Extrapolation-based Boundary Element Quadrature*

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(In *Rend. Semin. Mat. Univ. Pol, Torino*: fasc. spec. 1991, pp. 189–203)
Preprint MCS-P208-0191

1 Introduction

The classical Euler Maclaurin summation formula expresses the discretization error made by the trapezoidal rule approximation to a finite integral as an asymptotic expansion in the mesh ratio. As such, it plays a significant role in the theory of numerical quadrature. In particular, it forms the basis for Richardson's deferred approach to the limit, also known as Romberg integration and as quadrature by extrapolation. In this review, we draw attention to some more recent theory, based on the Euler Maclaurin summation formula (or expansion).

During the past 25 years, this theory has undergone significant development. Briefly, the expansion has been systematized to embrace all quadrature rules. It has been generalized to the N-dimensional cube and to the N-dimensional simplex. Moreover, a significant generalization has been made to cover functions having algebraic and logarithmic singularities at vertices. In this article we discuss some of these developments in the context of their possible use to analyze finite elements. In general, no proofs are provided, but references are given.

This article, and most of the available theory, is limited to the following class of problem (together with its higher-dimensional analogues). The integration regions are the square and the triangle. The integrand function is singular only at a vertex of the integration region. Taking this as the origin, we treat only integrand functions of the type

$$f(\vec{x}) = \rho^{\alpha} \ln \rho \; \Theta(\theta) h(r) g(x.y).$$

Here $\vec{x} = (x, y)$ are Cartesian coordinates with the corresponding polar coordinates r, θ. The functions $\Theta(\theta)$, $h(r)$, and $g(x, y)$ are regular, and rho is homogeneous about the origin. Examples of $\rho(\vec{x})$ include the following:

$$\begin{aligned} \rho &= r \\ \rho &= Ax + By, \qquad A, B > 0 \\ \rho &= Ax^{3/2} + By^{3/2}. \end{aligned}$$

The quadrature rules may be linear combinations of m_i-copies $Q^{(m_i)}f$ of any quadrature rule $Qf = Q^{(1)}f = \sum w_i f(\vec{x}_i)$.

*This work was supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy, under Contract W-31-109-Eng-38.

2 The Classical One-Dimensional Background

In one dimension, we treat the interval $[0,1]$ and an integrand $f(x) \in C^{p-1}[0,1]$. We denote the exact integral by

$$If = \int_0^1 f(x)dx, \tag{2.1}$$

and we consider any quadrature rule of the form

$$Qf = \sum_{j=1}^{\nu} w_j f(x_j), \qquad \sum_{j=1}^{\nu} w_j = 1. \tag{2.2}$$

The m-copy version $Q^{(m)}$ of this quadrature rule Q is obtained by subdividing the integration interval into m equal parts and applying the properly scaled version of Q to each. Specifically,

$$Q^{(m)}f = \sum_{k=0}^{m-1} \sum_{j=1}^{\nu} \frac{w_j}{m} f\left(\frac{x_j + k}{m}\right). \tag{2.3}$$

Theorem 2.4. (The Euler Maclaurin summation formula). *When* $f(x) \in C^p[0,1]$,

$$Q^{(m)}f - If = \sum_{s=1}^{p-1} \frac{B_s}{m^s} + R_p^{(m)}(Q;f), \tag{2.4}$$

where B_s *is independent of* m *and* $R_p^{(m)} = O(m^{-p})$.

The important feature of this expansion is that the coefficients $B_s = B_s(Q;f)$ are independent of m. In fact, they take the form

$$B_s(Q;f) = c_s(Q)If^{(s)}, \tag{2.5}$$

where $c_s(Q)$ is the result obtained by applying Q to the integrand function $B_s(x)/s!$ and $B_s(x)$ is the Bernoulli function. A simple integral representation for the remainder term is available.

We shall now demonstrate how the information that expansion (2.4) exists may be exploited to construct a quadrature technique or a new quadrature rule. In this demonstration, we choose Q to be the trapezoidal rule $Qf = \frac{1}{2}(f(0) + f(1))$ so that

$$Q^{(m)}f = \frac{1}{m}\left(\frac{1}{2}f(0) + \sum_{j=1}^{m-1} f(j/m) + \frac{1}{2}f(1)\right). \tag{2.6}$$

Let us write down the expansion (2.4) with $p = 6$ for $m = 1, 2$ and 4. (We may omit B_s for s odd since Q is symmetric and so $c_s(Q) = 0$.) We find

$$\begin{aligned} Qf &= If + B_2 + B_4 + R_6^1 \\ Q^{(2)}f &= If + B_2/2^2 + B_4/2^4 + B_6^2 \\ Q^{(4)}f &= If + B_2/4^2 + B_4/4^4 + R_6^4. \end{aligned} \tag{2.7}$$

Extrapolation comprises carrying out the following *ad hoc* procedure. We remove the remainder terms and solve the resulting set of three equations in three unknowns. Note that, after we have removed the remainder terms, the unknowns are only approximations to If, B_2, and B_4. For classical reasons we denote by $T_{0,2}$ this particular approximation to If. We find

$$T_{0,2} = \frac{1}{45}Q^{(1)}f - \frac{20}{45}Q^{(2)}f + \frac{64}{45}Q^{(4)}f, \tag{2.8}$$

which, using (2.6), we may reexpress in terms of function values

$$T_{0,2} = \frac{1}{90}\left\{7f(0) + 32f(1/4) + 12f(1/2) + 32f(3/4) + 7f(1)\right\}. \tag{2.9}$$

This is, in fact, the Newton Cotes equispaced rule of degree 5.

This straightforward example is given here to illustrate an important general point. Once the *form* (2.4) of the asymptotic expansion for $Q^{(m)}f - If$ is known, one may construct an integration rule using only linear algebra. No further explicit information about moments or about interpolation polynomials is necessary. Any required information of that type must have already been contained implicitly in the expansion.

We note that (other than $B_s = 0$ for s odd) we do not need to know the form of B_s at all, merely that it is independent of m. Thus, starting with a rule Q, the only restriction being that $\Sigma w_j = 1$, and using only $Q^{(m)}f = If + \Sigma B_s/m^s + R_p(m)$, we can find a respectable quadrature rule of moderate polynomial degree.

We make this point strongly here because, in many elementary descriptions, the theory becomes enmeshed in algebraic detail and the reader can be left with the impression that serious restrictions are in place. The only serious restriction we know is that the integrand must be well behaved.

Note that this technique cannot produce all quadrature rules. One could, however, develop the theory of (constant weight finite interval) quadrature if one took advantage of detailed expressions for the term $c_s(Q)$ which appears in (2.5) above. Such a development would be roughly equivalent to the standard development of numerical quadrature theory.

However, given a quadrature rule, one can sometimes find interesting properties or useful extensions simply using the Euler Maclaurin expansion. In one dimension, one discovers only already-known properties. The extension of these ideas to simplices and cubes with simple singularities at vertices can be helpful by providing new insight into standard intractable problems. This is the thrust of the rest of this paper.

3 Multidimensional Generalizations of Classical Results

The generalizations of the asymptotic expansion to the N-dimensional hypercube $[0, 1)^s$ is straightforward. In this new context, Qf stands for any rule that integrates the constant function correctly, while $Q^{(m)}f$ stands for the result of subdividing the hypercube into m^N subcubes each of side m^{-1} and applying the appropriately scaled version of Q to each. We find immediately that

$$Q^{(m)}f = If + \sum_{s=1}^{p-1} B_s/m^s + R_p^{(m)}(Q, f). \tag{3.1}$$

Here, B_s is an obvious generalization of (2.5), namely,

$$B_s = \sum_{\substack{\Sigma s_i = s \\ s_i \geq 0}} c_{s_1,s_2,\ldots,s_N}(Q) If^{(s_1,s_2,\ldots,s_N)}. \tag{3.2}$$

However, the corresponding integral representation for R_p (which is $O(m^{-p})$) is a somewhat lengthy uncompromising expression.

Turning to the simplex, we encounter further complications. There are variant definitions for $Q^{(m)}f$. We shall proceed as follows. We treat the isosceles simplex

$$S : x_i \geq 0; \quad \Sigma x_i < 1. \tag{3.3}$$

This forms a part of the hypercube $[0,1)^N$. We define our rule Qf as if for this hypercube. Then $Q^{(m)}f$ for the simplex is the result of applying the rule Q to a function θ that coincides with f within S and is zero outside S. (Certain conventions, not given here, apply on the boundary of S within H.) The reason for this somewhat complicated definition can be seen even in two dimensions. The subdivision of the triangle into four equal subtriangles by lines parallel to the edges of the triangle leaves the center triangle with a different orientation to the other three. If we simply defined a rule Q for the triangle S, we would leave ambiguity as to how to apply it in this center triangle. The approach given above provides a definite specification for this situation. The "rule" for the "other" triangle is the part of the rule for the square that lies in this "other" triangle.

An asymptotic expansion of precisely the same form as (3.1) appears to exist for the simplex. The coefficients B_s are, as before, independent of m but are given by an expression significantly more complicated than (3.2) above.

In the practice of higher-dimensional extrapolation, it is important to know whether any of these coefficients B_s vanish. In fact, certain cofactors $c_{s_1,s_2,\ldots,s_n}(Q)$ vanish when the rule Q enjoys certain symmetry and polynomial properties (SPP). This, in turn, causes terms B_s to vanish as specified in the following theorem.

Theorem 3.4 *When Q is symmetric (about $(\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$), then*

$$B_s = 0, \qquad \forall \quad s \text{ odd}. \tag{3.4}$$

When Q is of polynomial degree d (for the hypercube), then

$$B_s = 0, \qquad \forall \quad s \in [1,\delta], \tag{3.5}$$

where $\delta = d$ for the hypercube and $\delta = d+1-N$ for the N-dimensional simplex.

Conditions (3.4) and (3.5) are termed SPP conditions. It is important to note that in this theorem, one inspects the behavior of Q over the hypercube to determine its symmetry and degree. The result of this inspection determines which B_s may vanish when it is used over the simplex and which B_s may vanish when it is used over the hypercube. The possible properties of the truncated part of Q that may be applied to S may be of interest independently, but play no direct role in this theorem.

4 Singularities in One-Dimensional Quadrature

The error functional expansions given above are valid only when the integrand functions are sufficiently well behaved. In one dimension, for example, (2.4) is valid only when $f(x) \in C^{p-1}[0,1]$. This is not simply a technical restriction. The expression for B_s contains a factor $If^{(s)}$ that exists only when $f^{(s)}(x)$ is integrable.

In 1961, Navot published a generalization of the Euler Maclaurin expansion which applies to the function $f(x) = x^{\alpha}$ with noninteger α (as well as with integer α). His original proof involved somewhat heavy algebraic manipulation. Successively simpler proofs appeared later. A slightly more general version of his result is the following theorem.

Theorem 4.1 *Let $f_\alpha(x) = \kappa x^\alpha$, where $\alpha \neq -1$; then*

$$Q^{(m)} f_\alpha - I f_\alpha = \frac{A_{\alpha+1}}{m^{\alpha+1}} + \sum_{s=1}^{p-1} \frac{B_s}{m^s} + O(m^{-p}), \tag{4.1}$$

where $A_{\alpha+1}$ and B_s are independent of m.

Simple analytic forms for these coefficients are known. $A_{\alpha+1}$ involves the generalized zeta function.

To specify these new coefficients B_s, we extend the notation If defined in (2.1) (or (4.2) below). When dealing with f_α, which is simply κx^α, we define $If_\alpha^{(s)}$ as a finite part integral, namely,

$$If_\alpha^{(s)} = \int_0^1 f_\alpha^{(s)}(x)dx, \qquad \alpha - s > -1, \tag{4.2}$$

$$If_\alpha^{(s)} = -\int_1^\infty f_\alpha^{(s)}(x)dx, \qquad \alpha - s < -1. \tag{4.3}$$

This is not defined for $\alpha - s = -1$. For other values of $\alpha - s$, one and only one of the integrals (4.2) or (4.3) converges and this one is used for the definition. In the next section this definition is extended to N dimensions with f_α any homogeneous function of order α.

The coefficient in (4.1) is

$$B_s = c_s(Q) I f_\alpha^{(s)}. \tag{4.4}$$

Thus, B_s is given by its standard formula (2.5) when that is meaningful but is modified when that is not meaningful. We note in passing that the theorem applies, as written, when $\alpha < -1$, and so the conventional integral does not exist. The term If_α on the right of (4.1) is then simply a finite part integral as in (4.3) above. In fact, it is notationally convenient but initially bewildering to replace If_α in (4.1) by B_0 and to include this in the sum on the right. Finally, we note that the only value of α for which Theorem 4.1 is invalid is $\alpha = -1$. In that case the proper expansion includes other terms not given in (4.1).

Naturally, this expansion is not needed to integrate x^α whose integral is known to be $1/(\alpha + 1)$. It is used to construct an expansion for the more sophisticated function $x^\alpha g(x)$, where $g(x) \in C^p[0, 1]$. Using a Taylor expansion (with remainder term), we set

$$\begin{aligned} f(x) &= g(0)x^\alpha + g'(0)x^{\alpha+1} + \frac{g''(0)}{2!}x^{\alpha+2} \\ &\quad + \ldots + \frac{g^{(p-1)}(0)}{(p-1)!}x^{\alpha+p-1} + G_p(x)x^{\alpha+p}. \end{aligned} \tag{4.5}$$

One may apply Theorem 2.4 to the final term on the right. The other terms on the right are functions to which Theorem 4.1 applies (for the appropriate α).

Applying these theorems leads to the result

$$Q^{(m)} f - If \simeq \sum_{j=1} \frac{A_{\alpha+1+j}}{m^{\alpha+1+j}} + \sum_{s=1} \frac{B_s}{m^s}, \qquad \alpha \neq \text{ negative integer}. \tag{4.6}$$

The structure of B_s here is not trivial. It is composed of a weighted sum of integrals over two possible intervals. However, it is independent of m, and each integral has a cofactor $c_s(Q)$. Thus, these coefficients B_s also satisfy the SSP conditions (Theorem 3.4).

Note that, at this point, the restriction in Theorem 4.1 to the effect that $\alpha \neq -1$ has the effect of invalidating (4.6) for all negative integers α.

5 The Euler Maclaurin Expansion for a Vertex Singularity

The generalization to the N-dimensional hypercube of the result of the preceding section appeared in Lyness (1976a). Before stating this generalization, Theorem 5.10, we clarify some of the concepts on which it is based.

Definition 5.1 *The N-dimensional function $f(\vec{x})$ is said to be homogeneous of degree α (about the origin) if*

$$f(\lambda\vec{x}) = \lambda^{\alpha} f(\vec{x}), \qquad \forall \quad \lambda > 0 \text{ and } \vec{x} \neq \vec{0}. \tag{5.1}$$

We denote such a function by $f_\alpha(\vec{x})$. We have anticipated this notation in Section 4. The one-dimensional function $f_\alpha(x) = \kappa x^\alpha$ is clearly homogeneous of degree α.

A monomial $x^p y^q$ is homogeneous of degree $p + q$, and many properties relating to the polynomial degree of functions of monomials have direct analogues in the context of homogeneous degree. Thus, $(f_\alpha)^\beta$ and $f_\alpha f_\beta$ are of degree $\alpha\beta$ and $\alpha + \beta$, respectively, and $f_\alpha(M\vec{x})$ when $|M| \neq 0$ is also of degree α; $\partial^s f_\alpha / \partial x^s$ is of degree $\alpha - s$.

In more than one dimension, many more interesting functions are homogeneous. For example, $r = (x^2+y^2)^{1/2}$ is homogeneous of degree 1, and $\theta = \arctan y/x$ is homogeneous of degree zero, as is any function $\phi(\theta)$. Our basic result, (5.10) below, applies to a homogeneous function of specified degree.

We also require an N-dimensional generalization of the interval $[a, b)$ when $0 \leq a \leq b$.

Definition 5.2 $L[a, b); \{\vec{x} \mid a \leq \max_i x_i < b; \quad x_i \geq 0\}$.

This is the region remaining when $[0, a)$ is removed from $[0, b)$. In two dimensions, it has the shape of the capital letter L.

We shall be integrating homogeneous functions over L-shaped regions. To this end we note first that

$$\int_{L[a,b)} f_\gamma(\vec{x}) d^N x = m^{-\gamma-N} \int_{L[ma,mb)} f_\gamma(\vec{x}) d^N x. \tag{5.3}$$

This follows by elementary scaling. Next we express

$$L[1, \infty) = \bigcup_{j=0}^{\infty} L[m^j, m^{j+1}), \tag{5.4}$$

obtaining

$$\int_{L[1,\infty)} f_\gamma(\vec{x}) d^N x = (1 + m^{\gamma+N} + m^{2(\gamma+N)} + \ldots) \int_{L[1,m)} f_\gamma(\vec{x}) d^N x, \tag{5.5}$$

which, so long as $\gamma + N < 0$, can be put in the form

$$\int_{L[1,m)} f_\gamma(\vec{x}) d^N x = (1 - m^{\gamma+N}) \int_{L[1,\infty)} f_\gamma(\vec{x}) d^N x. \tag{5.6}$$

When $\gamma + N > 0$, a similar derivation leads to

$$\int_{L[1,m)} f_\gamma(\vec{x}) d^N x = -(1 - m^{\gamma+N}) \int_{[0,1)} f_\gamma(\vec{x}) d^N x. \tag{5.7}$$

These formulas are basic in establishing the main result (5.10) below. We note that the left-hand side of (5.7) appears at first sight to depend on m in an involved way. In fact, (5.7) shows that the dependence on m has a straightforward form; we write both (5.6) and (5.7) in the form

$$\int_{L[1,m)} f_\gamma(\vec{x}) d^N x = -(1 - m^{\gamma+N}) If_\gamma, \qquad \gamma + N \neq 0, \tag{5.8}$$

where we have used the following definition, which is an immediate generalization of the one given in (4.2) and (4.3) above.

Definition 5.9 *When f_γ is of homogeneous degree $\gamma \neq -N$,*

$$\begin{aligned} If_\gamma &= \int_{[0,1)} f_\gamma(\vec{x}) d^N x \qquad N + \gamma > 0, \\ &= -\int_{L[1,\infty)} f_\gamma(\vec{x}) d^N x \qquad N + \gamma < 0. \end{aligned} \tag{5.9}$$

Theorem 5.10 *Let $f_\alpha(\vec{x})$ be an N-dimensional homogeneous function of degree $\alpha \neq -N$ which has no singularity in $L[1,2)$. Let Qf be any integration rule for the hypercube $[0,1)^N$. Then*

$$Q^{(m)} f_\alpha - If_\alpha = \frac{A_{\alpha+N}}{m^{\alpha+N}} + \frac{C_{\alpha+N} \ln m}{m^{\alpha+N}} + \sum_{s=1}^{p-1} \frac{B_s}{m^s} + O(m^{-p}), \tag{5.10}$$

the constants A_j, B_j, and C_j being independent of m.

In addition, the coefficients B_j and C_j have relatively simple forms:

$$\begin{array}{ll} B_s = \sum_{\Sigma s_i = s} c_{s_1,s_2,\ldots,s_N}(Q) If_\alpha^{(s_1,s_2,\ldots,s_n)} & s \neq \alpha + N \\ B_s = 0 & s = \alpha + N \\ C_{\alpha+N} = 0 & \alpha + N \neq \text{ integer} \\ C_{\alpha+N} = \sum_{\Sigma s_i = s} c_{s_1,s_2,\ldots,s_N}(Q) \int_{L[1,2)} f_\alpha^{(s_1,s_2,\ldots,s_N)} d^N x / \ln 2 & s = \alpha + N. \end{array}$$

The proof of this theorem is given in Lyness (1976a). An expression for A_s and an integral representation of the remainder term is given there. In fact, although (5.10) is an asymptotic expansion, no use of this is made in the derivation. It may be treated as an identity.

We close this section by making some minor and detailed observations about the formula.

1. The coefficients B_s and C_s involve the factors $c_{s_1,s_2,\ldots,s_N}(Q)$ in the same way as in the standard case (where $f(\vec{x})$ does not have a singularity). Thus, both B_s and C_s obey the SPP conditions (Theorem 3.4 above); that is, specified coefficients vanish when Q is symmetric and when Q is of specified polynomial degree. (In fact, C_s also satisfies Theorem 3.4 with δ replaced by d.)

2. The condition that $f_\alpha(\vec{x})$ has no singularity in $L[1,2)$ is simply an alternative way of indicating that $f_\alpha(\vec{x})$ has no singularity in any finite region R of the first octant $0 \leq x_i$ which does not contain the origin.

3. For large α, this formula reduces to the standard N-dimensional Euler Maclaurin formula (3.1). While this is, of course, a convenient special case, the reader should bear in mind that (3.1) is used precisely in this context in the standard proof of (5.10).

4. We note that one can obtain the expansion for $F(\vec{x}) = f_\alpha(\vec{x})g(\vec{x})$, where $g(\vec{x})$ has low-order integrable derivatives, in precisely the same manner as in the one-dimensional case (see (4.6) above).

6 Further Generalizations of the Multidimensional Results

As mentioned above, when one sets into expansion (5.10) the forms of the constants A_s, B_s, and C_s and the integral representation for the remainder term, it becomes an identity. Because of this, it is not surprising to learn that the theorem is valid for all values of α and not only those for which If_α is a convergent integral. Intrinsically, it can be used to evaluate finite part integrals. For example, with $N = 2$ and $\alpha = -5/2$, we would find an expansion with terms in $m^{1/2}, m^0, m^{-1/2}, m^{-1}, \ldots$, and we might extrapolate to find the coefficient of m^0 in this expansion. This is not the leading coefficient; this approach, akin to numerical differentiation, is not generally recommended by this author, but we mention it to clarify the underlying situation. When α is a negative integer and $\alpha \leq -N$, even this approach is extraordinarily difficult because the coefficient denoted by A_0 is indistinguishable numerically from that denoted by B_0. The situation in one dimension has been clarified and is the subject of a forthcoming paper by Lyness and Monegato, but in higher dimensions it remains open.

Another consequence of the circumstance that this is an identity is that we may obtain a new identity by differentiating this one with respect to any incidental parameter. An obvious candidate is α. For example, when $F(x,y) = (Ax + By)^\alpha g(x,y)$, we find $\partial F/\partial\alpha = F(x,y)\ln(Ax + By)$; and it appears that, when $\alpha \neq$ integer, we may obtain from the expansion for F an expansion for $F(x,y)\ln(Ax + By)$. When $\alpha \neq$ integer, this derived expansion contains terms in $m^{-(\alpha+N+j)}, m^{-(\alpha+n+j)}\ln m$, and m^{-s}.

At first sight this procedure might appear to be limited to noninteger α. This is because, as α passes through an integer value, one of the coefficients B_s in (5.10) changes abruptly as $If_\alpha^{(s)}$ changes its region of integration, and another coefficient C_s appears and disappears. Further investigation leads to the conclusion that $Qf_\alpha - If_\alpha$ does not change abruptly and that these apparently abrupt changes result simply from the way the expansion is expressed. In Lyness (1976a), it is shown how to reformulate the expansion. It then appears that crude differentiation of form (5.10) gives, after all, a valid result. The reader may refer to that paper to see how Theorem 3.4 (about SSP conditions) can be applied in general in these cases. However, we recommend that potential users rederive any expansion they plan to use from (5.10) and carry out numerical experiments to confirm its validity.

Other generalizations of (5.10) are geometric in character. The formula as written applies equally to the simplex $S : 0 \leq x_i < 1; \Sigma x_i < 1$, though the coefficients B_s are different. This fact may be demonstrated geometrically. In two dimensions, for example, the square H may be considered to be the union of two triangles, S and $\bar{S}$, with S defined as above. The Euler Maclaurin expansion for S is then the difference of the appropriate expansions for H and for $\bar{S}$. Since f_α has no singularity in $\bar{S}$, the expansion for $\bar{S}$ contains only terms B_s/m^s. It follows that the expansion for S contains the same terms as the expansion for H. This brief synopsis of the two-dimensional derivation requires elaboration, and the N-dimensional result is significantly more difficult to derive; see Lyness (1978) and Lyness and Monegato (1980). It is again a result that is easy to state and is readily predictable, but is difficult to prove.

However, the next geometrical result is easy to prove and extraordinarily useful. The reader will be familiar with the circumstance that, by means of an affine transformation, one simplex may be transformed into another and that any quadrature rule of specified polynomial degree for one is transformed into one of the same degree for the other. The reader should also be aware that this remark refers only to situations where no weight function is involved. When a weight function is involved, the new rule applies with respect to a transformed and generally different weight function. One of the inconvenient aspects of the boundary element method seems to be that different Gaussian rules are required for differently shaped regions and only occasionally are such rules related.

Methods based on extrapolation do not suffer from this drawback. The underlying reason is that the affine transformation of f_α, a homogeneous function of degree α, is another homogeneous function of degree α. Furthermore, no weight function is involved in the formulation of the theory. To the author's knowledge, this is the first time this possibility has been mentioned in the literature. Properly exploited, it allows the straightforward construction of powerful quadrature results for these regions and function singularities.

7 Concluding Remarks

This review article covers a small area of numerical quadrature. It is designed to treat only those aspects of quadrature by extrapolation that might be of interest in the theory of boundary integration methods. While one or two remarks are original, with few exceptions the mathematical theory described here has already been published in the open literature. The author hopes that this article will be helpful in informing the user as to what is available and where to look for further details.

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