# On the Unboundedness of the Number of Solutions of a Dirichlet Problem* 

Man Kam Kwong<br>Mathematics and Computer Science Division<br>Argonne National Laboratory<br>Argonne, IL 60439-4801


#### Abstract

We confirm a conjecture raised by Lazer and McKenna on the number of Dirichlet solutions of the equation $u^{\prime \prime}+f(u)=s \sin (t)+h(t)$ in $[0, \pi]$, where the nonlinear function $f(u)$ satisfies $-\infty<f^{\prime}(-\infty)<$ $f^{\prime}(\infty)=\infty$. Our result asserts that given any positive integer $N$, there exists a real number $s_{N}$ such that for all $s>s_{N}$ there are at least $N$ Dirichlet solutions.


AMS(MOS) Subject Classification. Primary 34B15. Secondary 35 J 25.

Key Words and Phrases. Nonlinear boundary value problem, variation index, shooting method, multiplicity of solutions.

Proposed Running Head. Unbounded Number of Solutions

[^0]
## 1 Introduction

In an informative article [1], Lazer and McKenna proposed a modified mathematical model for the onset of large-amplitude oscillations in suspension bridges by wind with specific velocities. The study was motivated by the inadequacy of older theories to explain the collapse of the Tacoma Narrows Bridge of Seattle in 1941.

In the Lazer-McKenna model, the motion of the bridge is, as usual, governed by a system of differential equations, more specifically, semilinear elliptic differential equations, the complexity of which depends on the degree of approximation and simplifications one is willing to accept. One of the new ideas introduced is the asymmetry of the restoring force from a cable, with respect to expansion and compression. The authors' basic assumption is that the cable "strongly resists expansion, but does not resist compression." The study of elliptic equations involving a nonlinear restoring-force term of this type is still largely unexplored. In the same article, Lazer and McKenna posed many interesting open questions. Some of these have not been answered even in the one-dimensional case, when the elliptic equation becomes a second-order nonlinear ordinary differential equation. In an earlier article [5], we gave a counterexample to their Problem 4. In this article, we take up another one of these questions (Problem 2) concerning a nonlinear function that grows very fast at infinity.

The boundary value problem we are interested in is

$$
\begin{gather*}
u^{\prime \prime}(t)+f(u(t))=s \sin (t)+h(t)  \tag{1}\\
u(0)=u(\pi)=0 \tag{2}
\end{gather*}
$$

where $f(u)$ is a genuinely nonlinear continuously differentiable function on $(-\infty, \infty), h(x)$ is any continuous function on $[0, \pi]$, and $s$ is a real parameter. We shall refer to a solution of (1)-(2) as a $D$-solution and reserve the simpler term solution for one that satisfies (1), but not necessarily the Dirichlet boundary conditions. Our main objective is to determine upper and lower bounds for the number of distinct $D$-solutions when $f$ satisfies certain growth conditions.

For the question we are studying, the conditions imposed on $f$ are

$$
\begin{equation*}
-\infty<f^{\prime}(-\infty)<f^{\prime}(\infty)=\infty \tag{3}
\end{equation*}
$$

At the end of paper, we shall indicate how the second inequality can be relaxed.

In [2], Lazer and McKenna proved that under the condition

$$
\begin{equation*}
f^{\prime}(-\infty)<1<n^{2}<f^{\prime}(\infty)<\infty, \tag{4}
\end{equation*}
$$

there exist at least $2 n$ D-solutions, for all sufficiently large $s$. Based on this result, they conjectured that if the boundedness of $f^{\prime}(\infty)$ is replaced by $f^{\prime}(\infty)=\infty$, then for suitable choices of $s$, there can be any number of D-solutions. Our purpose here is to confirm this conjecture under condition (3). The first inequality in (3) differs from that of (4) by requiring that $f^{\prime}(-\infty)$ be bounded, but not necessarily by 1 .

Theorem 1 Under condition (3), for any positive integer $N$, there exists a real number $s_{N}$ such that for all $s>s_{N}$, (1) has at least $N$ D-solutions.

The function $\sin (t)$ on the righthand side of (1) can be replaced by any nontrivial symmetric (with respect to $t=\pi / 2$ ) function that is nondecreasing in $[0, \pi / 2]$.

An important tool in the proof of Theorem 1 is the variation index of a solution, a concept we used successfully in [5]. A definition of the index was recently given by Clemons in [4]. Given a solution $u(t)$, which may or may not be a D -solution, we substitute it into the derivative of $f$ to obtain the function $g(t)=f^{\prime}(u(t))$. Let $w(t)$ be the solution of the linear initial value problem (treating $g(t)$ as a known coefficient of the linear term)

$$
\begin{gather*}
w^{\prime \prime}(t)+g(t) w(t)=0, \quad t \in(0, \pi)  \tag{5}\\
w(0)=0, \text { and } w^{\prime}(0)=1 . \tag{6}
\end{gather*}
$$

Suppose $w(t)$ has $k$ zeros in the open interval $(0, \pi)$. We then say that the variation index, or in short the index, of $u(t)$ is $k^{+}$. The + in the notation is added to emphasize that we are a little beyond the degenerate situation in which the $k^{\text {th }}$ zero falls exactly at the endpoint $\iota$. The significance of the index in the study of the multiplicity of D-solutions is contained in the following lemma; its special case with $n=1$ was proved in Section 2 of [5].

Lemma 1 Suppose (3) holds, and $f^{\prime}(-\infty)<m^{2}$ for some positive integer $m$. If there exists a D-solution $u(t)$ with index $k^{+}$, where $k \geq 2 m$, then there are at least $2(k-2 m+2$ ) distinct $D$-solutions (including $u(t)$ ).

Thus, Theorem 1 is established if we can exhibit a D-solution with an arbitrarily large index. Instead of tackling directly the original equation (1), we shall work with the scaled equation, obtained by substituting $u(t)=s y(t)$ :

$$
\begin{equation*}
y^{\prime \prime}(t)+\frac{1}{s} f(s y)=\sin (t)+\frac{h(t)}{s} . \tag{7}
\end{equation*}
$$

In Section 2 we give the arguments in a heuristic manner, drawing on some yet unproved assertions. In Section 3 we complete the job by supplying the necessary lemmas. The proof of Lemma 1 contains some of the techniques used in the proofs of the other lemmas and is therefore given last.

## 2 The Heuristic Arguments

The formal and complete proof of our main result contains a lot of dry and technical estimations. For pedagogical purposes, we give in this section the main steps in outline form, appealing often to our intuition, and leaving certain rigorous details till the next section.

Recall that we are working with the scaled equation (7), with a sufficiently large parameter $s$. A first simplification, easily justified by intuition, is obtained by ignoring the small term $h(t) / s$. In the rest of this section, we consider the simplified equation

$$
\begin{equation*}
y^{\prime \prime}(t)+\frac{1}{s} f(s y)=\sin (t) . \tag{1}
\end{equation*}
$$

As already pointed out in the Introduction, it suffices to construct a Dsolution with a large index. The first question to consider is what properties must be present in the D-solution to guarantee a large index. A successful technique used in obtaining information on the index is the Sturm comparison principle; see, for instance, Coffman [3], McLeod and Serrin [10], Ni and Nussbaum [11], and Kwong et al. [6]-[9]. The index is a measure of the amount of oscillation of the function $w(t)$. The Sturm technique consists of finding a comparison function $v(t)$ with a suitable number of zeros. Substituting $v(t)$ for $w(t)$ into the lefthand side of (5) gives a differential equation, a careful study of which via the classical Sturm comparison theorem can reveal information concerning whether $v(t)$ oscillates faster or more slowly than $w(t)$. As a result, we obtain an upper or lower bound on the index of $u(t)$. In general, $v(t)$ is constructed from $u(t)$, its derivatives, and the independent variable $t$, in a manner that differs from one equation to another. Experience tells us that $v(t)=y^{\prime}(t)$ is often a good candidate to try. It turns out to be just what we need here. The differential equation for $v(t)$ is

$$
\begin{equation*}
v^{\prime \prime}(t)+g(t) v(t)=\cos (t) . \tag{2}
\end{equation*}
$$

Note that the righthand side is positive in $(0, \pi / 2)$ and negative in $(\pi / 2, \pi)$. The Sturm comparison theorem states that in any interval $[\gamma, \delta]$ in which $v(t)$ and the righthand side of (2) have the same sign, $v(t)$ oscillates more slowly than $w(t)$ - more precisely, between any two zeros of $v(t)$ in $[\gamma, \beta]$ there must be at least one zero of $w(t)$. See, for example, Theorem 1 in [6].

Suppose we can find a D-solution that oscillates wildly in $[0, \pi / 2]$, where there are many subintervals $\left[\gamma_{i}, \delta_{i}\right] \subset[0, \pi / 2]$ in which $y(t)$ is increasing,
each $\gamma_{i}$ being a local maximum and each $\delta_{i}$ a local minimum. In other words, all the $\gamma_{i}$ and $\delta_{i}$ are zeros of $v(t)$, and $v(t)$ has the same sign as the righthand side of (2) in each $\left[\gamma_{i}, \delta_{i}\right]$. The comparison principle implies now that $w(t)$ has at least one zero within each $\left[\gamma_{i}, \delta_{i}\right]$, and so the index of $u(t)$ is at least equal to the number of such subintervals. If the oscillations are in $[\pi / 2, \pi]$ instead, the arguments are modified by using subintervals in which $y(t)$ is decreasing.

Theorem 1 is, therefore, proved if we can construct a D-solution of (1) that has a large number of local maxima. The simplified equation (1) is symmetric with respect to $t=\pi / 2$. Even though not all D-solutions need to be symmetric, we can simplify our construction by asking for a symmetric one. We then have to work with only half of the interval; we choose the second half $[\pi / 2, \pi]$. We use the familiar shooting method. The differential equation (1) is solved as an initial value problem, with $t=\pi / 2$ as the starting time, initial height $\beta$, and initial slope zero. The solution, denoted by $y(t, \beta)$ is then a function of both $t$ and $\beta$. Any $\beta$ that makes $y(\pi, \beta)=0$ corresponds to a symmetric D-solution. It turns out that only negative $\beta$ need be used.

We next investigate what effect the nonlinear term has on a solution, for large values of $s$. The coefficient $f(s y) / s$ in (1) is derived from the function $f(u)$ using two compressions, one horizontal and the other vertical. Geometrically, the graph of the former is obtained by shrinking the graph of the latter with respect to the origin in a scale of $s$ to 1 . Any given bounded portion of the graph of $f(u)$ can be contracted to as small an area as we please, by making $s$ sufficiently large. As a consequence, the behavior of $f(u)$ for small $u$, as manifested by $f(s y) / s$, does not have any lasting effect on the solution. The predominant influence of $f(s y) / s$ on a solution is thus determined by the behavior of $f(u)$ as $u \rightarrow \infty$. In the extreme limit, when $s \rightarrow \infty, f(s y) / s$ is indistinguishable from $f^{\prime}(-\infty) y$ for $y<0$ and $f^{\prime}(\infty) y$ for $y>0$. That $f^{\prime}(\infty) y=\infty$ necessitates further interpretation.

To emphasize the special nature of this extreme limiting situation, we denote the corresponding limiting solution, $\lim _{s \rightarrow \infty} y(t)$, by $z(t)$. For $z<0$, the limiting function $f^{\prime}(-\infty) z$ can be interpreted as the restoring force exerted by a spring with Hooke's constant $f^{\prime}(-\infty)$ on a moving particle. The extremely large restoring force $f^{\prime}(\infty) z$ for $z>0$ means that at $z=0$, the particle encounters a hard, perfectly elastic wall. Upon impact, the motion is reversed, and the particle bounces back with speed equal to, but opposite
in direction from, the incoming speed. This interpretation will be backed up by lemmas proved in Section 3.

To avoid technical details, we discuss here only the simplest case when $f(-\infty)=0$; the general case requires only more computation. In the region $z<0$, the particle now moves without being attached to a spring, but under the action of an external force $\sin (t)$ in a direction towards the wall. Since the wall is impenetrable, $z(t)$ is never positive. One can flip the geometry upside down and imagine a bouncing ball. The magnitude of the solution describes the position of a ball dropped at time $t=\pi / 2$, with an initial height $|\beta|$. The gravitational force $\sin (t)$ is not a constant but decreases in time. This property has the effect of causing the ball to bounce higher and higher. When the ball hits the ground, $z(t)=0$, it bounces upwards. Gravity can bring it down again at a later time; the bouncing is then repeated until the time is up.

The number of bounces depends on the initial height. If $|\beta|$ is too large, the ball takes longer than the allotted time to reach the ground, and thus no bounces occur within $[\pi / 2, \pi]$. On the other hand, if $|\beta|$ is small, the ball can reach the ground quickly. A simple computation will convince one that by making $|\beta|$ small enough, any number of bounces can be attained. By finely tuning $|\beta|$, we can even arrange to have the ball hit the ground exactly at the final time $t=\pi$. We then have a D -solution.

The extreme limiting case described above is, of course, not attainable for any finite $s$. It serves, however, as a good approximation when $s$ is sufficiently large. In particular, there exist D-solutions that bounce at least as often as prescribed. For such solutions, the region $y>0$ is no longer impenetrable. Instead, after $y(t)$ reaches 0 , it becomes positive for a short while and then is deflected back. Furthermore, $y^{\prime}(t)$ is not discontinuous as in the limiting case, a fact needed to validate the Sturm comparison process.

## 3 Lemmas

The arguments given in Section 2 are now substantiated by a sequence of lemmas. Straightforward reasoning involving routine computation or wellknown and elementary arguments are omitted. Computations involving the inhomogeneous term $\sin (t)$ are often simplified by replacing the function with its lower and upper bounds, -1 and 1 , to yield estimations. Indeed, no specific properties of the inhomogeneous term, other than its boundedness and monotonicity, are needed.

To start, we assume that $h(t)=0$; hence, we are dealing with equation (1) on the half-interval $I=[\pi / 2, \pi]$. Define

$$
\begin{equation*}
q(y, s)=\frac{f(s y)}{s y} . \tag{1}
\end{equation*}
$$

With this notation we rewrite (1) as

$$
\begin{equation*}
y^{\prime \prime}(t)+q(y, s) y(t)=\sin (t) \tag{2}
\end{equation*}
$$

The solution $y(t, \beta)$ satisfies the initial conditions

$$
\begin{equation*}
y(\pi / 2, \beta)=\beta, \quad \text { and } \quad y^{\prime}(\pi / 2, \beta)=0 . \tag{3}
\end{equation*}
$$

In reality, $y(t, \beta)$ depends also on the choice of $s$, but since there is no risk of confusion, the dependence is not explicitly acknowledged in the notation. We always assume that $\beta<0$. The statement " $|\beta|$ is large" is thus another way of saying that " $\beta$ is very negative."

For convenience, we adopt the following convention. All $\epsilon$ that appear in inequalities refer to any given, arbitrarily small, positive constant. The constant can differ from one inequality to another. Two quantities are "close" if their difference is less than $\epsilon$, understood in the above sense. The phrase "for sufficiently large $s$ " means "there exists some number $s_{1}$ that depends on $\epsilon$, such that for all $s>s_{1}$." The various $s_{1}$ can be assumed equal, simply by increasing all of them if necessary.

Lemma 2 For sufficiently large s,

$$
\begin{equation*}
\left|q(y, s)-f^{\prime}(-\infty)\right|<\epsilon, \text { for } y<-\epsilon, \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
q(y, s)>\frac{1}{\epsilon}, \text { for } y>\epsilon \tag{5}
\end{equation*}
$$

Furthermore, the function $q(y, s)$ is uniformly bounded from below.

Proof. It is well known that for a function $\phi(x), \lim _{x \rightarrow \infty} \phi(x) \rightarrow A$ implies that the mean $\int_{0}^{u} \phi(x) d x / u$ converges to the same limit $A$ as $u \rightarrow \infty$. Hence, $f(u) / u=\int_{0}^{u} f^{\prime}(x) d x / u \rightarrow f^{\prime}( \pm \infty)$ as $u \rightarrow \pm \infty$. The conclusion follows easily.

Lemma 2 is the basis for using the bouncing ball solution to approximate $y(t, \beta)$.

Definition. Let $\left[\pi / 2, \tau_{1}\right] \subset[\pi / 2, \pi]$ be the maximal interval for the existence of a nonpositive solution of the differential equation

$$
\begin{equation*}
z^{\prime \prime}+f^{\prime}(-\infty) z=\sin (t) \tag{6}
\end{equation*}
$$

with initial conditions $z(\pi / 2, \beta)=\beta$ and $z^{\prime}(\pi / 2, \beta)=0$. Define $z(t, \beta)$ to be this solution in $\left[\pi / 2, \tau_{1}\right]$. If $\tau_{1}<\pi$, then $z\left(\tau_{1}, \beta\right)=0$. For $t>\tau_{1}$, continue $z(t, \beta)$ to be the solution of (6) with initial conditions $z\left(\tau_{1}, \beta\right)=0$ and $z^{\prime}\left(\tau_{1}+, \beta\right)=-z^{\prime}\left(\tau_{1}-, \beta\right)$. If $z(t, \beta)$ hits the $t$ axis again, the process is repeated until $z(t, \beta)$ is defined up to $t=\pi$. The points $\tau_{1}, \tau_{2}, \cdots$ at which the solution crosses the $t$ axis are called the zeros of $z(t, \beta)$.

That $z(t, \beta)$ can be successfully continued beyond each zero is justified by the following lemma, which can be verified by direct computation.

Lemma 3 At each $\tau_{i}$ where $z\left(\tau_{i}, \beta\right)=0, z^{\prime}\left(\tau_{i}, \beta\right) \neq 0$.

It is easy to see that $z(t, \beta)$ depends continuously on the initial height $\beta$. However, because of the discontinuity of $z^{\prime}$ at each zero, the continuous dependence of $z^{\prime}(t, \beta)$ holds only in subsets of $[\pi / 2, \pi]$ not containing any zeros of $z(t, \beta)$, for all the $\beta$ under consideration. The awkwardness of having to exclude the zeros from the domain of continuous dependce can be avoided by noting that $z(t, \beta)=-|\psi(t, \beta)|$, where $\psi(t, \beta)$ is the $C^{1}$ solution of the initial value problem

$$
\begin{equation*}
\psi^{\prime \prime}+f^{\prime}(-\infty) \psi=\operatorname{sign}(\psi) \sin (t), \quad \psi(\pi / 2, \beta)=\beta, \psi^{\prime}(\pi / 2, \beta)=0 . \tag{7}
\end{equation*}
$$

Lemma 4 There exists an infinite sequence $\beta_{0}<\beta_{1}<\beta_{2}<\cdots<0$ such that $z\left(t, \beta_{n}\right)$ has exactly $n$ zeros in the open interval $I$ and the right endpoint $\pi$ is also a zero.

Proof. When $f^{\prime}(-\infty)=0$, the lemma is obvious. Furthermore, $z(t, \beta)$ has $n$ zeros for $\beta \in\left(\beta_{n-1}, \beta_{n}\right)$. In the general case with $f^{\prime}(-\infty) \neq 0$, if we choose a sufficiently small $\beta, z$ will remain small. As a result, the linear term $f^{\prime}(-\infty) z$ can be ignored in comparison to the righthand side. Thus, $z(t, \beta)$ is close to the solution for the case $f^{\prime}(\infty)=0$, and so has an arbitrarily large number of zeros. A shooting argument gives us the desired D-solutions.

We now choose a fix $\beta_{n}$ from the sequence and consider a sufficiently small neighborhood $\left(\beta_{n}-\xi, \beta_{n}+\xi\right)$, which does not contain any other $\beta_{i}$. All subsequent estimates are understood to be uniform for $\beta$ in this neighborhood and for all sufficiently large $s$. Uniformity with respect to $\beta$ usually follows from the relative compactness of the neighborhood. Showing uniformity with respect to $s$ often requires more work.

Lemma 5 For all $\beta \in\left(\beta_{n}-\xi, \beta_{n}\right), z(t, \beta)$ has $n$ zeros in $(\pi / 2, \pi)$. For all $\beta \in\left(\beta_{n}, \beta_{n}+\xi\right), z(t, \beta)$ has $n+1$ zeros in $(\pi / 2, \pi)$, and the last zero $\tau_{n+1}$ is very close to $\pi$. Furthermore, if $\xi$, and $\delta$ is chosen sufficiently small, no zeros of the solutions will fall within $[\pi-2 \delta, \pi-\delta]$.

Our next step is to substantiate the claim that $z(t, \beta)$ is close to $y(t, \beta)$. From the way $z(t, \beta)$ is defined in steps over the subintervals between successive zeros, it is only natural to expect that the estimation of $y(t, \beta)$ is done in the same piecewise maner.

Let $\rho_{1}$ be the first zero of $y(t, \beta)$, or $\pi$ if there is no zero. Then in $J=\left[\pi / 2, \rho_{1}\right] \cap\left[\pi / 2, \tau_{1}\right]$, the inequality (4) is valid, for large $s$. Standard perturbation arguments show that $y(t, \beta)$ is close to $z(t, \beta)$ in $J$. If $\rho_{1}>\tau_{1}$, then $y\left(t_{1}, \beta\right)$ is close to $z\left(t_{1}, \beta\right)$ and $y^{\prime}\left(t_{1}, \beta\right)$ is close to $z^{\prime}\left(t_{1}, \beta\right)$. It follows that $y(t, \beta)$ has to cross the $t$ axis soon after $t_{1}$. Hence $\rho_{1}$ is a zero close to $\tau_{1}$. Similarly, if $\rho_{1}<\tau_{1}$, the two zeros must also be close. Furthermore, in all cases, $y^{\prime}\left(\rho_{1}-, \beta\right) \approx z^{\prime}\left(\tau_{1}-, \beta\right)$.

After $\rho_{1}, y(t, \beta)$ becomes positive until the next zero $\sigma_{1}$. In $\left(\rho_{1}, \sigma_{1}\right)$, $y(t, \beta)$ can no longer be approximated by $z(t, \beta)$; the coefficient $q(y, s)$ has
changed radically, now that $y$ becomes positive. The next lemma, however, shows that the duration of positivity is short.

Lemma 6 Let $\rho$ and $\sigma$ be two successive zeros of $y(t, \beta)$ and $y(t, \beta)>0$ in $(\rho, \sigma)$. Then for sufficiently large $s$,

$$
\begin{equation*}
\sigma-\rho<\epsilon \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
y^{\prime}(\rho, \beta) \leq-y^{\prime}(\sigma, \beta) \leq y^{\prime}(\rho, \beta)+\epsilon . \tag{9}
\end{equation*}
$$

Proof. First we claim that $y^{\prime}(\rho, \beta)$ is uniformly bounded, in both $\beta$ and $\sigma$. This is true if $\rho=\rho_{1}$ is the first zero, since $y^{\prime}\left(\rho_{1}, \beta\right)$ is close to $z^{\prime}\left(\rho_{1}, \beta\right)$, which is uniformly bounded in $\beta(z(t, \beta)$ does not depend on $s$ at all $)$. The same argument goes through by induction for subsequent zeros $\rho_{i}$; we anticipate the fact that $y^{\prime}\left(\rho_{i}, \beta\right)$ is still close to $z^{\prime}\left(\tau_{i}, \beta\right)$.

Lemma 3 implies that $z^{\prime}\left(\tau_{i}, \beta\right)$ is uniformly bounded away from zero for all $\beta \in\left(\beta_{n}-\xi, \beta_{n}+\xi\right)$; more precisely, there exists a $\delta$ such that $z^{\prime}(\rho, \beta)>$ $\delta>0$. The same argument used to prove the first claim shows that $y^{\prime}(\rho, \beta)$ is uniformly bounded away from zero, for all $\beta$ and $s$ under consideration.

Inequality (5) tells us that the coefficient $q(y)$, which measures the tendency to deflect $y(t, \beta)$, can be assumed arbitrarily large except for very small values of $y$. Since $y^{\prime}(\rho, \beta)>0$ is bounded away from zero, the duration for which $y(t, \beta)$ is small is arbitrarily short. Furthermore, the sign of $y^{\prime}(t, \beta)$ can be shown to remain positive. Once $y(t, \beta)$ becomes larger than $\epsilon$, it is subjected to a large spring coefficient of at least $1 / \epsilon$. By making this coefficient sufficiently large, we know that before long, $y(t, \beta)$ will start to decrease, say, at the point $\theta$. We have seen that $\theta-\rho$ is small and that in $[\rho, \theta), y^{\prime}(t, \beta)>0$.

We now compare the two portions of the solution $y(t, \beta)$ in the two subintervals $[\rho, \theta]$ and $[\theta, \sigma]$. Roughly speaking, $y$ descends in $[\theta, \sigma]$ at a faster rate than it ascends in $[\rho, \theta]$. More precisely, for any two points $t_{1} \in[\rho, \theta]$ and $t_{2} \in[\theta, \sigma]$, at which $y\left(t_{1}, \beta\right)=y\left(t_{2}, \beta\right)$,

$$
\begin{equation*}
y^{\prime}\left(t_{1}, \beta\right) \leq-y^{\prime}\left(t_{2}, \beta\right) . \tag{10}
\end{equation*}
$$

To prove this, we make use of the familiar energy functional

$$
\begin{equation*}
\phi(t)=\frac{y^{\prime 2}}{2}+Q(y)-\sin (t) y, \tag{11}
\end{equation*}
$$

where

$$
\begin{equation*}
Q(y)=\int_{0}^{y} q(y, s) d y \tag{12}
\end{equation*}
$$

For convenience, the dependence of $Q$ on $s$ is not explicitly shown in the notation. As usual, the energy is differentiated along the solution curve, and we have

$$
\begin{equation*}
\phi^{\prime}(t)=-y \cos (t) \tag{13}
\end{equation*}
$$

Since $y>0$; and $\cos (t)<0, \phi(t)$ is an increasing function of $t$ in $[\rho, \sigma]$. Inequality (10) now follows from the inequality $\phi\left(t_{1}\right) \leq f\left(t_{2}\right)$.

The first inequality in (9) follows from (10) by taking $t_{1}=\rho$ and $t_{2}=\sigma$. Another consequence of (10) is that $\sigma-\theta \leq \theta-\rho$, from which (8) follows.

The uniform boundedness of $y^{\prime}(\rho, \beta)$ and the boundedness from below of $q(y, s)$ imply the uniform boundedness of $y^{\prime}(t, \beta)$ throughout the entire interval, which in turn implies the uniform boundedness of $y(t, \beta)$, which in turn implies the boundedness of $f^{\prime}(t)$, by (13). The second inequality in (9) now follows from integrating (13) over the arbitrarily short interval $[\rho, \sigma]$.

We now continue our task of establishing $z(t, \beta)$ as an approximation of $y(t, \beta)$. The functions are not to be compared in the usual way, at the same values of $t$. A translation is made to shift the graph of $y(t, \beta)$ until $\sigma_{1}$ coincides with $\tau_{1}$. The shifted copy of $y(t, \beta)$ is then compared with $z(t, \beta)$ between $\sigma_{1}$ and the next zero $\rho_{2}$ or $\tau_{2}$, whichever occurs first. The translation means also that the inhomogeneous driving forces for $y(t, \beta)$ and $z(t, \beta)$, respectively, will no longer be the same at the corresponding (but in general distinct) points where the comparison is made. But since the translation is arbitrarily small, the difference introduced will also be arbitrarily small. The process of translation and comparison is then repeated for the rest of the interval. Since $n$ is fixed, the number of translations required is fixed; we therefore have control over the total amount of error introduced.

Our next step is the shooting argument. We investigate the deformation of $y(t, \beta)$ as $\beta$ varies from $\beta_{n}-\xi$ to $\beta_{n}+\xi$. For $s$ sufficiently large, $y(t, \beta)$ is close to $z(t, \beta)$ in the sense explained above. Since $z\left(t, \beta_{n}-\xi\right)$ has no zeros in $[\pi-2 \delta]$, where $\delta$ is as in Lemma 5 , by choosing the appropriate $\epsilon$ suitably small, we can guarantee that $y\left(t, \beta_{n}-\xi\right)$ has no zero in the same interval. Similarly, $y(t, \beta)$ (for all $\beta$ ), just like $z(t, \beta)$ as asserted in Lemma 5 , has no zero in $[\pi-2 \delta, \pi-\delta]$. On the other hand, $y\left(t, \beta_{n}+\xi\right)$ has a zero close to the
$n+1^{\text {st }}$ zero of $z\left(t, \beta_{n}+\xi\right)$. This zero must be in ( $\left.\pi-\delta, \pi\right]$. Thus, as $\beta$ is varied from $\beta_{n}-\xi$ to $\beta_{n}+\xi$, there is a net gain of one (or perhaps more) zero in the interval $[\pi-2 \delta, \pi]$. Such a gain can occur only in one of three ways: some $y(t, \beta)$ touches the $t$ axis and the tangent point later develops into two zeros; some zero slides continuously in from the left endpoint $\pi-2 \delta$; or some new zero appears at $\pi$ and slides continuously to the left. The first possibility is ruled out by the fact that at any zero $\rho, y^{\prime}(\rho, \beta)$ is uniformly bounded away from zero, as established in the proof of Lemma 6 . The second possibility is ruled out by the fact that no zeros can occur in $[\pi-2 \delta, \pi-\delta]$. The third case furnishes a D -solution for some intermediate value of $\beta$. As is wellknown, the shooting argument can be put into a rigorous basis by using the implicit function theorem.

Lemma 7 The D-solution obtained above has index at least $2 n^{+}$.

Proof. The basis of the proof has already been expounded in Section 2. First the solution has $n$ zeros in $(\pi / 2, \pi)$ close to the $n$ zeros of $z(t, \beta)$. By symmetry, there are at least $2 n$ zeros in $(0, \pi)$. Each zero is associated with an interval of positivity $[\rho, \sigma$ ], as in Lemma 6, and each interval contains a local maximum. Between any two successive local maxima, there is a local minimum. Hence, there are at least $n$ subintervals of $[0, \pi / 2]$ in which $y^{\prime}(t, \beta)$ is positive, and another $n$ subintervals of $[\pi / 2, \pi]$ in which $y^{\prime}(t, \beta)$ is negative, with $y^{\prime}(t, \beta)=0$ at the endpoints of all $2 n$ subintervals. As explained in Section 2, the Sturm comparison theorem gives at least one zero of $w(t)$, the solution of the variational equation (5), in each subinterval, and the lemma is proved.

Proof of Lemma 1. Let the D-solution with index $k^{+}$in the hypothesis be denoted by $u(t, \beta)$. As shown in Section 2 of [5], the number of intersection points in $(0, \pi)$ between $u(t, \beta)$ and a nearby solution $u(t, \alpha), a \approx b$, is $k$. We claim that if $\alpha$ is decreased to a sufficiently negative value, then the number of intersection points between $u(t, \beta)$ and $u(t, \alpha)$ becomes $2 m-2$, and if $\alpha$ is increased to a sufficiently large positive value, then the number of intersection points becomes $2 m-1$. As explained in [5], the shooting argument, by tracking the number of zeros lost when $\alpha$ is varied continuously, gives $k-2 m+2$ D-solutions for $\alpha<\beta$, and $k-2 m+1 \mathrm{D}$-solutions for $\alpha>\beta$. The lemma is thus proved.

The number of intersection points is the number of zeros in $(0, \pi)$ of the function $Z(t, \alpha)=u(t, \alpha)-u(t, \beta)$, which satisfies the differential equation

$$
\begin{equation*}
Z^{\prime \prime}+\frac{f(u(t, \alpha))-f(u(t, \beta))}{u(t, \alpha)-u(t, \beta)} Z=0 \tag{14}
\end{equation*}
$$

When, $\alpha$ is close to $\pm \infty$, the fraction in the equation is close to $f^{\prime}(=-\infty)$. Without going into detail, we just point out that the dynamics of the limiting $Z($ as $\alpha \rightarrow \pm \infty)$ is similar to that of the "bounding ball" solution, except that the ball is now moving under a spring with constant $f^{\prime}(-\infty)$ and no gravitational force (equation (14) has a zero righthand side). The number of times that $Z$ hits the ground is thus $m-1$. Each of these corresponds to two zeros of $Z(t, \alpha)$ for finite $\alpha$. That there is one more zero for large positive values of $\alpha$ is due to the fact that $Z(t, \alpha)$ starts out being positive, increases rapidly for $t$ near 0 , and is quickly deflected to give a first zero. This is not the case when $\alpha$ is sufficiently negative.

To complete the proof of the main theorem, it remains to consider the general situation in which $h(t)$ is nontrivial. Standard perturbation techniques are more than enough for the purpose.

Examining the proof of our main result reveals that the assumption $f^{\prime}(\infty)=\infty$ is needed only in Lemma 6 to show that the duration $[\rho, \theta]$ in which $y(t, \beta)$ is positive and increasing is arbitrarily short. In $[\rho, \theta], y(t, \beta)$ satisfies the differential inequality

$$
\begin{equation*}
y^{\prime \prime}+\frac{f(s y)}{s} \leq 1 \tag{15}
\end{equation*}
$$

The usual trick of multiplying the inequality by $y^{\prime}$ and integrating will furnish a bound on $y^{\prime 2}$, from which a bound on $\theta-\rho$ can be derived in terms of the indefinite integral of the function $f(u)$. Roughly speaking, Theorem 1 still holds if the integral of $f(u)$ grows fast enough.

## References

[1] Lazer, A. C., and McKenna, P. J., Large-amplitude periodic oscillation in suspension bridges: Some new connections with nonlinear analysis, SIAM Review, 32 (1990), 537-578
[2] Lazer, A. C., and McKenna, P. J., On a conjecture related to the number of solutions of a nonlinear Dirichlet problem, Proc. Royal Soc. of Edinburgh, 95A (1983), 275-283.
[3] Coffman, C. V., On the positive solutions of boundary value problems for a class of nonlinear differential equations, J. Diff. Eq., 3 (1967), 92-111.
[4] Clemons, C. B., Uniqueness results for semilinear elliptic equations, Ph. D. Dissertation, University of Maryland, 1990.
[5] Kaper, H. G., and Kwong, Man Kam, On a conjecture concerning the multiplicity of a Dirichlet problem, Preprint MCS-P211-0191, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Illinois.
[6] Kwong, Man Kam, On the Kolodner-Coffman method for the uniqueness problem of Emden-Fowler BVP, ZAMP (J. of Applied Math. and Phy.), 41 (1990), 79-104.
[7] Kwong, Man Kam, Uniqueness of positive solutions of $\Delta u-u+u^{p}=0$ in $R^{n}$, Arch. Rational Mech. Anal., 105 (1989), 243-266.
[8] Kwong, Man Kam, and Li, Yi, Uniqueness of radial solutions of semilinear elliptic equations, Preprint MCS-P156-0590, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Illinois (to appear in Trans. Amer. Math. Soc.).
[9] Kwong, Man Kam, and Zhang, L., Uniqueness of the positive solution of $\Delta u+f(u)=0$ in an annulus, Preprint MCS-P117-1289, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Illinois (to appear in Differential and Integral Equations).
[10] McLeod, K., and Serrin, J., Uniqueness of positive radial solutions of $\Delta u+f(u)=0$ in $R^{n}$, Arch. Rational Mech. Anal., 99 (1987), 115-145.
[11] Ni, W. M., and Nussbaum, R., Uniqueness and nonuniqueness for positive radial solutions of $\Delta u+f(u, r)=0$, Comm. Pure and Appl. Math., 38 (1985), 69-108.


[^0]:    *This work was supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U. S. Department of Energy, under Contract W-31-109-Eng-38.

