A Dirichlet Problem with Infinite Multiplicity*

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Dedicated to Professor V. Lakshmitkantham
on his 65th birthday

Abstract

We construct examples of strictly convex functions f on $(-\infty, \infty)$ satisfying $f'(-\infty) < n^2 < f'(\infty)$ such that the Dirichlet problem $u'' + f(u) = h(x)$ in $[0, \pi]$, $u(0) = u(\pi) = 0$, has an infinite number of solutions, for any choice of $h(x)$. Kaper and Kwong earlier have presented examples with five solutions to settle a conjecture raised by Lazer and McKenna. Here, we also give a sufficient condition for the number of solutions to be finite. Bounds for the number of solutions of a Dirichlet problem are of interest in the study of boundary value problems of semilinear elliptic equations.

AMS(MOS) Subject Classification. Primary 34B15. Secondary 35J25.

Key Words and Phrases. Nonlinear boundary value problems, semilinear elliptic equations, Dirichlet problems, multiplicity of solutions.

*This work was supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy, under Contract W-31-109-Eng-38.

In an informative article [18], Lazer and McKenna proposed a modified mathematical model for the onset of large-amplitude oscillations in suspension bridges by wind with specific velocities. The study was motivated by the inadequacy of older theories to explain the collapse of the Tacoma Narrows Bridge of Seattle in 1941.

In the Lazer-McKenna model, the motion of the bridge is, as usual, governed by a system of differential equations, more specifically, semilinear elliptic differential equations, the complexity of which depends on the degree of approximation and simplifications one is willing to accept. One of the new ideas introduced is the asymmetry of the restoring force from a cable, with respect to expansion and compression. The authors' basic assumption is that the cable "strongly resists expansion, but does not resist compression." The study of elliptic equations involving a nonlinear restoring-force term of this type is still largely unexplored. In the same article, Lazer and McKenna posed many interesting open questions. Some of these have not been answered even in the one-dimensional case, when the elliptic equation becomes a second-order nonlinear ordinary differential equation.

The study of the multiplicity of boundary value problems of semilinear elliptic equations has attracted much attention recently. The survey paper [18] is a good source of reference to previous work by Lazer and McKenna and others. We mention the related problem of the uniqueness of the positive solution (called the ground state when the solution exists in the entire space R^n), which we have actively worked on in the past few years. Many other authors have contributed to this area; see [2–8,12–14,16,17,20-22] and the references therein. It turns out that techniques used in studying this latter problem can be borrowed to tackle problems mentioned in [18].

In the present work, the boundary value problem we are interested in is

$$\Delta u(x) + f(u(x)) = h(x) \quad \text{in } \Omega \subset R^n, \tag{1}$$

subject to the Dirichlet boundary condition

$$u(x) = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where $f(u)$ is a genuinely, nonlinear continuously differentiable function on $(-\infty, \infty)$, and $h(x)$ is any continuous function in Ω.

The one-dimensional special case of the problem is

$$u''(x) + f(u(x)) = h(x), \quad x \in (0, \pi), \tag{3}$$

$$u(0) = u(\pi) = 0. \tag{4}$$

We shall refer to a solution of (3)–(4) as a *D-solution* and reserve the simpler term *solution* for one that satisfies (3), but not necessarily the Dirichlet boundary conditions.

A central problem in the theory is the determination of upper and lower bounds for the number of distinct D-solutions when f satisfies certain conditions. By continuity, the range of f' is a connected subinterval of $(-\infty, \infty)$. Of particular interest is the dependence of the bounds on I. Let λ_n denote the n^{th} Dirichlet eigenvalue of the Laplacian on Ω. In the one-dimensional case, $\lambda_n = n^2$.

It is a well-known result of Dolph [9] (and Hammerstein [10] for the case $n = 1$) that if the range of f' lies "strictly" between λ_n and λ_{n+1} (more precisely, if $I \subset (\lambda_n + \epsilon, \lambda_{n+1})$, for some $\epsilon > 0$), then the Dirichlet problem has a unique solution.

We say that the range of f' crosses the n^{th} eigenvalue if $\lambda_n \in I$. In such a case, multiple solutions are possible. A result of Lazer and McKenna in [19] shows that if the first few eigenvalues are crossed, then there can be twice as many solutions as there are eigenvalues crossed. It is easy to see that unless further conditions are imposed on f, there can be no finite upper bound for the number of solutions.

The first upper bound was obtained by Ambrosetti and Prodi [1], who showed that under the assumptions

$$f \text{ is strictly convex} \tag{5}$$

and

$$\text{only the first eigenvalue } \lambda_1 \text{ is crossed,} \tag{6}$$

there are at most two solutions.

When a higher eigenvalue is crossed instead, there can be three solutions. Earlier numerical evidence supported the belief that three was the upper bound when only one eigenvalue was crossed. Lazer and McKenna formulated this question and its generalization as

PROBLEM 5 (in [18]). *"If $n^2 < f'(-\infty) < (n+1)^2$, $(n+1)^2 < f'(+\infty) < (n+2)^2$, and $f''(s) > 0$, are there always at most three solutions to* (3)–(4)*?* $\cdots$

(much harder) If $n^2 < f'(-\infty) < f'(+\infty) < (n+k+1)^2$, and $f''(s) > 0$, are there at most $2k+1$ [solutions]?"

The first part of the conjecture was refuted recently in [11] when we constructed an example having five solutions. A natural question that ensues is whether five is now the upper bound. In this article, we show that there can be no finite upper bound. The example we give, in fact, has an infinite number of solutions. This result, incidentally, also settles the second part of Problem 5 in the negative.

The ultimate interest, of course, is in the corresponding problem for the higher-dimensional case, (1)–(2), but any information on the one-dimensional case can shed light on the higher-dimensional case.

Theorem 1 *For any integer $n > 1$ and any $\epsilon > 0$, there exists a strictly convex function f, such that*

$$n^2 - \epsilon < f'(x) < n^2 + \epsilon$$

and the Dirichlet problem (3)–(4) *has an infinite number of solutions for any choice of the function $h(x)$.*

Proof. In the example we construct below, the range of f' crosses the second eigenvalue $\lambda_2 = 4$. The example can be easily modified to take care of other eigenvalues.

The approach we use is the familiar shooting method. We study the one-parameter family of solutions $u(x;\gamma)$ of (3) determined by the initial conditions

$$u(0;\gamma) = 0, \quad u'(0;\gamma) = \gamma. \tag{7}$$

As we vary γ, we note the sign of $u(\pi;\gamma)$. A change in sign implies the existence of a D-solution.

We use an idea we exploited in [15] to confirm another conjecture (the one-dimensional analog of Problem 2 raised in [18]):

> *If $f'(\infty) = \infty$, then the number of solutions becomes unbounded as $h(x)$ becomes large in a certain sense.*

Suppose that for large positive values of u, $f'(u)$ is approaching a constant limit α^2. Let $u(x;\gamma)$ be a solution of (3) having a very large initial slope γ. Obviously, $u(x)$ will be positive in some subinterval $(0,\rho) \subset (0,\pi)$ and has a large amplitude $u_M = \max\{u(x) : x \in [0,\rho]\}$. We scale u vertically to a function of unit amplitude, $\overline{u} = u(x)/u_M$. It satisfies the differential equation

$$\overline{u}''(x) + \left(\frac{f(u)}{u}\right)\overline{u}(x) = \frac{h(x)}{u_M}. \tag{8}$$

Since γ is very large, $u(x)$ becomes very large within a short distance from the initial point $x = 0$, and stays large throughout most of $(0,\rho)$. Thus, $f(u)/u \approx \alpha^2$ for the majority of points in $(0,\rho)$, and the right-hand side of (8) is very small. We can argue (as in [15] using the classical Sturm-Liouville comparison theorem) that $\overline{u}(x)$ is approximately a solution of the linearized equation

$$\overline{U}''(x) + \alpha^2\overline{U}(x) = 0. \tag{9}$$

In particular, if ρ is taken to be the first zero of $u(x)$, then $\rho \approx \pi/\alpha$.

Likewise, if for large negative values of u, $f'(u)$ is approaching some constant limit β^2, then any negative solution $u(x)$ of (3) in a subinterval $(\rho,\sigma) \subset (0,\pi)$, with sufficiently large amplitude, is an approximate solution of

$$\underline{U}''(x) + \beta^2\underline{U}(x) = 0. \tag{10}$$

In particular, if ρ and σ are consecutive zeros of $u(x)$, then $\sigma - \rho \approx \pi/\beta$.

In other words, if we let γ be sufficiently large, then the solution $u(x;\gamma)$, after some vertical scaling, coincides approximately with a solution of

$$U''(x) + F(U(x)) = 0, \tag{11}$$

which is obtained by replacing $f(x)$ in (3) with the two-piece linear function

$$F(u) = \begin{cases} f'(\infty)u, & u \geq 0 \\ f'(-\infty)u, & u \leq 0 \end{cases}. \tag{12}$$

Let us now choose two numbers a and b such that

$$2-\epsilon < b < a < 2+\epsilon \quad \text{and} \quad \frac{1}{a}+\frac{1}{b}=1. \tag{13}$$

Without loss of generality, we may assume that $\epsilon < 1$. This will ensure that a solution $u(x;\gamma)$ of (3) cannot have more than three zeros in $(0,\pi)$.

We shall construct our function f so that

$$f'(\infty)=a^2 \quad \text{and} \quad f'(-\infty)=b^2. \tag{14}$$

It is easy to see that the corresponding differential equation (12) has an infinite number of D-solutions (they are identical modulo a constant multiple), each with exactly three zeros 0, $\rho=\pi/a$, and π. Hence by our observation, those solutions $u(x;\gamma)$ of (3) with large γ will almost be D-solutions. By manipulating the way $f'(u)$ converges to a^2 and b^2, respectively, as $u\to\infty$ and $-\infty$, one can make $u(\pi;\gamma)$ change sign infinitely many times, thus obtaining an infinite number of D-solutions.

We sketch the construction here; the proof of the details can be filled in by the reader without much difficulty.

Choose two sequences of numbers $\{a_n\}$ and $\{b_n\}$ such that

$$b<\cdots<b_2<b_1<2<a_1<a_2<\cdots<a, \tag{15}$$

$$\lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} b_n = b, \tag{16}$$

and

$$\frac{1}{a_n}+\frac{1}{b_n}\begin{cases} >1, & n \text{ odd} \\ <1, & n \text{ even}\end{cases}. \tag{17}$$

This can be achieved in the following way. The sequence $\{a_n\}$ can be any increasing sequence with limit a. Take $\hat{b}_n = a_n/(1-a_n)$. Then modify each $\hat{b}_n$, by either increasing or decreasing its value (a little is enough) according to whether n is odd or even, to get b_n.

For each n, construct $f(u)$ in an interval $[-u_n,u_n]$ for some sufficiently large u_n in the following inductive way. Assume that $f(u)$ has already been defined in $[-u_{n-1},u_{n-1}]$. First continue $f(u)$ outside this interval as if it were going to have the limits $\lim_{u\to\infty} f'(u)=a_n^2$ and $\lim_{u\to-\infty} f'(u)=b_n^2$, while maintaining the strict monotonicity of f'. Then, as the slope γ is gradually increased, $u(x;\gamma)$, after being scaled, will be more and more approximately equal to a solution of

$$U_n''(x) + F_n(U_n(x)) = 0, \tag{18}$$

where

$$F_n(U) = \begin{cases} a_n^2 U, & u \geq 0 \\ b_n^2 U, & u \leq 0 \end{cases}. \tag{19}$$

By (17), such a solution satisfies

$$U_n(\pi) \begin{cases} > 0, & n \text{ odd} \\ < 0, & n \text{ even} \end{cases}. \tag{20}$$

Thus, if γ is large enough, $u(\pi;\gamma)$ has the same sign as $U_n(\pi)$. Fix one such γ, and let u_n be any number larger than $\max\{|u(x;\gamma)| : x \in [0,\pi]\}$.

We summarize by giving an intuitive picture of our arguments. The behavior of $u(x;\gamma)$, as $\gamma \to \infty$, is influenced by the values of $f'(u)$ for large values of u. Our choice of a and b means that, *asymptotically*, $u(x;\gamma)$ approaches a D-solution, but the exact sign of $u(\pi;\gamma)$ depends delicately on the relative sizes of $f'(u)$ for positive and negative large values of u. We choose the sequence of $\{a_n\}$ and $\{b_n\}$ discretely, with the property (17), so that the positive side and the negative side of u win out alternatively, thus producing an infinite number of changes of sign for $u(\pi;\gamma)$. ∎

It is an interesting question to ask

> *what additional condition on f is required to restore* 3 *and* $2k+1$ *as upper bounds for the number of D-solutions, as stated in Problem 5 of* [18]*?*

A wild guess is $f''(s) > 0$.

The method of scaling used in the proof of Theorem 1 can be used to derive a simple sufficient condition for the finiteness of the number of D-solutions. We look at the limiting boundary value problem (11). If it has no D-solution, then for sufficiently large γ, both positive and negative, $u(x;\gamma)$ is not a D-solution of the original problem. Hence, all D-solutions of (3) occur within a bounded interval of initial slopes. As a consequence, there must be a finite number of D-solutions.

For the next result, we do not assume that f is convex, but only that

$$\lim_{u\to\pm\infty} f'(u) = f'(\pm\infty) \text{ exist and are finite.} \tag{21}$$

It is easy to give examples for which either $f'(-\infty)$ or $f'(\infty)$ is equal to 1 and an infinte number of D-solutions exist. Hence, we also assume that

$$f'(\pm\infty) \neq 1. \tag{22}$$

Lemma *Assume that* (21) *and* (22) *hold. If*

$$\textit{either } f'(\infty) \textit{ or } f'(-\infty) < 1, \tag{23}$$

then only a finite number of D-solutions exist.

Proof. First assume that both $f'(-\infty)$ and $f'(\infty)$ are < 1. Then no nontrivial solutions of (11) can have more than one zero in $[0, \pi]$. Hence, (11) has no D-solutions. Suppose that $f'(\infty) > 1$ and $f'(-\infty) < 1$. Any solution of (11) starting from the left endpoint $x = 0$ with a negative initial slope will not vanish again before or at $x = \pi$. Thus, it is not a D-solution. On the other hand, any solution starting from $x = 0$ with a positive initial slope must vanish once in $(0, \pi)$ and then becomes negative all the way up to $x = \pi$. Thus, it is not a D-solution either. The conclusion follows. ∎

To deal with the remaining cases, we assume that

$$\lim_{u\to\infty} f'(u) = a^2 \quad \text{and} \quad \lim_{u\to-\infty} f'(u) = b^2. \tag{24}$$

Theorem 2 *Suppose that* (21) *and* (22) *hold. The boundary value problem* (3)–(4) *has a finite number of solutions if either* (23) *is satisfied or*

$$\textit{none of the three numbers } \frac{ab}{a+b},\ \frac{ab+a}{a+b},\ \textit{or } \frac{ab+b}{a+b} \textit{ is an integer,} \tag{25}$$

where a and b are defined in (24).

Proof. In view of the lemma, it suffices to consider the second case. Again, we look at the limiting boundary value problem (11). The situation in which $ab/(a+b)$ is an integer corresponds to the existence of a D-solution with the same number of positive and negative *half-cycles*, while the other two situations correspond to the existence of one more negative half-cycle and the existence of one more positive half-cycle, respectively. ∎

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