)\|)  \tag{A2}\\
& \left.-G_{y}(x, y ; t-\eta) f(u(y, \eta))\right] d y d \eta \\
\equiv & u(x, t)
\end{align*}
$$

To show that (A2) is solvable for sufficiently small $T$, we use a contraction mapping argument. To this end, we define

$$
\begin{equation*}
u_{1}(x, t)=0 \tag{A3a}
\end{equation*}
$$

and

$$
\begin{equation*}
u_{n+1}(x, t)=u_{n}(x, t) \tag{A3b}
\end{equation*}
$$

Local Existence Theorem. Let the initial datum for problem (G) be continuous on $[0,1]$ and satisfy

$$
\begin{equation*}
0<m<\int_{0}^{1} \int_{0}^{1} G(x, y ; t) u_{0}(y) d y d x \tag{A4}
\end{equation*}
$$

for $0 \leq t \leq 1$, say. Then for sufficiently small $T$, (G) has a unique solution that satisfies

$$
\begin{equation*}
\|u\| \geq \frac{m}{2} \tag{A5}
\end{equation*}
$$

on $[0, T]$. The solution is $C^{1}$ in $t$ and $C^{2}$ in $x$ on $D_{T}$ and continuous on $\bar{D}_{T}$.

Proof. First, we define

$$
\begin{aligned}
M_{0} & =\sup _{0 \leq x \leq 1}\left|u_{0}(x)\right| \\
\mu(t) & =\sup _{\substack{0 \leq x \leq 1 \\
0 \leq \tau \leq t}} \int_{0}^{\tau} \int_{0}^{1} G(x, y ; \tau-\eta) d \eta \\
\nu(t) & =\sup _{\substack{0 \leq x \leq 1 \\
0 \leq \tau \leq t}} \int_{0}^{\tau} \int_{0}^{1} G_{y}(x, y ; \tau-\eta) d \eta
\end{aligned}
$$

Clearly, $\mu(t)$ and $\nu(t)$ tend to zero as $t \rightarrow 0^{+}$. For fixed $M>M_{0}$, choose $T$ so small that $T \leq 1$ and

$$
\begin{equation*}
\nu(T) f_{M}+\mu(T) g_{M}<\max \left(M-M_{0}, \frac{1}{2} m\right) \tag{A6}
\end{equation*}
$$

$\alpha \equiv \nu(T) \sup _{|\xi| \leq M}\left|f^{\prime}(\xi)\right|$

$$
+\mu(T)\left(\begin{array}{ll}
\sup _{\substack{ \\
|\xi| \leq M}}\left|g_{\xi}(\xi, \eta)\right|+\sup _{|\xi| \leq M}^{|\eta| \leq M}  \tag{A7}\\
|\eta| g_{\eta}(\xi, \eta) \left\lvert\,\left(\frac{m}{2}\right)^{\frac{1}{q}-1} M^{q-1}\right. \\
|\eta| \leq M
\end{array}\right)
$$

$<1$.

It then follows from (A3a), (A3b), and (A6) and induction that on $\bar{D}_{T}$

$$
\left\|u_{n}\right\|_{L^{\infty}\left(\bar{D}_{T}\right)} \leq M
$$

for all $n=1,2, \cdots$. Moreover, we have from (A3b) and (A6) that

$$
\begin{equation*}
\left\|u_{n}\right\| \geq\left\|u_{n}\right\|_{L^{1}\left(\bar{D}_{T}\right)}>\frac{m}{2} \tag{A8}
\end{equation*}
$$

for all $n=2,3, \cdots$ Using (A7) and letting $\beta_{n} \equiv\left\|u_{n+1}-u_{n}\right\|_{L^{\infty}\left(\bar{D}_{T}\right)}$, we can see that

$$
\beta_{n+1} \leq \alpha \beta_{n} \leq \cdots \leq \alpha^{n} \beta_{1} .
$$

Therefore, $\left\{u_{n}\right\}$ is uniformly convergent on $\bar{D}_{T}$, and

$$
\begin{equation*}
u(x, t)=\lim _{n \rightarrow \infty} u_{n}(x, t) \tag{A9}
\end{equation*}
$$

solves (A2) with $\|u\| \geq m / 2$.
The asserted interior regularity follows from the properties of $G$ and the continuity of $u$ in $\bar{D}_{T}$. We omit the standard arguments.

The above result also allows us to make the following continuation statement: If $u$ is a classical solution on $D_{T}$, bounded on $\bar{D}_{T}$, then $u$ may be extended to $\overline{D_{T+\delta}}$ for some $\delta>0$.

## Appendix B

Here we establish Lemma 3.2.

Lemma 3.2. The integral

$$
\int_{0}^{1} \frac{z^{1 / 2}}{\left(\frac{1-e^{-\sigma z}}{1-e^{-z}}-\sigma\right)^{1 / 2}} d \sigma
$$

is increasing in $z$ for $z \geq 0$.
We see that the statement is equivalent to showing that

$$
\begin{aligned}
& \int_{0}^{1 / 2}\left[\frac{z^{1 / 2}}{\left(\frac{1-e^{-\sigma z}}{1-e^{-z}}-\sigma\right)^{1 / 2}}+\frac{z^{1 / 2}}{\left(\frac{1-e^{-(1-\sigma) z}}{1-e^{-z}}-(1-\sigma)\right)^{1 / 2}}\right] d \sigma \\
& \quad=\int_{0}^{1 / 2}\left[\left(\frac{z}{f(z, \sigma)}\right)^{1 / 2}+\left(\frac{z}{g(z, \sigma)}\right)^{1 / 2}\right] d \sigma
\end{aligned}
$$

is increasing in $z$ for $z \geq 0$.
To establish this assertion, we shall show that

$$
\left(\frac{z}{f}\right)^{1 / 2}+\left(\frac{z}{g}\right)^{1 / 2}
$$

is increasing in $z$, for $\sigma \in[0,1 / 2]$.

Reduction 1. The presence of square roots in the integrand causes difficulties. By taking the square, we see that the lemma is a consequence of showing that

$$
\frac{z}{f}+\frac{z}{g} \quad \text { and } \quad \frac{z^{2}}{f g}
$$

are increasing.
Reduction 2. Let us assume that we already know that $a^{2} / f g$ is increasing. We have

$$
\begin{equation*}
\frac{z}{f}+\frac{z}{g}=\frac{z}{(f g)^{1 / 2}} \cdot\left(\left(\frac{f}{g}\right)^{1 / 2}+\left(\frac{g}{f}\right)^{1 / 2}\right) \tag{B1}
\end{equation*}
$$

The first factor is increasing by assumption. We therefore have (B1) once we can show that the second factor is also increasing. We claim that this is equivalent to showing that $f / g$ is increasing in $z$, for $\sigma \in[0,1 / 2]$. Let

$$
\begin{equation*}
F^{2}(z)=\frac{f}{g} . \tag{B2}
\end{equation*}
$$

Using L'Hopital's rule, we obtain

$$
\begin{equation*}
F(0)=\lim _{z \rightarrow 0} \frac{f}{g}=1 . \tag{B3}
\end{equation*}
$$

The derivative of the second factor in (B1) is

$$
\begin{equation*}
\frac{F^{\prime}(z)\left[F^{2}(z)-1\right]}{F^{2}(z)} \tag{B4}
\end{equation*}
$$

If we can show that $F(z)$ is increasing in $z$, then, in view of (B3), both factors in the numerator are nonnegative (and hence the second factor in ( B 1 ) is increasing).

We have thus reduced the problem to proving that $f / g$ is increasing and $z^{2} / f g$ is increasing in $z$ for all $\sigma \in[0,1 / 2]$.

## Proof that $f / g$ is increasing

After simplification and the substitution $y=e^{-z}$, we have

$$
\frac{f}{g}=\frac{1-y^{\sigma}-\sigma(1-y)}{1-y^{1-\sigma}-(1-\sigma)(1-y)},
$$

$y \in[0,1], \sigma \in[0,1 / 2]$. Note that $y$ decreases as $z$ increases. Hence, we have to show that $f / g$ is decreasing in $y$. The numerator of its derivative, namely, $f^{\prime} g-f g^{\prime}$, is
$X(y)=(4 \sigma-2)+\left(\sigma^{2}-2 \sigma+1\right) y^{\sigma}-\sigma^{2} y^{\sigma-1}+\left(\sigma^{2}-2 \sigma+1\right) y^{-\sigma}-\sigma^{2} y^{1-\sigma}$.
We need to show that it is nonpositive. Note that $X(1)=0$; hence, it suffices to show that $X^{\prime}(y) \geq 0$ in $[0,1]$, or equivalently that

$$
Y(y)=\frac{y X^{\prime}}{\sigma(1-\sigma)}=-\sigma y^{1-\sigma}-(1-\sigma) y^{-\sigma}+(1-\sigma) y^{\sigma}+\sigma y^{\sigma-1} \geq 0
$$

Again we see that $Y(1)=0$; hence, it remains to show that $Y^{\prime}(y) \leq 0$ in $[0,1]$. But

$$
Y^{\prime}(y)=-\frac{\sigma(1-\sigma)(1-y)\left(1-y^{1-2 \sigma}\right)}{y^{2-\sigma}} \leq 0,
$$

as desired (recall that $0<\sigma<1 / 2$ ).

Now it remains to prove that $z^{2} / f g$ is increasing, or rather its reciprocal

$$
\begin{equation*}
K(z, \sigma)=\frac{\left(1-e^{-\sigma z}-\sigma\left(1-e^{-z}\right)\right)\left(1-e^{-(1-\sigma) z}-(1-\sigma)\left(1-e^{-z}\right)\right)}{z^{2}\left(1-e^{-z}\right)^{2}} \tag{B5}
\end{equation*}
$$

is decreasing in $z \geq 0$ for $\sigma \in[0,1 / 2]$.
Reduction 3. Note that

$$
K(z, 0) \equiv 0
$$

Hence, trivially,

$$
K_{z}(z, 0)=0 .
$$

If we can show that

$$
\begin{equation*}
K_{z \sigma}(z, \sigma) \leq 0, \tag{B6}
\end{equation*}
$$

for $z \geq 0, \sigma \in[0,1 / 2]$, then we know that for a fixed $z, K_{z}$ is a decreasing function in $\sigma$. Since $K_{z}$ starts out at 0 when $\sigma=0, K_{z}$ must be nonpositive for all $\sigma>0$ up to $\sigma=1 / 2$, and (B5) is then proved. The inequality (B6) is equivalent to showing that $K_{\sigma}$ is decreasing in $z \geq 0$ for $\sigma \in[0,1 / 2]$. After differentiating $K$ with respect to $\sigma$, we can separate out those terms in the numerator that have the extra factor $z$ :

$$
K_{\sigma}=H+L,
$$

where

$$
\begin{equation*}
H=\frac{(1-2 \sigma)\left(1-e^{-z}\right)^{2}+e^{-(1-\sigma) z}+e^{-(1+\sigma) z}-e^{-\sigma z}-e^{-(2-\sigma) z}}{z^{2}\left(1-e^{-z}\right)^{2}}, \tag{B7}
\end{equation*}
$$

and

$$
\begin{equation*}
L=\frac{\sigma e^{\sigma z}-\sigma e^{-(2-\sigma) z}+(1-\sigma) e^{-(1+\sigma) z}-(1-\sigma) e^{-(1-\sigma) z}}{z\left(1-e^{-z}\right)^{2}} . \tag{B8}
\end{equation*}
$$

We have thus made the final reduction to proving that $H$ and $L$ are decreasing in $z$ for $z \geq 0$ and all $\sigma \in[0,1 / 2]$.

Proof that $H$ is decreasing. The crucial observation is that the numerator of $H$ has the factor $\left(1-e^{-z}\right)$. After cancellation, we have

$$
\begin{align*}
H & =\frac{(1-2 \sigma)\left(1-e^{-z}\right)+e^{-(1-\sigma) z}-e^{-\sigma z}}{z^{2}\left(1-e^{-z}\right)} \\
& =\frac{(1-2 \sigma)\left(e^{z / 2}-e^{-z / 2}\right)+e^{-(1-\sigma) z / 2}-e^{(1-\sigma) z / 2}}{z^{2}\left(e^{z / 2}-e^{-z / 2}\right)} . \tag{B9}
\end{align*}
$$

The next step involves an obvious change of variables

$$
t=1-\sigma, \quad y=z / 2
$$

and the use of hyperbolic functions. We have

$$
\begin{equation*}
H=\frac{t \sinh (y)-\sinh (t y)}{4 y^{2} \sinh ^{2}(y)}, \tag{B10}
\end{equation*}
$$

$y \geq 0, t \in[0,1]$. Note that

$$
H(y, 0)=H(y, 1)=0 .
$$

Hence,

$$
\begin{equation*}
H_{y}(y, 0)=H_{y}(y, 1)=0 . \tag{B11}
\end{equation*}
$$

Let us fix a $y$ and see how $R(t)=H_{y}(\cdot, t)$ changes with $t$. We need to show that $R(t) \leq 0$ in order to establish that $H$ is decreasing. By (B11),

$$
\begin{equation*}
R(0)=R(1)=0 . \tag{B12}
\end{equation*}
$$

By differentiating (B10) twice with respect to $t$ and once with respect to $y$, we see that

$$
\begin{equation*}
R^{\prime \prime}(t)=\left(-\frac{\sinh (t y)}{4 \sinh (y)}\right)_{y}=\frac{\cosh (y) \sinh (t y)-t \sinh (y) \cosh (t y)}{4 \sinh ^{2}(y)}, \tag{B13}
\end{equation*}
$$

where the subscript $y$ outside the parentheses denotes partial differentiation in $y$. If we can show that $R^{\prime \prime}(t) \geq 0$, then $R(t)$ is a convex function of $t$ and
then by virtue of $(\mathrm{B} 12), R(t) \leq 0$ as required. The nonnegativity of $R^{\prime \prime}$ is implied by

$$
\cosh (y) \sinh (t y) \geq t \sinh (y) \cosh (t y)
$$

or equivalently

$$
\begin{equation*}
\frac{y \cosh (y)}{\sinh (y)} \geq \frac{(t y) \cosh (t y)}{\sinh (t y)}, \quad t \in[0,1] . \tag{B14}
\end{equation*}
$$

This is a consequence of the elementary fact that $y \cosh (y) / \sinh (y)$ is an increasing function in $y$.

Proof that $L$ is decreasing in $z$. We follow the same line of attack. Denote $S(\sigma)=L_{z}(z, \sigma)$ for a fixed $z$. Note that

$$
\begin{equation*}
L(z, 0)=L(z, 1 / 2) \equiv 0, \tag{B15}
\end{equation*}
$$

implies that

$$
\begin{equation*}
S(0)=S(1 / 2)=0 . \tag{B16}
\end{equation*}
$$

Unlike the function $R(t)$, the second derivative of $S(\sigma)$ does not have a simple form from which we can deduce $S^{\prime \prime}(\sigma) \geq 0$. We first rewrite $L$ in the following form:

$$
\begin{equation*}
L(z, \sigma)=\frac{\sigma \sinh (1-\sigma) z-(1-\sigma) \sinh (\sigma z)}{z \sinh ^{2}(z / 2)} . \tag{B17}
\end{equation*}
$$

Taking the derivative twice with respect to $\sigma$, we have

$$
\begin{equation*}
L_{\sigma \sigma}=-2 \frac{\cosh (1-\sigma) z-\cosh \sigma z}{\sinh ^{2}\left(\frac{z}{2}\right)}+z^{2} L . \tag{B18}
\end{equation*}
$$

Using the sum and product formula on the first term, we obtain the simplified equation:

$$
\begin{equation*}
L_{\sigma \sigma}=-4 \frac{\sinh \left(\left(\frac{1}{2}-\sigma\right) z\right)}{\sinh \left(\frac{z}{2}\right)}+z^{2} L . \tag{B19}
\end{equation*}
$$

The first term on the righthand side is an increasing function of $z$, since it is really the same function occurring in $R^{\prime \prime}$ in (B13) (take $y=z / 2$, and $t=(1-2 \sigma)<1$ ). Now take the $z$ derivative of (B19) to obtain

$$
\begin{equation*}
S^{\prime \prime}(\sigma)=\left[\left(-\frac{\sinh \left(\left(\frac{1}{2}-\sigma\right) z\right)}{\sinh \left(\frac{z}{2}\right)}\right)_{z}+2 z L\right]+z^{2} S(\sigma) \tag{B20}
\end{equation*}
$$

The terms inside the square brackets are positive. Now suppose that $L$ is not decreasing. Then for some $z, S(\sigma)$ must have a strictly positive maximum in $[0,1 / 2]$. At this maximum, the lefthand side of (B20) is nonpositive while the righthand side is strictly positive, giving a contradiction.

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