# On the Positive Solutions of the Free-Boundary Problem for Emden-Fowler Type Equations

Yi Li[*], Hans G. Kaper, and Man Kam Kwong[†]

## 1 Introduction

Let $\Omega$ be a smooth, bounded and connected domain in $\Re^n$. In this paper, we consider the following boundary value problem:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \qquad \boxed{\text{de1}} \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $n$ denotes the unit outer normal to $\partial\Omega$. (See [KK] for existence and uniqueness results for (1.1).) We prove the following two theorems.

**Theorem 1** *Let $f$ be such that*

$$f(s) = f_1(s) + f_2(s), \qquad \boxed{\text{f}} \tag{1.2}$$

*where $f_1$ is nondecreasing and $f_2$ Lipschitz continuous. If $u \in C^2(\overline{\Omega})$ be a classical solution of* (1.1)*, then $\Omega$ is an open ball, $\Omega = B_R(x_0)$ say, in $\Re^n$ and $u$ is radially symmetric about the center $x_0$. Furthermore,*

$$\frac{\partial u}{\partial r} < 0 \quad \textit{for} \quad 0 < r \equiv |x - x_0| < R.$$

**Theorem 2** *Let $B_R(0)$ be a ball of radius $R > 0$. Let $u$ be a classical solution of the boundary value problem,*

$$\begin{cases} \Delta u + f(u) = 0 & \textit{in } B_R(0), \\ u > 0 & \textit{in } B_R(0), \qquad \boxed{\text{de2}} \\ u = 0 & \textit{on } \partial B_R(0). \end{cases} \tag{1.3}$$

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*If* $f \in C^{0,1}_{loc}((0,\infty))$ *and there exists an* $s_0 > 0$ *such that* $f(s)$ *is strictly decreasing in* $[0, s_0]$, *then* $u$ *is radially symmetric about* 0. *Furthermore,*

$$\frac{\partial u}{\partial r} < 0 \quad \textit{for} \quad 0 < r < R.$$

We use the well-known moving-plane method, which was first proposed by Alexandrov. In 1971, Serrin used this method to prove the symmetry result for (1.1) in the case where $f(s)$ is real and constant. Since we are dealing with nonsmooth functions like $f_1$, some stronger version of the Hopf near-boundary theorem has to be used. In fact, we use the moving-plane method, in combination with a result (Lemma 4) of Gidas, Ni and Nirenberg [GNN], to prove Theorem 1. To prove Theorem 2, we need to analyze the locations of possible minima of the difference $u - u^\lambda$ in order to continue the moving-plane process.

## 2 Preliminaries

To prove Theorems 1 and 2, we need a series of technical lemmas, whose proofs can be found in [GNN], [H], [PW] and [S].

Let $u$ be a nonnegative classical solution in $\Omega$ of the following differential inequality:

$$Lu \equiv a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u \le 0 \quad \text{in} \quad \Omega, \qquad \boxed{\text{di}} \tag{2.1}$$

where $(a^{ij}(x)) \ge \lambda I$ in $\Omega$ for some fixed $\lambda > 0$ and $a^{ij}, b^i, c \in L^\infty(\Omega)$.

**Lemma 1 (Strong Maximum Principle)** *If* $u \ge 0$ *in* $\Omega$ *and* $u$ *vanishes at some point inside* $\Omega$, *then* $u \equiv 0$ *in* $\Omega$.

**Lemma 2 (Hopf Boundary Lemma)** *If* $x_0 \in \partial\Omega$, $u > 0$ *in* $\Omega$, *and* $u(x_0) = 0$, *then*

$$\frac{\partial u}{\partial n}(x_0) < 0$$

*and*

$$\lim_{x \to x_0, x \in \Omega} \frac{u(x) - u(x_0)}{|x - x_0|} > 0 \tag{2.2}$$

*for any non-tangential limit.*

**Lemma 3 ([GNN])** *Let* $\Omega$ *be a domain in* $\Re^n$. *Let* $y_0 \in \partial\Omega$ *and assume that, near* $y_0$, $\partial\Omega$ *consists of two transversally intersecting* $C^2-$*hypersurfaces* $\varphi = 0$ *and* $\psi = 0$. *Suppose that* $\varphi, \psi < 0$ *in* $\Omega$. *Let* $u$ *satisfy* (2.1), $u > 0$ *in* $\Omega$, *and* $u(y_0) = 0$. *Assume that*

$$\ell(y_0) \equiv a^{ij}(y_0)D_i\varphi(y_0)D_j\psi(y_0) \ge 0$$

*and, if $\ell(y_0) = 0$, assume furthermore that $a^{ij} \in C^2$ in some $\Omega_\varepsilon = \Omega \cap B_\varepsilon(y_0)$ $(\varepsilon > 0)$, and that $\nabla_t(\ell(y)) = 0$ at $y_0$ for any tangential derivatives $\nabla_t$ along $\{\varphi = 0\} \cap \{\psi = 0\}$. Then*

$$\frac{\partial u}{\partial s} > 0 \quad at \quad y_0 \quad if \;\; \ell(y_0) > 0,$$

$$\frac{\partial u}{\partial s} > 0 \quad or \quad \frac{\partial^2 u}{\partial s^2} > 0 \quad at \quad y_0 \quad if \;\; \ell(y_0) = 0,$$

*for any direction $s$ entering $\Omega$ at $y_0$ transversally to the hypersurfaces $\varphi = 0$ and $\psi = 0$.*

**Lemma 4 ([GNN])** *Let $x_0 \in \partial\Omega$ with $n_1(x_0) > 0$, where $n_1$ is the first component of $n$. Assume that $u > 0$ in $\Omega_\varepsilon$, $u \equiv 0$ on $\partial\Omega \cap B_\varepsilon(x_0)$, and*

$$\Delta u + f(u) = 0 \quad in \ \Omega,$$

*where $f$ satisfies* (1.2). *Then there exists a $\delta > 0$ such that $D_1 u < 0$ in $\Omega_\delta$.*

# 3 Proof of Theorem 1

Before we can use the moving-plane method, we introduce a few definitions. Let $e_1 = (1, 0, \ldots, 0)$ be the unit vector along the $x_1$-axis, and let $T_\lambda$ be the hyperplane $\{x_1 = \lambda\}$. Since $\Omega$ is bounded and smooth, $T_\lambda \cap \overline{\Omega} = \emptyset$ for large $\lambda$. Now, let $\lambda$ decrease until $T_\lambda$ touches $\overline{\Omega}$ at $\lambda_0$, say.

For $\lambda < \lambda_0$, let $\Sigma_\lambda^+ = \Omega \cap \{x_1 > \lambda\}$, and let $\Sigma_\lambda^-$ be the reflection of $\Sigma_\lambda^+$ about the plane $T_\lambda$. Let $x^\lambda$ be the reflection point of $x$ about $T_\lambda$, i.e.,

$$x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n).$$

If $\lambda_0 - \lambda$ is small, $\Sigma_\lambda^-$ will be inside $\Omega$. But as $\lambda$ decreases continuously, $\Sigma_\lambda^-$ will be in $\Omega$ until one of the following occurs:

1. $\overline{\Sigma_\lambda^-}$ becomes internally tangent to $\partial\Omega$ at some point $x_0$ not on $T_\lambda$;
2. $T_\lambda$ becomes orthogonal to $\partial\Omega$ at some point $y_0 \in \partial\Omega \cap T_\lambda$.

We let $T_{\lambda_1}$ denote the plane that first reaches one of these two possibilities and call $\Sigma_{\lambda_1}^+$ the maximal cap.

**Lemma 5** *If*

$$u(x) \le u(x^\lambda) \ \ in \ \Sigma_\lambda^+$$

*for some* $\lambda \in (\lambda_1, \lambda_0)$*, then*

$$\begin{cases} D_{e_1} u(x) < 0 & \text{on } \Omega \cap T_\lambda, \\ u(x) < u(x^\lambda) & \text{in } \Sigma_\lambda^+. \end{cases} \qquad \boxed{\text{iq2}} \tag{3.1}$$

**Proof**. Let $v(x) = u(x^\lambda)$, $x \in \Sigma_\lambda^+$. Then

$$\Delta(v(x) - u(x)) + f_1(v(x)) + f_2(v(x)) - f_1(u(x)) - f_2(u(x)) = 0$$

and

$$v(x) - u(x) \geq 0 \quad \text{in } \Sigma_\lambda^+,$$

with

$$v(x) - u(x) \not\equiv 0 \text{ in } \Sigma_\lambda^+.$$

Now, $f_1(v(x)) \geq f_1(u(x))$ and $f_2(v(x)) - f_2(u(x)) = c_\lambda(x)(v(x) - u(x))$, where $c_\lambda(x)$ is bounded, because $f_1$ is nondecreasing and $f_2$ is Lipschitz. Hence,

$$\Delta(v(x) - u(x)) + c_\lambda(x)(v(x) - u(x)) \leq 0, \qquad \boxed{\text{iq3}} \tag{3.2}$$

$$v(x) - u(x) \geq 0 \quad \text{in} \quad \Sigma_\lambda^+, \qquad \boxed{\text{iq4}} \tag{3.3}$$

and

$$v(x) - u(x) \not\equiv 0 \quad \text{in} \quad \Sigma_\lambda^+. \tag{3.4}$$

Then Lemma 1 implies that $v(x) - u(x) > 0$ in $\Sigma_\lambda^+$, while Lemma 2 gives us that

$$\frac{\partial}{\partial x_1}(v(x) - u(x))\bigg|_{x_1=\lambda} > 0, \qquad \boxed{\text{iq5}} \tag{3.5}$$

because $v(x) - u(x)|_{x_1=\lambda} \equiv 0$, which is the minimum. From (3.5), we obtain the inequality

$$-\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_1}\bigg|_{x_1=\lambda} > 0,$$

which completes the proof of Lemma 5. ∎

**Lemma 6** *Let* $u$ *be a classical solution of* (1.1). *Then* (3.1) *holds for all* $\lambda \in (\lambda_1, \lambda_0)$.

**Proof**. From Lemma 4, we know that (3.2) and (3.3) hold for all $\lambda \in (\lambda_1, \lambda_0)$ with $\lambda_0 - \lambda$ sufficiently small.

Suppose the lemma is false. That is, suppose that there exists a $\lambda_2 \in (\lambda_1, \lambda_0)$ such that (3.1) holds for $\lambda \in (\lambda_2, \lambda_0)$ but not for $\lambda < \lambda_2$.

On the other hand, the continuity of $u$ implies that

$$u(x) \le u(x^{\lambda_2}), \quad \lambda \in \Sigma^+_{\lambda_2},$$

and since $\lambda_2 \in (\lambda_1, \lambda_0)$, Lemma 5 implies that (3.2) and (3.3) also hold for $\lambda = \lambda_2$.

Since $\lambda_2 > \lambda_1$, $n_1(x_0) > 0$ for each point $x_0 \in \partial\Sigma^+_{\lambda_2} \backslash (T_{\lambda_2} \cap \Omega)$. And hence Lemma 4 concludes that there exists $\varepsilon_{x_0} > 0$ such that

$$D_{e_1} u(x) < 0 \quad \text{in} \quad \Omega \cap B_{\varepsilon_{x_0}}(x_0).$$

Since $D_{e_1} u(x) < 0$ on $T_{\lambda_2} \cap \Omega$, we have that there exists some $\varepsilon > 0$, such that

$$D_{e_1} u(x) < 0 \quad \text{in} \quad \Omega \cap \{x_1 > \lambda_2 - \varepsilon\}, \qquad \boxed{\text{iq6}} \tag{3.6}$$

because $T_{\lambda_2} \cap \overline{\Omega}$ is compact.

Therefore, if (3.2) and (3.3) fail to hold in $(\lambda_1, \lambda_2)$, we must have a sequence $\{\lambda^i\}$ such that

$$\lambda^i > \lambda_1 \quad \text{and} \quad \lambda^i \nearrow \lambda_2 \text{ as } i \to \infty,$$

with

$$u(x^i) \ge u(x^{i\lambda^i}) \quad \text{for some } x^i \in \Sigma^+_{\lambda_i}.$$

But $\Omega$ is bounded, so we can find a subsequence of $\{x^i\}$, say $\{x^i\}$ itself, which converges to some point $x_0 \in \overline{\Sigma^+_{\lambda_2}}$ as $i \to \infty$ with $u(x_0) \ge u(x_0^{\lambda_2})$. Therefore, $x_0 \in \partial\Sigma^+_{\lambda_2}$, because (3.1) holds for $\lambda_2$. Thus, we have either of two possibilities:

1. $x_0 \in \partial\Sigma^+_{\lambda_2} \backslash (T_\lambda \cap \overline{\Omega})$. But then $x_0 \in \partial\Omega$ with $x_0^{\lambda_2} \in \Omega$, since $\lambda_2 > \lambda_1$, which implies that $0 \ge u(x_0^{\lambda_2}) > 0$. This is impossible.
2. $x_0 \in T_{\lambda_2} \cap \overline{\Omega}$. Therefore, $x_0^{\lambda_2} = x_0$.

Since $\lambda^i > \lambda_1$, we have that the line segment $P_i$ joining $x^i$ and $x^{i\lambda^i}$ lies in $\Omega$. Therefore, $u(x^i) \ge u(x^{i\lambda^i})$ implies that

$$D_{e_1} u(y^i) \ge 0 \quad \text{for some } y^i \in P_i.$$

But $x^i \to x_0$ and $x^{i\lambda^i} \to x_0^{\lambda_2} = x_0$, so $P_i$ shrinks into the single point $x_0$. This gives us a contradiction with (3.6), because $y^i \in \Omega \cap \{x_1 > \lambda_2 - \varepsilon\}$ for $i$ large enough. This completes the proof of Lemma 6. ∎

**Proof of Theorem 1.** By Lemma 6, (3.2) and (3.3) hold for all $\lambda \in (\lambda_1, \lambda_0)$. Let us discuss the following two possible cases.

Case 1. $\Sigma_{\lambda_1}^- \cup \Sigma_{\lambda_1}^+ \cup (T_{\lambda_1} \cap \Omega) = \Omega$.

Then $\Omega$ is symmetric about $T_{\lambda_1}$, in which case we have shown that

$$\begin{cases} u(x) \le u(x^{\lambda_1}), & x \in \Sigma_{\lambda_1}^+, \\ D_{e_1} u < 0, & x \in \Sigma_{\lambda}^+, \end{cases} \forall \lambda \in (\lambda_1, \lambda_0), \qquad \boxed{\text{iq7}} \tag{3.7}$$

or

$$\begin{cases} u(x) = u(x^{\lambda_1}), & x \in \Omega, \\ D_{e_1} u > 0, & \text{if } x \in \Omega \cap \Sigma_{\lambda_1}^-, \\ D_{e_1} u < 0, & \text{if } x \in \Omega \cap \Sigma_{\lambda_1}^+. \end{cases} \qquad \boxed{\text{iq8}} \tag{3.8}$$

Case 2. $\Sigma_{\lambda_1}^- \cup \Sigma_{\lambda_1}^+ \cup (T_{\lambda_1} \cap \Omega) \subsetneqq \Omega$.

Then $u(x^{\lambda_1}) - u(x) \ge 0$ in $\Sigma_{\lambda_1}^+$ and not identically zero. Therefore, the same argument as in Lemma 5 implies that

$$u(x^{\lambda_1}) > u(x) \quad \text{in } \Sigma_{\lambda_1}^+.$$

at some point $x_0 \notin T_{\lambda_1}$. As in the proof of Lemma 5, we find by letting $v(x) = u(x^{\lambda_1})$ that

$$\begin{cases} \Delta(v(x) - u(x)) + c_{\lambda_1}(x)(v(x) - u(x)) \le 0 & \text{in } \Sigma_{\lambda_1}^+, \\ v(x) - u(x) > 0 & \text{in } \Sigma_{\lambda_1}^+, \\ v(x_0) - u(x_0) = 0. \end{cases} \qquad \boxed{\text{iq9}} \tag{3.9}$$

Since $x_0 \notin T_{\lambda_1}$, $\Sigma_{\lambda_1}^+$ is smooth near $x_0$. Hence we may use Lemma 2 to conclude that

$$\frac{\partial}{\partial n}(v - u)(x_0) < 0,$$

which is in contradiction with the boundary condition $\frac{\partial v}{\partial n} = \frac{\partial u}{\partial n}(x_0) = 0$. Hence, $T_{\lambda_1}$ must become orthogonal to $\partial\Omega$ at some point $y_0 \in \partial\Omega \cap T_{\lambda_1}$.

However, $u(x^{\lambda_1}) - u(x)$ satisfies (3.8) in $\Sigma_{\lambda_1}^+$, $y_0 \in \partial\Sigma_{\lambda}^+$, and, near $y_0$, $\partial\Sigma_{\lambda}^+$ consists of two transversally intersecting hypersurfaces $x_1 = \lambda_1$ and $\partial\Omega$, which become orthogonal at $y_0$. A simple computation shows that $\ell(y_0) = 0$ and, for any tangential direction $t$ along $T_{\lambda_1} \cap \partial\Omega$ at $y_0$,

$$\nabla_t(\ell(y)) = 0 \quad \text{at } y_0,$$

which implies by Lemma 3 that for any $s$ entering $\Omega$ at $y_0$ transversally to $T_\lambda$ and $\partial\Omega$,

$$\frac{\partial(v-u)}{\partial s} > 0 \quad \text{or} \quad \frac{\partial^2(v-u)}{\partial s^2} > 0 \text{ at } y_0.$$

On the other hand, it follows from (1.1) that

$$(v-u)(y_0) = 0, \quad \nabla(v-u)(y_0) = 0, \quad \text{and } D^2(v-u)(y_0) = 0.$$

This again leads us to a contradiction, so it must be the case that

$$\Sigma_{\lambda_1}^- \cup \Sigma_{\lambda_1}^+ \cup (T_{\lambda_1} \cap \Omega) = \Omega.$$

On the other hand, since we can start moving the plane from the left to the right along the $x_1$-axis as well, we conclude that

$$\begin{cases} u(x) = u(x^{\lambda_1}) & x \in \Omega, \\ D_{e_1} u > 0 & \text{if} \quad x \in \Omega \cap \Sigma_{\lambda_1}^-, \\ D_{e_1} u < 0 & \text{if} \quad x \in \Omega \cap \Sigma_{\lambda_1}^+. \end{cases} \qquad \boxed{\text{iq10}} \tag{3.10}$$

But equation (1.1) is rotationally invariant. Therefore $\Omega$ is symmetric in every direction. We thus find that $\Omega$ must be a ball, because it is connected. Then (3.10) gives the conclusions of Theorem 1.

# 4 Proof of Theorem 2

Gidas, Ni, and Nirenberg proved in [GNN] that the solutions of

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } B_R(0), \\ u > 0 & \text{in } B_R(0), \\ u = 0 & \text{on } \partial B_R(0), \end{cases} \qquad \boxed{\text{de3}} \tag{4.1}$$

with

$$f(s) = f_1(s) + f_2(s), \qquad \boxed{\text{f2}} \tag{4.2}$$

where $f_1$ is Lipschitz continuous and $f_2$ nondecreasing, must be radially symmetric about 0 and, furthermore, $\frac{\partial u}{\partial r} < 0$ for $0 < r < R$. On the other hand, if a decomposition like (4.2) does not exist, in particular if $f$ is not smooth, then it is an open problem whether positive solutions of (1.1) are radially symmetric. Actually, some examples given in [GNN, pp. 220] show that these cases could be very delicate.

In this part, we will try to treat a family of nonlinear terms $f$ which are neither Lipschitz nor nondecreasing. Such situations arise, for example, in the study of free-boundary problems for Emden-Fowler type equations (see [KK]), where

$$f(u) = u^{1/p} - u^{1/q} \text{ with } 1 \le p < q \le \infty. \qquad \boxed{\text{f3}} \tag{4.3}$$

**Remark 1.** If $q = \infty$, then $f(s) = s^{1/p} - 1$ is an increasing function in $s$ and therefore [GNN]'s result implies that $u$ must be radially symmetric. Therefore, the difficult parts occur when $1 \le p < q < \infty$. For such cases,

$$f(s) = \begin{cases} \text{strictly decreasing in } \left[0, (p/q)^{pq/(q-p)}\right], \\ \text{strictly increasing in } \left[(p/q)^{pq/(q-p)}, \infty\right). \end{cases}$$

**Remark 2.** Recently, new symmetry results have been obtained in [GL] and [LV] for equations on nonsmooth domains.

**Proof of Theorem 2.** First we define

$$\Lambda = \left\{ \lambda \in (0,R) | u(x) < u(x^\lambda) \quad \text{if } x \in \Sigma_\lambda^+ \right\}.$$

Because $u|_{\partial B_R} = 0$, there exists a $\lambda_0 \in (0,R)$ such that

$$u(B_R(0) \backslash B_{\lambda_0}(0)) \subset (0, s_0). \qquad \boxed{\text{incl}} \tag{4.4}$$

Step 1. $(\frac{1}{2}(\lambda_0 + R), R) \in \Lambda$.

For any $\lambda \in (\frac{1}{2}(\lambda_0 + R), R)$, $f$ is a strictly decreasing function in teh interval $[0, \max\{\sup_{\Sigma_\lambda^+} u, \sup_{S_\lambda^+} u^\lambda\}]$, where $u^\lambda = u(x^\lambda)$, because of (4.4), and

$$\begin{cases} \Delta(u^\lambda - u)(x) + f(u^\lambda(x)) - f(u(x)) = 0 & \text{in } \Sigma_\lambda^+, \\ u^\lambda - u = 0 & \text{on } \overline{T}_\lambda, \\ u^\lambda - u > 0 & \text{on } \partial\Sigma_\lambda^+ \setminus \overline{T}_\lambda. \end{cases}$$

Claim 1. If $u^\lambda - u \geq 0$ in $\Sigma_\lambda^+$, then $u^\lambda - u > 0$ in $\Sigma_\lambda^+$ and $\frac{\partial u}{\partial x_1} < 0$ on $T_\lambda$.

Suppose the claim is false, i.e. there exists a $y_0 \in \Sigma_\lambda^+$, such that $(u^\lambda - u)(y_0) = 0$.

On the other hand, both $u$ and $u^\lambda$ are strictly positive in $\Sigma_\lambda^+$, so

$$\Delta(u^\lambda - u)(x) + \frac{f(u^\lambda(x)) - f(u(x))}{u^\lambda(x) - u(x)}(u^\lambda - u)(x) = 0,$$

where $\frac{f(u^\lambda(x)) - f(u(x))}{u^\lambda(x) - u(x)}$ is locally bounded, because $f \in C^{0,1}_{loc}((0,\infty))$.

Hence the strong maximum principle implies a contradiction. Therefore, if $u^\lambda - u \geq 0$ in $\Sigma_\lambda^+$, then $u^\lambda - u > 0$ there.

Claim 2. $u^\lambda - u \geq 0$ in $\Sigma_\lambda^+$.

Otherwise, because $u^\lambda - u \geq 0$ on $\partial\Sigma_\lambda^+$, $u^\lambda - u$ would have a strictly interior negative minimum, say at $y_0 \in \Sigma_\lambda^+$. But at $y_0$ we have $\Delta(u^\lambda - u)(y_0) \geq 0$ and, since $s_0 > u(y_0) > u^\lambda(y_0) > 0$ by (4.4), $f(u(y_0)) < f(u^\lambda(y_0))$. Therefore,

$$\Delta(u^\lambda - u)(y_0) + f(u^\lambda(y_0)) - f(u(y_0)) > 0,$$

a contradiction.

Thus Step 1 is proved.

Step 2. $\Lambda$ is closed w.r.t. $(0, R)$.

If $\{\lambda^i\}$ is a sequence in $\Lambda$ which converges to some $\lambda$ in $(0, R)$, then, since

$$u(x) < u(x^{\lambda_i}), \quad x \in \Sigma^+_{\lambda_i},$$

letting $i \to \infty$, we find

$$u(x) \leq u(x^{\lambda}), \quad x \in \Sigma^+_{\lambda}.$$

But then 'Claim 1' in 'Step 1' shows that $u(x) < u(x^{\lambda})$ in $\Sigma^+_{\lambda}$, i.e. $\lambda \in \Lambda$.

Step 3. $\Lambda$ is open in $(0, R)$.

Suppose that $\Lambda$ is not open. Then there exists a $\lambda \in \Lambda$ and a sequence $\{\lambda^i\} \in (0, R)$ s.t. $\lambda^i \to \lambda$ with $\lambda^i \notin \Lambda$. That is, for each $i$ there exists $x^i \in \Sigma^+_{\lambda^i}$ with

$$0 > u(x^{i\lambda^i}) - u(x^i) = \min_{x \in \Sigma^+_{\lambda^i}} (u^{\lambda^i} - u)(x), \qquad \boxed{\text{min}} \tag{4.5}$$

and the mean-value theorem implies

$$\frac{\partial u}{\partial x_1}(y^i) \geq 0 \text{ for some } y^i \in \overline{x^{i\lambda^i} x^i}. \qquad \boxed{\text{mvt}} \tag{4.6}$$

Because $x^i \in B_R(0)$, there exists a subsequence, say $\{x^i\}$ itself, converging to a point $x_0 \in \overline{B}_R(0)$ and (4.5) implies that

$$u(x_0) \geq u(x_0^{\lambda}). \qquad \boxed{\text{iq11}} \tag{4.7}$$

But $\lambda \in \Lambda$, therefore (4.7) could only occur on $\partial \Sigma^+_{\lambda}$. Hence $x_0 \in \partial \Sigma^+_{\lambda}$.

On the other hand,

$$u < u^{\lambda} \quad \text{on } \partial \Sigma^+_{\lambda} \backslash \overline{T}_{\lambda}.$$

Therefore $x_0 \in \overline{T}_{\lambda}$.

In this case, since $x^i \to x_0 \in \overline{T}_{\lambda}$ and $\lambda^i \to \lambda$, we have $x^{i\lambda^i} \to x_0$. Therefore (4.6) implies that

$$\frac{\partial u}{\partial x_1}(x_0) \geq 0.$$

Hence, $x_0 \in \overline{T}_{\lambda} \cap \partial B_R(0)$ , because

$$\frac{\partial u}{\partial x_1} < 0 \quad \text{in } \overline{T}_{\lambda}.$$

Now, since $x_0 \in \partial B_R(0)$ and $\lim_{i \to \infty} x^i = \lim_{i \to \infty} x^{i\lambda^i} = x_0$, we have $0 < u(x^{i\lambda^i}) < u(x^i) < s_0$ if $i$ is large enough. Therefore, $f(u(x^i)) < f(u(x^{i\lambda^i}))$ for all $i$ large enough.

But $\Delta(u^{\lambda^i} - u)(x^i) \geq 0$, since $x^i$ is a minimum point of $u^{\lambda^i} - u$, so we reach a contradiction, because

$$\Delta(u^\lambda - u)(x^i) + f(u(x^{i\lambda^i})) - f(u(x^i)) = 0.$$

Therefore $\Lambda$ is open.

Step 4. Since $\Lambda$ is non-empty and both open and closed in $(0, R)$, it must be the case that $\Lambda = (0, R)$, so letting $\lambda \to 0$, we find that

$$u(x_1, \cdots, x_n) \leq u(-x_1, \cdots, x_n)$$

for $x \in \Sigma_0^+$. But both $B_R(0)$ and $\Delta$ are invariant under the symmetry group, so

$$u \quad \text{radially symmetric}$$

and

$$\frac{\partial u}{\partial r} < 0 \quad \text{for} \quad 0 < r < R,$$

by 'Claim 1'. Thus, the proof of Theorem 2 is complete.

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