Automated Discovery of New Axiomatizations of the Left Group and Right Group Calculi[1]

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Abstract. This paper shows how the automated theorem-proving program OTTER was used to discover new axiomatizations, including single axioms, for the left group and right group calculi. J. A. Kalman's original axiomatizations of the two calculi each contain five axioms. Three of Kalman's axioms (L1, L4, and L5) for the left group calculus were shown to be dependent on the remaining two axioms. Four of Kalman's axioms (R1, R3, R4, and R5) for the right group calculus were shown to be dependent on the remaining axiom. Alternative simpler axiomatizations were discovered for both calculi, including a single axiom for the left group calculus and five additional single axioms for the right group calculus. The program OTTER was vital in discovering candidate axiomatizations as well as in finding proofs of new axiomatizations. All of the relevant OTTER proofs are included.

Key Words. Automated deduction, condensed detachment, group calculi, single axioms.

1 Introduction

In [4], J. A. Kalman presents the following deductive axiomatization of left group tautologies, which we call the left group (LG) calculus:

$$E(E(E(x,E(E(y,y),x)),z),z) \tag{L1}$$
$$E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u) \tag{L2}$$
$$E(E(E(E(E(E(x,y),E(x,z)),u),E(E(y,z),u)),v),v) \tag{L3}$$
$$E(E(E(E(x,y),z),u),E(E(E(x,v),z),E(E(y,v),u))) \tag{L4}$$
$$E(E(E(x,E(E(y,x),z)),E(E(u,x),v)),E(E(E(E(x,y),u),z),v)) \tag{L5}$$

The term $E(\alpha,\beta)$ corresponds to left division $\alpha^{-1}\beta$ in groups. The axiomatization is deductive in the sense that the only rules of inference are substitution and detachment (modus ponens).

An equality axiomatization of group theory, in terms of the same operation E, is [2, 9]

$$E(E(x,E(y,y)),E(z,z)) = x \tag{D1}$$
$$E(E(x,y),E(x,z)) = E(y,z). \tag{D2}$$

Kalman proves that a formula $E(\alpha,\beta)$ follows from (L1,...,L5) by substitution and detachment (i.e., is a theorem in the LG calculus) if and only if $\alpha=\beta$ follows from (D1) and (D2)

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by equality deduction.

D. Scott observed [17] that the key steps of Kalman's proofs are first order and suggested that the program OTTER [11] search for proofs of the those steps. After I experimented with various search strategies, OTTER found the desired proofs. Scott also suggested searching for simpler axiomatizations of the LG calculus. Nine new and simpler axiomatizations were discovered. One of the new axiomatizations consists of a single axiom (of length 27), and two of the others are shown to consist of independent axioms. Section 4 shows how OTTER was used in discovering the new axiomatizations.

Kalman also gives [4] an axiomatization of the right group (RG) calculus, in which the operation $E(\alpha, \beta)$ is right division $\alpha\beta^{-1}$. Six single axioms (each of length 15) were discovered. Section 5 describes the use of OTTER in obtaining those results.

C. A. Meredith's *condensed detachment* [10, 15] is an inference rule that combines substitution and detachment by making use of unification. If $E(\alpha, \beta)$ and γ are both theorems, and if α and γ unify with most general unifier σ, then one can infer $\beta\sigma$. The formula $E(\alpha, \beta)$ is the *major premise*, and γ is the *minor premise*. Every formula that can be derived by detachment and substitution either can be derived by condensed detachment or is a substitution instance of a formula that can be derived by condensed detachment [8].

The LG and RG calculi (as well as other logic calculi) can be studied as first-order theories [8]. Each axiom of the calculus becomes the argument of a unary predicate symbol P, meaning "is a theorem", and condensed detachment becomes an axiom of the theory:

$$\forall x \forall y (P(E(x, y)) \,\&\, P(x) \;\rightarrow\; P(y)).$$

In the context of the LG calculus, condensed detachment can be interpreted as "if $x^{-1}y$ is the identity and x is the identity, then y is the identity".

An application of the inference rule hyperresolution [20] with the axiom condensed detachment as nucleus corresponds to an application of the inference rule condensed detachment.

All of the condensed detachment proofs, which make up the vast majority of the collection of proofs related to this study, were discovered by the program OTTER. However, the following key operations were *not* carried out by OTTER:

- Some of the candidates for axiom systems were chosen by the user.
- The user specified the search strategies and experimented with different search strategies.
- The two independence results in Section 6 are not first order and were not proved by OTTER. However, the proofs involve a simple induction argument that was suggested by examining the output of OTTER jobs.

Section 3 describes the features of OTTER and the search strategies that were used in the study.

The conclusion summarizes of the results of the study. A summary of the results also appears in [12].

2 Related Calculi and Related Work

The theorems of the classical equivalential calculus (EC) [18] are the formulas constructed from one binary operator and variables in which each variable has an even number of occurrences. If the EC operation is interpreted as the group operation, then the theorems of EC are the formulas that are equal to the identity in Boolean groups (groups in which the square of every element is the identity). The L calculus [13] is also of interest. If the L calculus operation is interpreted as $\alpha^{-1}\beta$ in groups, then the L theorems are the formulas equal to the identity in Abelian groups [3]. The relationship among the LG calculus, the L calculus, and EC is the following:

LG theorems $\subset$ L theorems $\subset$ EC theorems.

An analogous relationship exists among the RG calculus, the R calculus (with operation corresponding to $\alpha\beta^{-1}$), and EC:

RG theorems $\subset$ R theorems $\subset$ EC theorems.

A formula has the *2-property* [1, 7] if each of its variables has exactly two occurrences. A calculus has the 2-property if it has an axiomatization in which each axiom has the 2-property. EC and the L, R, LG, and RG calculi have the 2-property. Condensed detachment preserves the 2-property. Every theorem in a calculus with the 2-property either has the 2-property or is an instance of a theorem with the 2-property. In the remainder of this paper, when I write “LG theorem”, I generally mean “LG theorem with the 2-property”.

Note that L5 in Kalman's axiomatization of the LG calculus does not have the 2-property. However, L5 can be generalized and replaced with L5a, also an LG theorem, which does have the 2-property.

$$P(E(E(E(x,E(E(y,x),z)),E(E(u,x),v)),E(E(E(E(x,y),u),z),v))) \tag{L5}$$
$$P(E(E(E(x,E(E(y,w),z)),E(E(u,x),v)),E(E(E(E(w,y),u),z),v))) \tag{L5a}$$

There has been much interest in axiomatizations of those calculi, particularly in single axioms and other simple axiom systems [18, 13, 3, 4, 16, 22, 19, 21]. The more recent work has made heavy use of theorem-proving programs to search for and to find proofs and more general automated reasoning programs to help search for counterexamples and to help find independence proofs.

Single axioms were known for EC [18, 6, 16, 21], the L calculus [13, 7], and the R calculus [3, 7]. Prior to the work reported in this paper and in [12], no single axioms were known for the LG calculus or the RG calculus. The new single axioms answer questions raised by C. A. Meredith and A. N. Prior in [14, p. 222].

3 Otter

Otter [11] is a resolution/paramodulation theorem-proving program for first-order logic with equality. Its basic algorithm restricted to hyperresolution with condensed detachment is shown in Figure 1.

Start with *sos* list containing all axioms and with *usable* list empty.
Loop:

1. G = select-given-clause(*sos*);
2. move G from *sos* to *usable*;
3. apply condensed detachment as much as possible, with G as one premise, taking the other premise from *usable*; append to *sos* the results that are not subsumed by anything in *sos* or *usable*;

end loop.

Figure 1: OTTER's Basic Algorithm with Condensed Detachment

The computer on which the jobs were run is a SPARCstation 1+ with 16 MB of memory. In that environment, OTTER can deduce several thousand results per second (most of which are subsumed and deleted) and store about 20,000 theorems.

In this study (as in most other studies with OTTER) it was necessary to run many jobs, varying the axioms, search strategy, and other parameters. The following subsections show features of OTTER that were particularly useful.

3.1 Selecting the Given Clause

Selection of the given clause G from *sos* in the first step of the loop has a great effect on the search. Three methods were used:

(1) Select the smallest (fewest symbols) theorem in *sos*. If there is more than one of minimum length, select the first of those.

(2) Select the first theorem in *sos* (first-in-first-out).

(3) The user specifies a ratio n. Through n iterations of the main loop, the smallest theorem is selected; in the next iteration, the first theorem is selected; then through the following n iterations, the smallest theorem is selected; etc. This method allows large theorems to enter into the search while focusing mainly on small theorems.

Each of the three methods is complete when used with the basic algorithm: given unlimited time and space, every pair of *sos* theorems will be considered for application of condensed detachment, and every theorem will either be inferred or be an instance of one that is inferred. Kalman used similar strategies in one of his programs in 1976 [5].

3.2 Discarding Complex Theorems

Memory limitations forced the use of the following technique for conserving memory. Start with an initial bound $4n$, for some small n, on the length of kept clauses. If *sos* becomes empty, increase the bound by 4 and restart the job, etc. (n is the number of distinct variables. All 2-property EC theorems, and therefore all 2-property L, R, LG, and RG theorems, have $4n-1$ symbols, excluding parentheses, commas, and the predicate symbol P. OTTER includes P in the symbol count.)

Without this technique, the *sos* list typically grows very fast, much faster than given clauses are removed from it. Since most *sos* clauses—especially big ones—never enter the search, memory is wasted. When the smallest theorem is always selected as the given clause (method 1), this technique with a particular bound generates the same clauses in the same order as without a bound, until the *sos* list becomes empty (with a bound) or memory is exhausted (without). If the given clause is selected by either of the other methods, then use of a bound can alter the search by preventing the entrance of big theorems.

In several cases, the following refinement was used. The search is started with a high bound; then the bound is reduced to a specified value after a specified number of given clauses has been selected and used. The refinement enables the search to take advantage of the power and richness of some big formulas; then it prevents other big formulas from wasting memory.

3.3 Discarding Theorems with Instances of $E(x,x)$

In previous experiments on EC and various implicational calculi, L. Wos and I observed that theorems containing subformulas that are instances of $E(x,x)$ (or $i(x,x)$, or whatever the name of the operator) are generally not as powerful as those of the same length without such instances. In addition, most theorems in the calculi we have studied do have such instances. (Of the 560 EC theorems of length 15 with the 2-property, 501 have instances of $E(x,x)$.) Although many OTTER proofs contain formulas with instances of $E(x,x)$, we found that OTTER could discover proofs of those theorems without such instances. When OTTER was directed to discard such formulas, it usually found proofs much faster and with much smaller search spaces, although the proofs are sometimes longer. I do not know to what extent this strategy is incomplete. (Lukasiewicz was interested in *organic* theorems [18], which do not contain theorems as subformulas. Theorems without instances of $E(x,x)$ are in the spirit of organic theorems.)

The strategy can be implemented for OTTER by including the following list of rewrite rules.

$E(x,x) = junk.$
$E(junk,x) = junk.$
$E(x,junk) = junk.$
$P(junk) = \$T.$

Any formula containing an instance of $E(x,x)$ rewrites to the tautology $\$T$, which is immediately discarded. (Tricks like this can cause inconsistency. In this case, one can verify a proof by checking that the proof does not depend on any of the rewrite rules.)

3.4 Finding More Than One Refutation

One can tell OTTER not to stop its search if it finds a refutation. Different proofs of a theorem and proofs of different theorems can be found in a single OTTER search.

One use of this ability is the following. If the goal is to prove $\alpha\&\beta$, input three denials: $\neg\alpha$, $\neg\beta$, and $\neg(\alpha\&\beta)$. If OTTER fails to prove both, it is very useful to know whether OTTER proved one of them. If OTTER *does* prove both, it is useful to have separate proofs of α and β to easily see the axioms on which each proof depends. This feature was used extensively in all of the experiments.[2]

The ability to find and report more than one proof was also vital to the search for new axiomatizations. When experimenting with a candidate axiomatization for the LG calculus, denials of many LG theorems were included in the input along with known axiomatizations. If a search failed to derive a known axiomatization but proved many interesting LG theorems, the candidate axiomatization was studied further.

3.5 Common OTTER Features *Not* Used

Back subsumption is an OTTER option that causes previously kept clauses to be deleted if they are subsumed by a newly kept clause. Back subsumption was not used for the experiments, because initial experiments showed that it consumed substantial time and had little or no effect on the results.

Users of OTTER can input weight templates for measuring the complexity of formulas. For example, one can penalize formulas with a certain structure and cause them to be deleted or cause their selection as given clauses to be delayed. Weight templates were not used in the experiments.

Finally, the set of support strategy was not used. The initial *sos* list always contained all axioms.

3.6 Notes on OTTER Proofs

- Variables are s, t, u, v, w, x, y, z. (This is different from the normal OTTER convention of having variables start with u–z.)
- The justification for the condensed detachment inferences is given as a triple of integers [i,j,k]. Clause i is condensed detachment, j is the major premise, and k is the minor premise.
- All of the proofs were regenerated during the writing of this paper to make them more uniform, but they are essentially the same as the original proofs. The clause numbers are different from those in the original proofs and do not necessarily reflect the sizes of the searches.

[2] To OTTER users: Another reason for including the denials of α and β separately is that the proof for the combined denial might be delayed a long time after the individual proofs are reported. Refutation with a unit clause (unit conflict) is reported immediately, but refutation with a nonunit must be by hyperresolution; in particular, all parts must first be selected as given clauses.

4 The Left Group Calculus

4.1 Enumerating LG Theorems

At several points in the study, I wished to have all LG theorems with the 2-property with n variables. (Such formulas have length $4n-1$, where length includes variables and E, but not commas, parentheses, or P.) To build those sets, OTTER was used as a symbolic calculator rather than as a theorem prover.

First, the set of formulas with the 2-property (all of which are EC theorems) was constructed by considering all strings of variables with the 2-property of length $2n$, deleting alphabetic variants, and then, for each of those, building all associations with binary functor E. For example, if $n = 3$, there are 15 variable strings of length 6, and each can be associated in 42 ways, so the result is 630 formulas.

Then, the following procedure was used to decide which of the EC theorems are also LG theorems. Rewrite an EC theorem with the following rewrite rules (demodulators) until no rewrite rule can be applied:

$$
\begin{array}{ll}
f(e,x) = x & g(g(x)) = x \\
f(x,e) = x & f(y, f(g(y),x)) = x \\
f(g(x),x) = e & f(g(y), f(y,x)) = x \\
f(x,g(x)) = e & g(f(y,x)) = f(g(x),g(y)) \\
f(f(x,y),z) = f(x,f(y,z)) & \\
g(e) = e & E(x,y) = f(g(x),y)
\end{array}
$$

The formula is an LG theorem if and only if the result is the identity e. (The first ten rewrite rules are the complete set of reductions for free groups.)

Table 1 gives the sizes of the sets of interest. The sets "LG $-$ $E(x,x)$" are obtained from "LG 2-property" by deleting those containing instances of $E(x,x)$. The set corresponding to the position marked with "?" was not constructed, and its size is not known.

Table 1: Sizes of Formula Sets by Number of Variables

Number n of variables	1	2	3	4	5
Length of formulas	3	7	11	15	19
Variable strings	1	3	15	105	945
Associations of $2n$	1	5	42	429	4862
EC 2-property	1	15	630	45045	4594590
LG 2-property	1	4	40	560	?
LG $-$ $E(x,x)$	0	0	4	59	800

4.2 Discovering Dependencies among L1–L5

The first OTTER searches for dependencies among L1–L5 failed. Each of the five was considered in turn, and a proof of that one was sought from the remaining four. The only strategies used were to select the given clause by symbol count and to discard complex theorems. Many hours of CPU time were consumed, and no dependencies were found. The main reason for the failures is that key formulas were excluded from the search (were not selected as given clauses) because they are too complex. Selection of given clauses by the ratio strategy (Section 3.1), which allows complex formulas to enter the search, led to the first successes and was used for most of the remaining experiments in the study.

4.2.1 The Dependence of L1

The dependence of L1 on {L2,L3,L4} was discovered by accident when seeking a replacement for L1 in the L1–L5 axiomatization. A job was set up starting with L2–L5, several LG calculus lemmas (L6–L11) from Kalman's paper [4], and the denial of L1. Given clauses were selected with ratio 2, and the following proof of L1 was found in 179 seconds:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
2		$P(E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u))$	(L2)
3		$P(E(E(E(E(E(E(x,y),E(x,z)),u),E(E(y,z),u)),v),v))$	(L3)
4		$P(E(E(E(E(x,y),z),u),E(E(E(x,v),z),E(E(y,v),u))))$	(L4)
19	[1,4,3]	$P(E(E(E(E(E(E(x,y),E(x,z)),u),v),w),E(E(E(E(y,z),u),v),w)))$	
20	[1,4,2]	$P(E(E(E(E(E(x,y),E(x,z)),u),v),E(E(E(y,z),u),v)))$	
137	[1,19,3]	$P(E(E(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))),v),v))$	
9176	[1,20,137]	$P(E(E(E(x,E(E(y,y),x)),z),z))$	(L1)

Although the proof is short, it is difficult for OTTER to find without the ratio strategy, because there are many paths involving less complex formulas to explore. Note that the proof of L1 depends only on L2–L4.

4.2.2 The Dependence of L4

I do not have a record of the motivation for the search that discovered the dependence of L4 on {L2,L3}. I believe the job was a shot in the dark to see what could be derived from {L2,L3,L5} and Kalman's lemmas L6–L11. In any case, given clauses were selected with ratio 2. The following proof of L4 was found in 3 seconds:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
2		$P(E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u))$	(L2)
3		$P(E(E(E(E(E(E(x,y),E(x,z)),u),E(E(y,z),u)),v),v))$	(L3)
15	[1,3,3]	$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$	
16	[1,3,2]	$P(E(E(x,y),E(x,y)))$	
20	[1,3,15]	$P(E(E(x,y),E(E(E(E(z,u),E(z,v)),x),E(E(u,v),y))))$	
24	[1,15,16]	$P(E(E(x,y),E(E(z,x),E(z,y))))$	
33	[1,3,24]	$P(E(E(x,E(E(E(y,z),E(y,u)),v)),E(x,E(E(z,u),v))))$	
508	[1,33,20]	$P(E(E(E(E(x,y),z),u),E(E(E(x,v),z),E(E(y,v),u))))$	(L4)

The preceding two proofs show that L1 and L4 together are dependent on L2 and L3. It was later learned that L4 can be derived from L3 alone, as the following proof demonstrates:

1 $\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$ (CD)
2 $P(E(E(E(E(E(E(x,y),E(x,z)),u),E(E(y,z),u)),v),v))$ (L3)
26 [1,2,2] $P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$
30 [1,2,26] $P(E(E(x,y),E(E(E(E(z,u),E(z,v)),x),E(E(u,v),y))))$
43 [1,30,2] $P(E(E(E(E(x,y),E(x,z)),E(E(E(E(E(u,v),E(u,w)),s),$
$E(E(v,w),s)),t)),E(E(y,z),t)))$
292 [1,43,30] $P(E(E(E(E(x,y),z),u),E(E(E(x,v),z),E(E(y,v),u))))$ (L4)

4.2.3 The Dependence of L5

The study then turned to finding a simple replacement for L5. Formula L5 has length 23. A job was set up to collect LG theorems of lengths 15 and 19 to serve as candidate replacements for L5. Axioms L1–L5 were input, and given clauses were selected with ratio 2. The job was run for several minutes, and all LG theorems of length 15 (390 theorems) and length 19 (284 theorems) were extracted from the output.

Rather than try each of those 674 theorems as a possible replacement for L5, the entire set, along with L1–L4 and the 44 LG theorems of length 7 and 11 (Table 1), was input in an attempt to derive L5. Given clauses were selected with ratio 2. The following proof of L5a (a generalization of L5) was found in about 10 hours. (The clause numbers have no relationship to the size of the search in this proof.)

1 $\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$ (CD)
48 $P(E(E(x,y),E(E(z,x),E(z,y))))$ (Q2)
52 $P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$ (P1)
164 $P(E(E(E(x,y),E(E(z,x),u)),E(E(z,y),u)))$ (N3)
178 $P(E(E(E(E(x,y),z),E(y,u)),E(z,E(x,u))))$ (N4)
357 $P(E(E(x,E(E(y,z),u)),E(E(E(z,y),x),u)))$ (N5)
391 $P(E(x,E(y,E(E(E(E(z,u),y),E(u,z)),x))))$ (N6)
740 [1,52,164] $P(E(E(x,E(E(y,z),u)),E(E(E(z,v),x),E(E(y,v),u))))$
741 [1,48,164] $P(E(E(x,E(E(y,z),E(E(u,y),v))),E(x,E(E(u,z),v))))$
795 [1,48,357] $P(E(E(x,E(y,E(E(z,u),v))),E(x,E(E(E(u,z),y),v))))$
822 [1,178,391] $P(E(x,E(y,E(E(E(E(z,u),v),E(u,z)),E(E(y,v),x)))))$
903 [1,795,740] $P(E(E(x,E(E(y,z),u)),E(E(E(v,y),E(E(z,v),x)),u)))$
923 [1,741,822] $P(E(x,E(E(E(E(y,z),u),v),E(E(E(E(z,y),v),u),x))))$
1018 [1,903,923] $P(E(E(E(x,E(E(y,z),u)),E(E(v,x),w)),$
$E(E(E(E(z,y),v),u),w)))$ (L5a)

OTTER had found an axiomatization of the LG calculus in which all axioms are smaller than L5, namely, {L2,L3,Q2,P1,N3,N4,N5,N6}.

It was surprising to me that just 6 of the 722 input formulas occur in the proof, especially since all input clauses had actively participated in the search (had been selected as given clauses). I observed that none of the six formulas contains an instance of $E(x,x)$, and I guessed that formulas with such instances are somehow weaker and not as useful in finding proofs. (That guess is consistent with observations made in collaboration with L. Wos in other logic calculi.) It was then determined that 638 of the 722 formulas contain instances of $E(x,x)$.

A job similar to the preceding one was then set up; the difference was that derived formulas containing instances of $E(x,x)$ were discarded (Section 3.3). A proof similar to the preceding proof was found in about 36 minutes. This (incomplete) deletion strategy, along with the ratio strategy for selecting given clauses, was indispensible for the remainder of the study.

The next step was to see which of {Q2,P1,N3,N4,N5,N6} could be derived from L1–L4 (and therefore from {L2,L3}). Given clauses were selected with ratio 2, and the following proof of all of the goals was found within 17 seconds:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
3		$P(E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u))$	(L2)
4		$P(E(E(E(E(E(E(x,y),E(x,z)),u),E(E(y,z),u)),v),v))$	(L3)
5		$P(E(E(E(E(x,y),z),u),E(E(E(x,v),z),E(E(y,v),u))))$	(L4)
30	[1,4,4]	$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$	(P1)
36	[1,5,3]	$P(E(E(E(E(E(x,y),E(x,z)),u),v),E(E(E(y,z),u),v)))$	
38	[1,5,30]	$P(E(E(E(x,y),z),E(E(u,y),E(E(v,u),E(E(x,v),z)))))$	
40	[1,3,30]	$P(E(E(x,E(y,z)),E(E(E(y,u),x),E(u,z))))$	
77	[1,3,36]	$P(E(E(E(x,y),E(E(z,x),u)),E(E(z,y),u)))$	(N3)
85	[1,40,77]	$P(E(E(E(E(x,y),z),E(E(u,y),E(E(x,u),v))),E(z,v)))$	
90	[1,77,40]	$P(E(E(E(x,y),E(x,z)),E(y,z)))$	
178	[1,90,38]	$P(E(x,E(E(y,z),E(E(z,y),x))))$	
203	[1,77,178]	$P(E(E(x,y),E(E(z,x),E(z,y))))$	(Q2)
235	[1,40,203]	$P(E(E(E(E(x,y),z),E(y,u)),E(z,E(x,u))))$	(N4)
245	[1,3,203]	$P(E(E(x,E(E(y,z),E(y,u))),E(x,E(z,u))))$	
392	[1,235,178]	$P(E(x,E(y,E(E(z,u),E(E(y,E(u,z)),x)))))$	
795	[1,245,392]	$P(E(x,E(y,E(E(E(E(z,u),y),E(u,z)),x))))$	(N6)
1876	[1,85,77]	$P(E(E(E(x,y),E(E(y,x),z)),z))$	
1924	[1,30,1876]	$P(E(E(x,E(E(y,z),u)),E(E(E(z,y),x),u)))$	(N5)

All of the goals had been achieved, showing that {L2,L3} axiomatizes the LG calculus. The total number of symbols had been reduced from 87 (in L1–L5) to 34.

The obvious next step was to see whether either of {L2,L3} could be derived from the other. Nothing can be derived from L2 alone, which answers half of the question. A search of nearly 16 hours starting from L3 derived L4 and P1 but nothing else of obvious interest.

4.3 New Multiformula Axiomatizations of LG

The next sequence of experiments was to search for axiomatizations of the LG calculus with fewer symbols than in {L2,L3}. In all of the Otter jobs described in this subsection, given clauses were selected with ratio 2, and derived formulas containing instances of $E(x,x)$ were discarded.

The first job was to search for a small replacement for L3 (similar to the search for a replacement for L5). Starting with L2, the 44 LG theorems of lengths 7 and 11 (Table 1), and the 390 LG theorems of length 15 (Section 4.2.3), the following proof of L3 was derived in about 20 minutes. (The clause numbers have no relationship to the size of the search in this proof.)

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
49		$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$	(P1)
61		$P(E(E(x,E(y,z)),E(E(E(y,u),x),E(u,z))))$	(P2)
363		$P(E(E(E(x,E(y,z)),E(x,E(E(z,y),u))),u))$	(P3)
417		$P(E(x,E(E(E(E(y,z),E(y,u)),E(z,u)),x)))$	(P4)
464	[1,61,49]	$P(E(E(E(E(x,y),z),E(E(u,y),v)),E(z,E(E(u,x),v))))$	
469	[1,49,363]	$P(E(E(x,E(y,E(E(z,u),v))),E(E(E(y,E(u,z)),x),v)))$	
482	[1,464,417]	$P(E(x,E(E(E(E(y,z),E(y,u)),v),E(E(v,E(z,u)),x))))$	
497	[1,469,482]	$P(E(E(E(E(E(E(x,y),E(x,z)),u),E(E(y,z),u)),v),v))$	(L3)

An axiomatization of LG had been found in which all axioms have 15 symbols, namely, {L2,P1,P2,P3,P4}. (Again I was surprised that so few of the input formulas appear in the proof.)

The next three searches derived P4 from {L2,P1} in 37 seconds, derived P3 from {L2,P1,P2} in 9 seconds, and derived P2 from {L2,P1} in less than 1 second, establishing that {L2,P1} axiomatizes LG. The following proof (obtained later) shows derivations of P2, P3, and P4 from {L2,P1}:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
2		$P(E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u))$	(L2)
3		$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$	(P1)
33	[1,2,3]	$P(E(E(x,E(y,z)),E(E(E(y,u),x),E(u,z))))$	(P2)
36	[1,33,33]	$P(E(E(E(E(E(x,y),z),u),E(z,E(x,v))),E(u,E(y,v))))$	
38	[1,2,33]	$P(E(E(E(x,y),E(E(z,x),E(z,u))),E(y,u)))$	
39	[1,33,3]	$P(E(E(E(E(x,y),z),E(E(u,y),v)),E(z,E(E(u,x),v))))$	
40	[1,33,2]	$P(E(E(E(x,y),E(E(E(E(z,u),E(z,v)),E(u,v)),E(x,w))),E(y,w)))$	
63	[1,36,2]	$P(E(E(E(x,y),E(E(z,x),u)),E(E(z,y),u)))$	
81	[1,33,63]	$P(E(E(E(E(x,y),z),E(E(u,y),E(E(x,u),v))),E(z,v)))$	
87	[1,39,3]	$P(E(x,E(E(y,z),E(E(z,y),x))))$	
90	[1,63,87]	$P(E(E(x,y),E(E(z,x),E(z,y))))$	
93	[1,39,87]	$P(E(x,E(E(y,z),E(E(u,y),E(E(z,u),x)))))$	
98	[1,33,87]	$P(E(E(E(E(x,y),z),u),E(z,E(E(y,x),u))))$	
113	[1,90,90]	$P(E(E(x,E(y,z)),E(x,E(E(u,y),E(u,z)))))$	
147	[1,36,93]	$P(E(x,E(y,E(E(z,u),E(E(E(E(u,v),y),E(v,z)),x)))))$	
207	[1,98,38]	$P(E(E(E(x,y),E(x,z)),E(E(u,y),E(u,z))))$	
219	[1,40,113]	$P(E(E(E(x,y),z),E(E(E(u,x),E(u,y)),z)))$	
1091	[1,81,63]	$P(E(E(E(x,y),E(E(y,x),z)),z))$	
1104	[1,219,1091]	$P(E(E(E(x,E(y,z)),E(x,E(E(z,y),u))),u))$	(P3)
1106	[1,207,1091]	$P(E(E(x,E(E(y,z),E(E(z,y),u))),E(x,u)))$	
3098	[1,1106,147]	$P(E(x,E(E(E(E(y,z),E(y,u)),E(z,u)),x)))$	(P4)

Attempts to show that {L2,P2} and {L2,P3} axiomatize LG failed, but the following proof of P1 from {L2,P4}, obtained in less than one second, shows that {L2,P4} axiomatizes LG:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
2		$P(E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u))$	(L2)
3		$P(E(x,E(E(E(E(y,z),E(y,u)),E(z,u)),x)))$	(P4)
31	[1,3,3]	$P(E(E(E(E(x,y),E(x,z)),E(y,z)),$ $E(u,E(E(E(E(v,w),E(v,s)),E(w,s)),u))))$	
32	[1,2,3]	$P(E(E(E(E(x,y),E(x,z)),E(y,z)),E(E(E(u,v),E(u,w)),E(v,w))))$	
34	[1,2,32]	$P(E(E(E(x,y),E(x,z)),E(y,z)))$	
36	[1,34,31]	$P(E(E(x,y),E(E(E(E(z,u),E(z,v)),E(u,v)),E(E(w,x),E(w,y)))))$	
38	[1,2,36]	$P(E(E(E(E(x,y),E(x,z)),E(y,z)),$ $E(E(u,E(E(v,w),E(v,s))),E(u,E(w,s)))))$	
62	[1,2,38]	$P(E(E(x,E(E(y,z),E(y,u))),E(x,E(z,u))))$	
86	[1,62,36]	$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$	(P1)

Section 6 contains a proof that {L2,P4} is independent.

Each of L2, P1, and P4 has length 15. The next sequence of experiments was a successful attempt to replace P1 (or P4) with shorter axioms. Starting with L2 and the 44 LG theorems of lengths 7 and 11 (Table 1), Otter found the following proof of P1 in about 14 seconds:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
2		$P(E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u))$	(L2)
39		$P(E(x,E(E(y,z),E(E(z,y),x))))$	(Q1)
46		$P(E(E(x,y),E(E(z,x),E(z,y))))$	(Q2)
79	[1,46,46]	$P(E(E(x,E(y,z)),E(x,E(E(u,y),E(u,z)))))$	
81	[1,2,46]	$P(E(E(x,E(E(y,z),E(y,u))),E(x,E(z,u))))$	
82	[1,46,39]	$P(E(E(x,y),E(x,E(E(z,u),E(E(u,z),y)))))$	
95	[1,46,81]	$P(E(E(x,E(y,E(E(z,u),E(z,v)))),E(x,E(y,E(u,v)))))$	
98	[1,81,79]	$P(E(E(E(E(x,y),z),E(y,u)),E(z,E(x,u))))$	
113	[1,81,82]	$P(E(E(E(E(x,y),z),u),E(z,E(E(y,x),u))))$	
248	[1,98,2]	$P(E(E(E(x,y),z),E(E(E(u,x),E(u,y)),z)))$	
490	[1,113,248]	$P(E(x,E(E(y,z),E(E(E(u,z),E(u,y)),x))))$	
1237	[1,95,490]	$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$	(P1)

Thus, {L2,Q1,Q2} axiomatizes the LG calculus. Note that each of Q1 and Q2 has 11 symbols. Starting from just {L2,Q1}, nothing interesting could be derived. Starting with {L2,Q2}, the theorems P2, P3, N3, N4, N5, and several of Kalman's lemmas were derived, but not enough to show that {L2,Q2} axiomatizes LG.

The system was becoming narrower (shorter axioms) but longer (more axioms). The next move was to back up to the axiomatizations {L2,P1} and {L2,P4} and try to replace L2 with simpler axioms. A search starting with P1 and the 45 LG theorems of lengths 3, 7, and 11 (Table 1) produced the following proof of L2 in about 19 seconds:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
2		$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$	(P1)
32		$P(E(E(E(x,y),E(E(y,x),z)),z))$	(Q3)
39		$P(E(x,E(E(y,z),E(E(z,y),x))))$	(Q1)
44		$P(E(E(E(x,y),E(x,z)),E(y,z)))$	(Q4)
46		$P(E(E(x,y),E(E(z,x),E(z,y))))$	(Q2)
76	[1,2,32]	$P(E(E(x,E(E(y,z),u)),E(E(E(z,y),x),u)))$	
83	[1,2,44]	$P(E(E(x,E(y,z)),E(E(E(y,u),x),E(u,z))))$	
94	[1,46,76]	$P(E(E(x,E(y,E(E(z,u),v))),E(x,E(E(E(u,z),y),v))))$	
110	[1,83,39]	$P(E(E(E(E(x,y),z),u),E(z,E(E(y,x),u))))$	
111	[1,83,32]	$P(E(E(E(x,y),E(E(z,u),E(E(u,z),E(x,v)))),E(y,v)))$	
382	[1,110,2]	$P(E(x,E(E(y,z),E(E(u,y),E(E(z,u),x)))))$	
808	[1,94,382]	$P(E(x,E(E(E(y,z),E(y,u)),E(E(u,z),x))))$	
1421	[1,94,808]	$P(E(x,E(E(E(y,z),E(E(u,y),E(u,z))),x)))$	
1781	[1,111,1421]	$P(E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u))$	(L2)

The preceding proof shows that {P1,Q1,Q2,Q3,Q4} axiomatizes LG.

Axiom P1 was then tried with each of Q1–Q4. The search starting with {P1,Q3} was successful, deriving Q1, Q2, and Q4 in less than 3 seconds:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
2		$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$	(P1)
3		$P(E(E(E(x,y),E(E(y,x),z)),z))$	(Q3)
9	[1,2,2]	$P(E(E(x,y),E(E(E(z,u),x),E(E(v,u),E(E(z,v),y)))))$	
10	[1,2,3]	$P(E(E(x,E(E(y,z),u)),E(E(E(z,y),x),u)))$	
22	[1,3,9]	$P(E(E(x,y),E(E(z,x),E(z,y))))$	(Q2)
28	[1,9,2]	$P(E(E(E(x,y),E(E(z,u),v)),$ $E(E(w,y),E(E(x,w),E(E(s,u),E(E(z,s),v))))))$	
30	[1,10,22]	$P(E(E(E(x,y),E(x,z)),E(y,z)))$	(Q4)
39	[1,22,30]	$P(E(E(x,E(E(y,z),E(y,u))),E(x,E(z,u))))$	
118	[1,39,30]	$P(E(E(E(x,E(y,z)),E(x,E(y,u))),E(z,u)))$	
611	[1,118,28]	$P(E(x,E(E(y,z),E(E(z,y),x))))$	(Q1)

The preceding proof shows that {P1,Q3} axiomatizes LG. Searches starting with {P1,Q1} and {P1,Q2} derived nothing interesting. The search starting with {P1,Q4} derived Q1, Q2, P2, N4, and N6, but not enough to show axiomatization of LG.

The next few experiments paralleled the previous few, but with P4 instead of P1. The results were similar. The sets {P4,Q1,Q2,Q3,Q4} and then {P4,Q3} were shown to axiomatize LG. Rather than include proofs analogous to the preceding two, I shall show a derivation of P1 from {P4,Q3}, which was obtained later in less than 2 seconds:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
2		$P(E(x,E(E(E(E(y,z),E(y,u)),E(z,u)),x)))$	(P4)
3		$P(E(E(E(x,y),E(E(y,x),z)),z))$	(Q3)
37	[1,3,2]	$P(E(E(x,y),E(E(z,x),E(z,y))))$	
41	[1,37,37]	$P(E(E(x,E(y,z)),E(x,E(E(u,y),E(u,z)))))$	
43	[1,37,3]	$P(E(E(x,E(E(y,z),E(E(z,y),u))),E(x,u)))$	
55	[1,43,41]	$P(E(E(E(x,y),E(x,z)),E(y,z)))$	
64	[1,37,55]	$P(E(E(x,E(E(y,z),E(y,u))),E(x,E(z,u))))$	
118	[1,64,2]	$P(E(E(E(E(x,y),E(x,z)),u),E(E(y,z),u)))$	
198	[1,118,118]	$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$	(P1)

The preceding proof shows that {P4,Q3} axiomatizes LG. Section 6 contains a proof that {P4,Q3} is independent.

Continuing the search for simpler axiomatizations of LG, I next started a search with the 45 LG theorems of lengths 3, 7, and 11. The following proof of L2 was derived in about 56 minutes. (The clause numbers have no relationship to the size of the search in this proof.)

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
31		$P(E(E(E(x,y),E(E(y,x),z)),z))$	(Q3)
38		$P(E(x,E(E(y,z),E(E(z,y),x))))$	(Q1)
43		$P(E(E(E(x,y),E(x,z)),E(y,z)))$	(Q4)
45		$P(E(E(x,y),E(E(z,x),E(z,y))))$	(Q2)
81	[1,45,45]	$P(E(E(x,E(y,z)),E(x,E(E(u,y),E(u,z)))))$	
83	[1,45,43]	$P(E(E(x,E(E(y,z),E(y,u))),E(x,E(z,u))))$	
84	[1,45,38]	$P(E(E(x,y),E(x,E(E(z,u),E(E(u,z),y)))))$	
85	[1,45,31]	$P(E(E(x,E(E(y,z),E(E(z,y),u))),E(x,u)))$	
91	[1,81,38]	$P(E(x,E(E(y,E(z,u)),E(y,E(E(u,z),x)))))$	
97	[1,83,81]	$P(E(E(E(E(x,y),z),E(y,u)),E(z,E(x,u))))$	
99	[1,83,43]	$P(E(E(E(x,E(y,z)),E(x,E(y,u))),E(z,u)))$	
102	[1,83,84]	$P(E(E(E(E(x,y),z),u),E(z,E(E(y,x),u))))$	
116	[1,45,85]	$P(E(E(x,E(y,E(E(z,u),E(E(u,z),v)))),E(x,E(y,v))))$	
126	[1,43,91]	$P(E(x,E(y,E(E(z,u),E(E(y,E(u,z)),x)))))$	
139	[1,83,97]	$P(E(E(E(E(x,y),E(x,z)),E(y,u)),E(z,u)))$	
140	[1,81,97]	$P(E(E(E(E(x,y),z),E(y,u)),E(E(v,z),E(v,E(x,u)))))$	
158	[1,102,99]	$P(E(E(x,E(y,z)),E(E(E(y,u),x),E(u,z))))$	
172	[1,116,45]	$P(E(E(x,E(E(y,z),u)),E(E(E(z,y),x),u)))$	
186	[1,97,126]	$P(E(x,E(y,E(E(z,u),E(E(v,E(u,z)),E(E(y,v),x))))))$	
209	[1,102,139]	$P(E(E(x,y),E(E(E(z,u),E(z,x)),E(u,y))))$	
211	[1,85,139]	$P(E(E(E(E(x,y),E(x,E(z,u))),E(y,E(E(u,z),v))),v))$	
229	[1,116,158]	$P(E(E(x,E(y,E(E(z,u),v))),E(E(E(y,E(u,z)),x),v)))$	
298	[1,97,186]	$P(E(x,E(y,E(E(z,u),E(E(v,E(u,z)),E(E(w,v),E(E(y,w),x)))))))$	
321	[1,209,172]	$P(E(E(E(x,y),E(x,E(z,E(E(u,v),w)))),E(y,E(E(E(v,u),z),w))))$	
339	[1,140,211]	$P(E(E(x,E(y,E(E(z,u),E(E(v,E(u,z)),w)))),E(x,E(E(v,y),w))))$	
430	[1,339,298]	$P(E(x,E(E(y,z),E(E(u,y),E(E(z,u),x)))))$	
448	[1,321,430]	$P(E(x,E(E(E(y,z),E(y,u)),E(E(u,z),x))))$	
493	[1,229,448]	$P(E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u))$	(L2)

Thus, {Q1,Q2,Q3,Q4} axiomatizes LG.

All of the proper subsets of Q1–Q4 were considered as candidate axiomatizations of LG, and two yielded successes. The following proof of Q4 from {Q2,Q3} was found in less than one second:

1	$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
3	$P(E(E(x,y),E(E(z,x),E(z,y))))$	(Q2)
4	$P(E(E(E(x,y),E(E(y,x),z)),z))$	(Q3)
23 [1,3,3]	$P(E(E(x,E(y,z)),E(x,E(E(u,y),E(u,z)))))$	
26 [1,3,4]	$P(E(E(x,E(E(y,z),E(E(z,y),u))),E(x,u)))$	
67 [1,26,23]	$P(E(E(E(x,y),E(x,z)),E(y,z)))$	(Q4)

The preceding proof shows that {Q1,Q2,Q3} axiomatizes LG. The following proof, which shows that {Q1,Q3,Q4} axiomatizes LG, was also found in less than 1 second:

1	$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
2	$P(E(x,E(E(y,z),E(E(z,y),x))))$	(Q1)
4	$P(E(E(E(x,y),E(x,z)),E(y,z)))$	(Q4)
22 [1,2,2]	$P(E(E(x,y),E(E(y,x),E(z,E(E(u,v),E(E(v,u),z))))))$	
26 [1,22,4]	$P(E(E(E(x,y),E(E(z,x),E(z,y))),E(u,E(E(v,w),E(E(w,v),u)))))$	
37 [1,4,26]	$P(E(E(E(x,y),E(x,z)),E(E(u,v),E(E(v,u),E(y,z)))))$	
39 [1,4,37]	$P(E(E(x,y),E(E(z,x),E(z,y))))$	(Q2)

The final set of experiments included the 5 LG theorems of lengths 3 and 7 (Table 1) along with the subsets of Q1–Q4. No new axiomatizations were found.

If the measure of simplicity of an axiom system is total symbol count, then {P1,Q3} and {P4,Q3} are the simplest axiomatizations of the LG calculus that were found. Each system contains 26 symbols. If the measure is the length of the longest axiom, then {Q1,Q2,Q3} and {Q1,Q3,Q4} are simplest. Each of Q1–Q4 has 11 symbols. The single axiom presented in Section 4.4 contains 27 symbols.

4.4 The Search for a Single Axiom for LG

At the time when the intensive search for a single axiom began, the known axiomatizations of the the LG calculus were {L2,L3}, {L2,P1}, {L2,P4}, {L2,Q1,Q2}, {P1,Q3}, {P4,Q3}, {Q1,Q2,Q3}, and {Q1,Q3,Q4}. The overall strategy was to investigate LG theorems with the 2-property, but without instances of $E(x,x)$. (The limitation to 2-property theorems is safe, because every non-2-property theorem is an instance of a 2-property theorem, but the $E(x,x)$ limitation is a heuristic gleaned from previous proofs and known axiomatizations.) As listed in Table 1, there are 4 such theorems of length 11 (Q1–Q4), 59 of length 15 (including L2 and P1–P4), and 800 of length 19.

A simple UNIX C shell program was written to run a sequence of OTTER jobs. A header file contained OTTER input except for a prospective single axiom, and a second file contained all of the candidates. The C shell program simply iterated through the candidates, appending each to the header file and sending the results to OTTER. The only output was the proofs, if any, that were found and the reason the search stopped. The header file specified a simple search strategy: select given clauses with ratio 2, delete derived theorems containing instances of $E(x,x)$, delete theorems with length greater than 39, and stop the search after 2 minutes. (A limit of 2 minutes was reasonable, because with nearly all candidates, little or nothing within the constraints could be derived, and the search terminated within a few seconds.) Denials of the known axiomatizations were included. Denials of individual theorems L2, L3, P1–P4, and Q1–Q4 were also included so that candidates that derived some interesting theorems but no known axiomatization could be studied further.

The candidates of lengths 11 and 15 yielded nothing more than the set of experiments described in Section 4.3. The 800 candidates of length 19 yielded several promising theorems such as the following:

$$P(E(E(E(E(E(x,y),E(x,z)),u),v),E(E(E(y,z),u),v))) \qquad \text{(LG-19-12)}$$
$$P(E(E(E(E(E(E(x,y),E(E(z,x),E(z,y))),u),u),v),v)) \qquad \text{(LG-19-128)}$$
$$P(E(E(E(E(E(E(x,y),z),E(y,u)),E(z,E(x,u))),v),v)) \qquad \text{(LG-19-538)}$$

Theorem LG-19-12 (the twelfth theorem of length 19) derived L2, P3, Q3, and Q4. Theorem LG-19-128 derived L3 and P1. Theorem LG-19-538 derived L4, P4, and Q1. However, additional searches with those and other promising candidates failed to derive any of the known axiomatizations.

I could not find an effective way to generate the analogous complete sets of candidates of length 23 theorems or of length 27 theorems. Instead, I simply used OTTER to generate LG theorems without instances of $E(x,x)$ and collected 6456 of length 23 and 2552 of length 27.

Running the C shell program on the length 23 candidates yielded many promising theorems, including the following:

$$P(E(E(E(E(x,y),z),E(E(u,v),w)),E(E(E(E(y,x),E(v,u)),z),w))) \qquad \text{(LG-23-288)}$$
$$P(E(E(E(x,y),E(E(y,x),E(z,u))),E(E(v,E(w,z)),E(v,E(w,u))))) \qquad \text{(LG-23-566)}$$

Theorem LG-23-288 derived Q1 and Q3, and theorem LG-23-566 derived Q2, Q3, and Q4. However, additional searches with those and other promising candidates failed to derive any of the known axiomatizations.

One success occurred with one of the 2552 candidates of length 27:

$$P(E(E(E(E(x,y),z),E(E(u,v),E(E(E(w,v),E(w,u)),s))),E(z,E(E(y,x),s)))) \qquad \text{(LG-27-1690)}$$

The following proof (from the OTTER job invoked by the C shell program) derives P1 and Q3 from LG-27-1690 in less than 2 seconds:

1		$\neg P(E(x,y))\ \mid\ \neg P(x)\ \mid\ P(y)$	(CD)
10		$P(E(E(E(E(x,y),z),E(E(u,v),E(E(E(w,v),E(w,u)),s))),$ $E(z,E(E(y,x),s))))$	(LG-27-1690)
15	[1,10,10]	$P(E(E(E(x,y),E(E(E(z,y),E(z,x)),u)),$ $E(E(E(v,w),E(E(s,v),E(s,w))),u)))$	
16	[1,10,15]	$P(E(E(E(E(x,y),E(x,z)),E(E(E(u,E(E(v,w),E(v,s))),$ $E(u,E(w,s))),t)),E(E(y,z),t)))$	
17	[1,16,16]	$P(E(E(E(E(x,E(E(y,E(z,u)),E(y,E(z,v)))),E(x,E(u,v))),w),w))$	
19	[1,16,10]	$P(E(E(E(E(x,E(E(y,z),E(y,u))),E(x,E(z,u))),$ $E(E(E(v,w),E(v,s)),t)),E(E(w,s),t)))$	
20	[1,17,17]	$P(E(E(E(x,E(y,z)),E(E(u,E(v,x)),E(u,E(v,E(y,w))))),E(z,w)))$	
24	[1,17,10]	$P(E(E(E(x,E(y,E(E(z,u),E(z,v)))),E(x,E(y,w))),E(E(u,v),w)))$	
31	[1,20,15]	$P(E(E(E(x,y),E(E(z,x),E(E(E(u,y),E(u,z)),v))),v))$	
50	[1,19,10]	$P(E(E(E(E(x,y),E(x,z)),u),E(E(y,z),u)))$	
66	[1,50,50]	$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z))))$	(P1)
167	[1,24,31]	$P(E(E(E(x,y),E(E(y,x),z)),z))$	(Q3)

The preceding proof shows that LG-27-1690 is a single axiom for LG, because {P1,Q3} axiomatizes LG.

Including initial searches to adjust parameters and several false starts with the C shell program, well over 10,000 OTTER jobs were run during the search for a single axiom. The jobs were run over a two-day period, and about 12 hours of CPU time were used.

5 The Right Group Calculus

Let the *mirror image* of a formula constructed from variables and binary functor E be obtained by replacing all occurrences of $E(\alpha,\beta)$ with $E(\beta,\alpha)$. A formula is a *right group* (RG) theorem if and only if its mirror image is an LG theorem. The inference rule *reverse detachment* derives α from $E(\alpha,\beta)$ and β. The first-order form of reversed condensed detachment (RCD) is

$$\forall x \forall y (P(E(x,y))\ \&\ P(y) \quad\rightarrow\quad P(x))$$

If the inference rule for the RG calculus were reverse condensed detachment, then the RG calculus would be a trivial variation of the LG calculus, because all aspects of the two calculi would be mirror images. However, the inference rule for the RG calculus is ordinary condensed detachment (CD). (In fact, I first studied the RG calculus by considering the LG theorems with RCD.) The following observation follows from remarks in the preliminary version of [4].

Observation 1. If a set S of formulas axiomatizes the LG calculus, then the corresponding set of mirror images $M(S)$ axiomatizes the RG calculus.

Outline of Proof. The proof is from the first-order point of view, in which CD is an axiom rather than a rule of inference. By remarks in the preceding paragraph, the conclusion of the observation is equivalent to "S and RCD derive all LG theorems". It is sufficient to show Q3 & RCD $\Rightarrow$ CD, because S derives Q3, and S & CD derive all LG theorems. The following OTTER refutation shows that CD follows from Q3 and RCD. (A and B are Skolem constants.)

1	$\neg P(E(x,y)) \mid \neg P(y) \mid P(x)$	(RCD)
4	$P(E(E(E(x,y),E(E(y,x),z)),z))$	(Q3)
5	$P(E(A,B))$	
6	$P(A)$	
7	$\neg P(B)$	
12 [1,4,6]	$P(E(E(x,y),E(E(y,x),A)))$	
17 [1,12,4]	$P(E(E(E(x,y),A),E(y,x)))$	
21 [1,17,5]	$P(E(E(B,A),A))$	
29 [1,21,6]	$P(E(B,A))$	
32 [1,29,6]	$P(B)$	
33 [32,7]	$\square$	

The converse of Observation 1 is false: a counterexample is given at the end of this section.

Notation. If Θ is the name of an LG theorem (example Q3), then Θ' is the name of its mirror image (Q3$'$).

Kalman's axiomatization of the RG calculus [4] is the mirror image of his axiomatization of the LG calculus L1–L5. It follows from Observation 1 that L1$'$–L5$'$ axiomatizes RG.

$$P(E(x,E(x,E(E(y,E(z,z)),y)))) \quad \text{(L1}'\text{)}$$
$$P(E(x,E(x,E(E(y,z),E(E(y,u),E(z,u)))))) \quad \text{(L2}'\text{)}$$
$$P(E(x,E(x,E(E(y,E(z,u)),E(y,E(E(z,v),E(u,v))))))) \quad \text{(L3}'\text{)}$$
$$P(E(E(E(x,E(y,z)),E(u,E(y,v))),E(x,E(u,E(z,v))))) \quad \text{(L4}'\text{)}$$
$$P(E(E(x,E(y,E(z,E(u,v)))),E(E(x,E(v,z)),E(E(y,E(v,u)),v)))) \quad \text{(L5}'\text{)}$$

From Observation 1 and the results in Section 4.3, we can conclude that each of the sets {L2$'$,L3$'$}, {L2$'$,P1$'$}, {L2$'$,P4$'$}, {L2$'$,Q1$'$,Q2$'$}, {P1$'$,Q3$'$}, {P4$'$,Q3$'$}, {Q1$'$,Q2$'$,Q3$'$}, and {Q1$'$,Q3$'$,Q4$'$} axiomatizes the RG calculus.

$$P(E(x,E(x,E(E(y,z),E(E(y,u),E(z,u)))))) \quad \text{(L2}'\text{)}$$
$$P(E(E(E(x,E(y,z)),E(u,y)),E(x,E(u,z)))) \quad \text{(P1}'\text{)}$$
$$P(E(E(x,E(E(y,z),E(E(y,u),E(z,u)))),x)) \quad \text{(P4}'\text{)}$$
$$P(E(E(E(x,E(y,z)),E(z,y)),x)) \quad \text{(Q1}'\text{)}$$
$$P(E(E(E(x,y),E(z,y)),E(x,z))) \quad \text{(Q2}'\text{)}$$
$$P(E(x,E(E(x,E(y,z)),E(z,y)))) \quad \text{(Q3}'\text{)}$$
$$P(E(E(x,y),E(E(x,z),E(y,z)))) \quad \text{(Q4}'\text{)}$$

A sequence of OTTER jobs easily verified those axiomatizations. However, when attempting to further simplify those axiomatizations, the following proof was discovered by OTTER:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
2		$P(E(x,E(E(x,E(y,z)),E(z,y))))$	(Q3′)
18	[1,2,2]	$P(E(E(E(x,E(E(x,E(y,z)),E(z,y))),E(u,v)),E(v,u)))$	
20	[1,18,2]	$P(E(E(x,y),E(E(z,E(E(z,E(u,v)),E(v,u))),E(y,x))))$	
25	[1,20,2]	$P(E(E(x,E(E(x,E(y,z)),E(z,y))),E(E(E(u,E(v,w)),E(w,v)),u)))$	
30	[1,25,2]	$P(E(E(E(x,E(y,z)),E(z,y)),x))$	(Q1′)

Thus, each of {Q2′,Q3′} and {Q3′,Q4′} axiomatizes the RG calculus.

5.1 Single Axioms for the RG Calculus

The search for single axioms for the RG calculus was similar to the LG calculus search (Section 4.4). A C shell program iterated through the RG theorems with the 2-property but without instances of $E(x,x)$. No single axioms were found among the 4 theorems of length 11 (Q1′–Q4′), but six single axioms were found among the 59 theorems of length 15:

$$P(E(x,E(x,E(E(y,z),E(E(y,u),E(z,u)))))) \quad \text{(L2′)}$$
$$P(E(E(x,E(y,z)),E(x,E(E(y,u),E(z,u))))) \quad \text{(S2′)}$$
$$P(E(x,E(x,E(E(E(y,z),E(u,z)),E(y,u))))) \quad \text{(S3′)}$$
$$P(E(E(x,E(y,z)),E(E(x,E(u,z)),E(y,u)))) \quad \text{(S4′)}$$
$$P(E(E(x,E(E(y,z),E(E(y,u),E(z,u)))),x)) \quad \text{(P4′)}$$
$$P(E(E(x,E(E(E(y,z),E(u,z)),E(y,u))),x)) \quad \text{(S6′)}$$

Note that L2′ is a member of Kalman's original axiomatization of RG.

The following six Otter proofs show that each of the preceding formulas is a single axiom. The first proof derives axiomatization {Q2′,Q3′} from S4′, and the remaining proofs derive S4′. All were found within a few seconds.

Formula S4′ is a single axiom for the RG calculus:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
11		$P(E(E(x,E(y,z)),E(E(x,E(u,z)),E(y,u))))$	(S4′)
29	[1,11,11]	$P(E(E(E(x,E(y,z)),E(u,E(y,v))),E(E(x,E(v,z)),u)))$	
31	[1,29,11]	$P(E(E(x,E(y,z)),E(x,E(y,z))))$	
32	[1,29,31]	$P(E(E(x,E(y,y)),x))$	
35	[1,32,32]	$P(E(E(x,x),E(y,y)))$	
38	[1,11,32]	$P(E(E(E(E(x,y),E(z,z)),E(u,y)),E(x,u)))$	
41	[1,29,35]	$P(E(E(E(x,y),E(z,y)),E(x,z)))$	(Q2′)
64	[1,38,11]	$P(E(x,E(E(x,E(y,z)),E(z,y))))$	(Q3′)

Formula L2′ is a single axiom for the RG calculus:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
9		$P(E(x,E(x,E(E(y,z),E(E(y,u),E(z,u))))))$	(L2′)
31	[1,9,9]	$P(E(E(x,E(x,E(E(y,z),E(E(y,u),E(z,u))))),$ $E(E(v,w),E(E(v,s),E(w,s)))))$	
32	[1,31,9]	$P(E(E(x,y),E(E(x,z),E(y,z))))$	
34	[1,32,32]	$P(E(E(E(x,y),z),E(E(E(x,u),E(y,u)),z)))$	
38	[1,34,34]	$P(E(E(E(E(x,y),z),E(u,z)),E(E(E(x,v),E(y,v)),u)))$	
41	[1,34,32]	$P(E(E(E(x,y),E(z,y)),E(E(x,u),E(z,u))))$	
60	[1,38,41]	$P(E(E(E(x,y),E(z,y)),E(x,z)))$	
65	[1,32,60]	$P(E(E(E(E(x,y),E(z,y)),u),E(E(x,z),u)))$	
67	[1,60,31]	$P(E(E(x,E(E(y,z),E(u,z))),E(x,E(y,u))))$	
97	[1,65,67]	$P(E(E(x,E(y,z)),E(E(x,E(u,z)),E(y,u))))$	(S4′)

Formula S2′ is a single axiom for the RG calculus:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
9		$P(E(E(x,E(y,z)),E(x,E(E(y,u),E(z,u)))))$	(S2′)
27	[1,9,9]	$P(E(E(x,E(y,z)),E(E(x,u),E(E(E(y,v),E(z,v)),u))))$	
30	[1,27,9]	$P(E(E(E(x,E(y,z)),u),E(E(E(x,v),E(E(E(y,w),E(z,w)),v)),u)))$	
50	[1,30,27]	$P(E(E(E(x,y),E(E(E(z,u),E(v,u)),y)),$ $E(E(x,w),E(E(E(z,s),E(v,s)),w))))$	
51	[1,30,9]	$P(E(E(E(x,y),E(E(E(z,u),E(v,u)),y)),E(x,E(E(z,w),E(v,w)))))$	
163	[1,51,50]	$P(E(E(E(x,y),E(z,y)),E(E(x,u),E(z,u))))$	
177	[1,163,163]	$P(E(E(E(x,y),z),E(E(x,y),z)))$	
228	[1,27,177]	$P(E(E(E(E(x,y),z),u),E(E(E(E(x,y),v),E(z,v)),u)))$	
229	[1,9,177]	$P(E(E(E(x,y),z),E(E(E(x,y),u),E(z,u))))$	
243	[1,229,9]	$P(E(E(E(x,E(y,z)),u),E(E(x,E(E(y,v),E(z,v))),u)))$	
342	[1,228,177]	$P(E(E(E(E(x,y),z),E(u,z)),E(E(x,y),u)))$	
368	[1,342,177]	$P(E(E(x,y),E(x,y)))$	
399	[1,27,368]	$P(E(E(E(x,y),z),E(E(E(x,u),E(y,u)),z)))$	
400	[1,9,368]	$P(E(E(x,y),E(E(x,z),E(y,z))))$	
436	[1,399,368]	$P(E(E(E(x,y),E(z,y)),E(x,z)))$	
442	[1,400,436]	$P(E(E(E(E(x,y),E(z,y)),u),E(E(x,z),u)))$	
453	[1,436,368]	$P(E(x,x))$	
798	[1,243,453]	$P(E(E(x,E(E(y,z),E(u,z))),E(x,E(y,u))))$	
832	[1,442,798]	$P(E(E(x,E(y,z)),E(E(x,E(u,z)),E(y,u))))$	(S4′)

Formula S3′ is a single axiom for the RG calculus:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
9		$P(E(x,E(x,E(E(E(y,z),E(u,z)),E(y,u)))))$	(S3′)
27	[1,9,9]	$P(E(E(x,E(x,E(E(E(y,z),E(u,z)),E(y,u)))),$ $E(E(E(v,w),E(s,w)),E(v,s))))$	
28	[1,27,9]	$P(E(E(E(x,y),E(z,y)),E(x,z)))$	
30	[1,9,28]	$P(E(E(E(E(x,y),E(z,y)),E(x,z)),E(E(E(u,v),E(w,v)),E(u,w))))$	
32	[1,28,30]	$P(E(E(E(x,y),E(z,y)),E(E(x,u),E(z,u))))$	
34	[1,28,32]	$P(E(E(x,y),E(x,y)))$	
38	[1,32,9]	$P(E(E(x,y),E(E(x,E(E(E(z,u),E(v,u)),E(z,v))),y)))$	
39	[1,28,34]	$P(E(x,x))$	
49	[1,9,39]	$P(E(E(x,x),E(E(E(y,z),E(u,z)),E(y,u))))$	
54	[1,28,49]	$P(E(E(x,y),E(E(x,z),E(y,z))))$	
59	[1,54,28]	$P(E(E(E(E(x,y),E(z,y)),u),E(E(x,z),u)))$	
131	[1,38,39]	$P(E(E(x,E(E(E(y,z),E(u,z)),E(y,u))),x))$	
202	[1,59,131]	$P(E(E(x,E(E(y,z),E(u,z))),E(x,E(y,u))))$	
238	[1,59,202]	$P(E(E(x,E(y,z)),E(E(x,E(u,z)),E(y,u))))$	(S4′)

Formula P4′ is a single axiom for the RG calculus:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
9		$P(E(E(x,E(E(y,z),E(E(y,u),E(z,u)))),x))$	(P4′)
27	[1,9,9]	$P(E(E(E(x,y),E(E(x,z),E(y,z))),E(E(u,v),E(E(u,w),E(v,w)))))$	
28	[1,9,27]	$P(E(E(x,y),E(E(x,z),E(y,z))))$	
30	[1,28,28]	$P(E(E(E(x,y),z),E(E(E(x,u),E(y,u)),z)))$	
32	[1,28,9]	$P(E(E(E(x,E(E(y,z),E(E(y,u),E(z,u)))),v),E(x,v)))$	
69	[1,32,9]	$P(E(x,x))$	
73	[1,30,69]	$P(E(E(E(x,y),E(z,y)),E(x,z)))$	
80	[1,32,73]	$P(E(E(x,E(E(y,z),E(u,z))),E(x,E(y,u))))$	
82	[1,28,73]	$P(E(E(E(E(x,y),E(z,y)),u),E(E(x,z),u)))$	
133	[1,82,80]	$P(E(E(x,E(y,z)),E(E(x,E(u,z)),E(y,u))))$	(S4′)

Formula S6′ is a single axiom for the RG calculus:

1		$\neg P(E(x,y)) \mid \neg P(x) \mid P(y)$	(CD)
9		$P(E(E(x,E(E(E(y,z),E(u,z)),E(y,u))),x))$	(S6′)
27	[1,9,9]	$P(E(E(E(E(x,y),E(z,y)),E(x,z)),E(E(E(u,v),E(w,v)),E(u,w))))$	
28	[1,9,27]	$P(E(E(E(x,y),E(z,y)),E(x,z)))$	
30	[1,9,28]	$P(E(E(E(E(x,y),E(z,y)),u),E(E(x,z),u)))$	
35	[1,30,9]	$P(E(E(x,E(E(y,z),E(u,z))),E(x,E(y,u))))$	
58	[1,30,35]	$P(E(E(x,E(y,z)),E(E(x,E(u,z)),E(y,u))))$	(S4′)

Note that the preceding six proofs contain many instances of $E(x,x)$. The strategy of deleting such formulas was not used, because it resulted in longer proofs in those cases. In addition, the strategy appears to block *all* interesting theorems when starting the search with the single axiom S4′.

Counterexample to Converse of Observation 1. Formula L2′ is a single axiom for the RG calculus, but its mirror image L2 cannot be a single axiom for the LG calculus, because nothing can be derived from L2 alone by ordinary condensed detachment.

6 Independence

Independence of axiomatizations was not emphasized in the study described in this paper, but several results follow easily from failed OTTER searches. This section contains a proof that each of the axiomatizations {L2,P4} and {P4,Q3} of the LG calculus is independent.

$$P(E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u)) \tag{L2}$$
$$P(E(x,E(E(E(E(y,z),E(y,u)),E(z,u)),x))) \tag{P4}$$
$$P(E(E(E(x,y),E(E(y,x),z)),z)) \tag{Q3}$$

Lemma 6.1. *Let T be the formula $E(E(E(y,z),E(y,u)),E(z,u))$ (which is a subformula of P4). Starting with P4 and condensed detachment, every derived formula has the form $E(T',F)$, for some formula F, where T' is a variant of T.*

Proof. The proof is by induction on the number of given clauses. The base case holds because, on the first given clause, P4 condensed-detaches only with itself to derive $E(T',\mathrm{P4})$. Assume that the lemma holds for n given clauses, and consider given clause $n+1$, say G. Condensed detachment with minor premise G and major premise P4 derives $E(T',G)$. Assume that another formula is derived. By the induction hypothesis, G must be of the form $E(T,F)$. There are two cases. In Case 1, G is the major premise, and P4 is the minor premise. The occurs-check (Fig. 2) prevents application of condensed detachment. In Case 2, G condensed-detaches with a derived formula, say H. By the induction hypothesis, H must also be of the form $E(T,F)$. No matter which is the major premise, $E(T,F)$ must unify with a variant of T. However, the occurs-check (Fig. 2) prevents the unification. This finishes the proof of the two cases, the induction step, and the lemma.

```
Case 1.
    E(x,                E(E(E(E(y,z),E(y,u)),E(z,u)),x))         : P4
  E( E(E(E(v,w),E(v,t)),E(w,                             t)) ,F)     : E(T,F)

Case 2.
    E(E(E(E(x,y),E(x,z)),E(y,z)),F)                                   : E(T,F)
  E( E(E(E(v,     w),     E(v,t)),E(w,t)), F')                        : E(T',F')
```

Figure 2: Lemma 6.1 Occurs-Check Failures

Theorem 6.1. *In the LG calculus,* {L2,P4} *is independent, and* {P4,Q3} *is independent.*

Proof. Neither L2 nor Q3 condensed-detaches with itself, so neither can derive P4. Neither L2 nor Q3 is a variant of $E(E(E(E(y,z),E(y,u)),E(z,u)),F)$, for any F, so by Lemma 6.1, neither can be derived from P4. This completes the proof of the theorem.

Theorem 6.1 is the direct result of failed searches with OTTER. In the output of a search starting with just P4, it was observed that each given clause derived exactly one new theorem, and the pattern was as described in Lemma 6.1.

7 Summary

Kalman's axiomatization L1–L5 of the left group (LG) calculus was used as the starting point for the search for new axiomatizations of the LG calculus. Section 4 contains a proof that LG-27-1690 is a single axiom for the LG calculus.

$$P(E(E(E(E(x,y),z),E(E(u,v),E(E(E(w,v),E(w,u)),s))),E(z,E(E(y,x),s)))) \qquad \text{(LG-27-1690)}$$

Section 4 also shows that each of the sets {L2,L3}, {L2,P1}, {L2,P4}, {L2,Q1,Q2}, {P1,Q3}, {P4,Q3}, {Q1,Q2,Q3}, and {Q1,Q3,Q4} also axiomatizes the LG calculus.

$$P(E(E(E(E(E(x,y),E(x,z)),E(y,z)),u),u)) \qquad \text{(L2)}$$
$$P(E(E(E(E(E(E(x,y),E(x,z)),u),E(E(y,z),u)),v),v)) \qquad \text{(L3)}$$
$$P(E(E(E(x,y),z),E(E(u,y),E(E(x,u),z)))) \qquad \text{(P1)}$$
$$P(E(x,E(E(E(E(y,z),E(y,u)),E(z,u)),x))) \qquad \text{(P4)}$$
$$P(E(x,E(E(y,z),E(E(z,y),x)))) \qquad \text{(Q1)}$$
$$P(E(E(x,y),E(E(z,x),E(z,y)))) \qquad \text{(Q2)}$$
$$P(E(E(E(x,y),E(E(y,x),z)),z)) \qquad \text{(Q3)}$$
$$P(E(E(E(x,y),E(x,z)),E(y,z))) \qquad \text{(Q4)}$$

In addition, the sets {L2,P4} and {P4,Q3} are shown to be independent in Section 6.

Kalman's axiomatization L1′–L5′ (named R1–R5 by Kalman) of the right group (RG) calculus was used as the starting point for the search for new axiomatizations of the RG calculus. Section 5 contains proofs that each of the following formulas is a single axiom for the RG calculus:

$$P(E(x,E(x,E(E(y,z),E(E(y,u),E(z,u)))))) \qquad \text{(L2}'\text{)}$$
$$P(E(E(x,E(y,z)),E(x,E(E(y,u),E(z,u))))) \qquad \text{(S2}'\text{)}$$
$$P(E(x,E(x,E(E(E(y,z),E(u,z)),E(y,u))))) \qquad \text{(S3}'\text{)}$$
$$P(E(E(x,E(y,z)),E(E(x,E(u,z)),E(y,u)))) \qquad \text{(S4}'\text{)}$$
$$P(E(E(x,E(E(y,z),E(E(y,u),E(z,u)))),x)) \qquad \text{(P4}'\text{)}$$
$$P(E(E(x,E(E(E(y,z),E(u,z)),E(y,u))),x)) \qquad \text{(S6}'\text{)}$$

Section 5 also shows that each of the pairs {Q2′,Q3′} and {Q3′,Q4′} also axiomatizes the RG calculus.

$$P(E(E(E(x,y),E(z,y)),E(x,z))) \tag{Q2$'$}$$
$$P(E(x,E(E(x,E(y,z)),E(z,y)))) \tag{Q3$'$}$$
$$P(E(E(x,y),E(E(x,z),E(y,z)))) \tag{Q4$'$}$$

This paper has shown how the automated theorem-proving program OTTER was used to discover proofs that candidate sets of theorems axiomatize the LG and RG calculi. OTTER was also used in several ways that might not be considered standard first-order logic theorem proving. First, axiomatizations were found by starting a search with hundreds of input theorems, OTTER deriving a known axiomatization, then the user simply looking at the proof to find the new axiomatization (the input theorems on which the derivation depends). In most cases, the user then directed OTTER to search for dependencies in the new axiomatization. Second, single axioms were found by automatically running thousands of separate OTTER jobs with different candidates, which were extracted from the output of a previous OTTER run. Third, OTTER was used to enumerate all of the equivalential calculus (EC) theorems of a given length and to extract the LG theorems from the EC theorems by rewriting (Section 4.1, Table 1). Finally, OTTER produced unexpected results, for example, the dependence of L1 and L4 (Section 4.2) in Kalman's original axiomatization of the LG calculus.

The variation in OTTER jobs was much more in the set of input formulas than in the search strategy. The variation in search strategy (Section 3) involved selection of given clauses, whether or not derived formulas containing instances of $E(x,x)$ were discarded, and the weight threshold for derived clauses. The most successful combination for these experiments was to select given clauses with ratio 2, to discard derived formulas with instances of $E(x,x)$, and to discard derived formulas with weight greater than 28.

The study described in this paper consumed about 4 CPU days on a SPARCstation 1+ computer. The dependence of L1 and L4 was discovered on October 9, 1990. The rest of the results were obtained January 10–25, 1991.

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