Symmetry Results for Reaction-Diffusion Equations*

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Abstract

This article is concerned with symmetry properties of the solutions of the reaction-diffusion equation $\Delta u + f(u) = 0$ in a bounded connected domain Ω in $\mathbf{R}^N$ ($N = 2, 3, \ldots$). Of especial interest are nonlinear source terms f of the type $f(u) = u^p - u^q$ with $0 \leq q < p \leq 1$. Two results are presented.

The first result concerns the solution of a free boundary problem, where the domain Ω is unknown and u and its normal derivative $\partial_n u$ are required to vanish on the boundary $\partial\Omega$ of Ω. It is shown that, if f is the sum of a continuous nondecreasing function and a Lipschitz continuous function on $[0, \infty)$, then the free boundary problem does not have a positive solution unless Ω is a ball; in this case, any positive solution is radially symmetric around the center of the ball and decreasing with the radial distance from the center.

The second result concerns the solution of the Dirichlet problem on a ball in $\mathbf{R}^N$, when the nonlinear source term f is continuous, but not necessarily Lipschitz continuous at 0. It is shown that, if f is the sum of a locally Lipschitz continuous function on $(0, \infty)$ that is nonincreasing near 0 and a function that is Lipschitz continuous on $[0, \infty)$, then any positive solution u is radially symmetric around the center of the ball and decreasing with the radial distance from the center.

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1 Statement of the Problem

In this article we present two symmetry results for positive solutions of the reaction-diffusion equation

$$\Delta u + f(u) = 0, \quad x \in \Omega, \tag{1.1}$$

where Ω is a bounded connected domain in $\mathbf{R}^N$ $(N = 2, 3, \ldots)$. In the first problem, Ω is not specified, but both u and its normal derivative $\partial_n u$ are required to vanish on the boundary $\partial\Omega$ of Ω. In the second, Ω is a ball in $\mathbf{R}^N$, and u is required to vanish on the boundary of the ball.

The first problem would clearly be overdetermined if Ω were specified. However, as Ω is left unspecified, the extra degree of freedom may be enough to allow for a solution—which, in this case, consists of the pair (Ω, u). We refer to this problem as the *free boundary problem*. Of particular interest is the question whether a solution of the free boundary problem, if it exists, has any symmetry properties.

The second problem is a standard boundary value problem with Dirichlet data. We refer to it as the *Dirichlet problem*. The question here is whether a solution u is radially symmetric if f is continuous, but not necessarily Lipschitz continuous at the origin.

These problems are discussed in more detail in the following subsections. Our results are presented in Theorems 1 and 2 below. Section 2 contains some preliminary material for the proofs of these theorems; the proofs are given in Sections 3 and 4. This article extends and generalizes our earlier communication [1].

1.1 Free Boundary Problem

In a recent discussion of some computational problems in plasma physics, Miller et al. [2] proposed an interesting free boundary problem for reaction-diffusion equations. The physical problem, which concerns the existence of equilibrium configurations with magnetic islands in a Tokamak fusion device, can be posed as a free boundary problem for the partial differential equation

$$\Delta u + \sqrt{u} - 1 = 0, \quad x \in \Omega, \tag{1.2}$$

where Ω is a bounded connected domain in $\mathbf{R}^N$ $(N = 2, 3, \ldots)$. The function u must be positive throughout Ω, and u and its normal derivative $\partial_n u$ must vanish on $\partial\Omega$. The expression $\sqrt{u} - 1$ in the differential equation (1.2) is the simplest form the source term can take; another typical expression is $u^p - u^q$, where $0 \le q < p \le 1$. Therefore, the general form of the free boundary problem is

$$\begin{cases} \Delta u + f(u) = 0, & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0,\ \partial_n u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where the domain $\Omega \subset \mathbf{R}^N$ is unspecified and f is a nonlinear function that is defined and continuous on $[0, \infty)$, but generally not Lipschitz continuous at 0.

In [3], Kaper and Kwong investigated the free boundary problem for the *radially symmetric* case, assuming that Ω is a ball of (unknown) radius R centered at the origin, where the solution u depends only on the distance $r = |x|$ from the origin. They proved that, for a broad class of functions f, there exists a pair (R, u) with $R > 0$ and $u \in C^2((0,R)) \cap C^1([0,R])$, such that

$$\begin{cases} u'' + \frac{N-1}{r} u' + f(u) = 0, & 0 < r < R; \\ u(r) > 0, & 0 < r < R; \\ u'(0) = 0, & \\ u(R) = 0, \ u'(R) = 0. & \end{cases} \tag{1.4}$$

Specifically, they showed that the existence of (R, u) follows if f and its integral F,

$$F(u) = -\int_0^u f(s)\, ds, \tag{1.5}$$

satisfy the following conditions:

(F1) f is continuous on $[0, \infty)$ and locally Lipschitz continuous on $(0, \infty)$.

(F2) There exists a $\beta > 0$, such that $F(u) > 0$ for $0 < u < \beta$, $F(\beta) = 0$, and $f(u) > 0$ for $u \geq \beta$.

(F3) With β as in **(F2)**, $\int_0^\beta F^{-1/2}(u)\, du < \infty$.

(F4) $\liminf_{u \to \infty} f(u) > 0$.

The solution of (1.4) is unique if f satisfies the additional condition

(F5) With β as in **(F2)**, $u \mapsto f(u)/(u - \beta)$ is nonincreasing for $u > \beta$.

The uniqueness follows immediately from an earlier result of Kaper and Kwong [4], which is itself an extension of an earlier result of Peletier and Serrin [5]. The conditions **(F1)** through **(F5)** are satisfied for functions f of the type $f(u) = u^p - u^q$ for most, but not all, values of p and q in the range $0 \leq q < p \leq 1$.

In this article we turn again to the original problem (1.3) and address the issue whether a classical solution (Ω, u), if it exists, is necessarily radially symmetric. Numerical results presented for the case $N = 2$ in [2] seem to indicate that the implication is false if $f(u) = \sqrt{u} - 1$; the function u in [2, Fig. 6] is clearly not radially symmetric.

We prove the following theorem.

Theorem 1 *If f is such that*

$$f(u) = f_1(u) + f_2(u), \tag{1.6}$$

where f_1 is continuous and nondecreasing and f_2 is Lipschitz continuous on $[0, \infty)$, then any classical solution (Ω, u) of the free boundary problem (1.3) that satisfies the condition

$u \in C^2(\overline{\Omega})$ is necessarily radially symmetric—that is, Ω is an open ball in $\mathbf{R}^N$, $\Omega = B_R(x_0)$ say, and u is radially symmetric about the center x_0 of the ball. Furthermore,

$$\frac{\partial u}{\partial r} < 0, \quad 0 < r = |x - x_0| < R. \tag{1.7}$$

The conditions of the theorem, together with the conditions **(F1)** through **(F5)** above, guarantee that the free boundary problem (1.3) has one and only one classical solution (Ω, u) and that this solution is necessarily radially symmetric.

The conditions of Theorem 1 are satisfied, for example, if $f(u) = u^p - 1$, where $0 < p \leq 1$. (Take $f_1(u) = u^p$ and $f_2(u) = -1$.) They are not satisfied, however, if $f(u) = u^p - u^q$, where $0 < q < p \leq 1$.

The proof of Theorem 1 is given in Section 3.

1.2 Dirichlet Problem

Our second result is for the Dirichlet problem for the reaction-diffusion equation (1.1) in a ball of radius R ($R > 0$) in $\mathbf{R}^N$ ($N = 2, 3, \ldots$). Without loss of generality, we may assume that the ball is centered at the origin, so $\Omega = B_R(0)$.

According to the celebrated result of Gidas, Ni, and Nirenberg [6, Section 1.1, Theorem 1, and Section 2.3, Remark 1], any classical solution of the Dirichlet problem,

$$\begin{cases} \Delta u + f(u) = 0, & x \in B_R(0), \\ u(x) > 0, & x \in B_R(0), \\ u(x) = 0, & x \in \partial B_R(0), \end{cases} \tag{1.8}$$

is necessarily radially symmetric about the origin if f is the sum of a function that is continuous and nondecreasing and a function that is Lipschitz continuous on $[0, \infty)$; furthermore, $\partial u / \partial r < 0$ for $0 < r < R$. If such a decomposition of f is not possible, then it is an open problem whether classical solutions of (1.8), if they exist, are indeed radially symmetric. Actually, the examples given in [6, Section 2.4] show that these cases could be very delicate.

We prove the following theorem.

Theorem 2 *If f is such that*

$$f(u) = f_1(u) + f_2(u), \tag{1.9}$$

where f_1 is continuous on $[0, \infty)$, locally Lipschitz continuous on $(0, \infty)$, and nonincreasing near 0, and f_2 is Lipschitz continuous on $[0, \infty)$, then any classical solution u of (1.8) is necessarily radially symmetric about 0. Furthermore,

$$\frac{\partial u}{\partial r} < 0, \quad 0 < r = |x| < R. \tag{1.10}$$

The theorem covers the case $f(u) = u^p - u^q$, where $0 < q < p \leq 1$, as f is decreasing on $[0, (q/p)^{1/(p-q)})$.

The proof of Theorem 2 is given in Section 4.

1.3 Discussion

The following discussion clarifies the relationship between Theorems 1 and 2.

The central question of interest to us was whether the existence of a positive solution (Ω, u) of the free boundary problem (1.3) implies its radial symmetry when the nonlinear source term f is continuous, but not necessarily Lipschitz continuous at 0. In particular, our interest focused on functions f of the type $f(u) = u^p - u^q$, where $0 \leq q < p \leq 1$.

The case $f(u) = u^p - 1$ $(0 < p \leq 1)$ is indeed covered by Theorem 1, but $f(u) = u^p - u^q$ $(0 < q < p \leq 1)$ is not. In fact, we were unsuccessful in our attempts at extending Theorem 1 to include functions of the latter type. However, observing that such functions are Lipschitz continuous everywhere except at 0 and decreasing near 0, we found that the moving-plane method could again be used to prove the radial symmetry of any positive solution of the Dirichlet problem for the reaction-diffusion equation (1.1) *if* Ω is specified to be a ball. This is the result expressed in Theorem 2.

Theorem 1 bears an interesting relationship to the celebrated result of Gidas, Ni, and Nirenberg [6] quoted in the beginning of Subsection 1.2. Gidas, Ni, and Nirenberg showed that, *if* Ω is a ball, *then* any solution of the Dirichlet problem is necessarily radially symmetric, *provided* f is the sum of a function that is continuous nondecreasing and a function that is Lipschitz continuous on $[0, \infty)$. Theorem 1 shows that, by imposing the extra condition $\partial_n u = 0$ in (1.3), we have forced ourselves into a situation where there is no (positive) solution *unless* Ω is a ball.

We remark that new symmetry results for solutions of reaction-diffusion equations on nonsmooth domains have been obtained recently by Garofalo and Lewis [7] and Lewis and Vogel [8].

2 Maximum Principle and Boundary Lemmas

In this section, we summarize the maximum principle and two boundary lemmas for elliptic differential expressions in a form suitable for the subsequent analysis. Throughout this subsection, D is an arbitrary domain (i.e., an open connected set) in $\mathbf{R}^N$ and L denotes a uniformly elliptic differential expression on D,

$$Lu \equiv a^{ij}(x)\partial_{ij}u + b^i(x)\partial_i u + c(x)u,$$

where $a^{ij}, b^i, c \in L^\infty(D)$. (We use the notation $\partial_i = \partial/\partial x_i$ and $\partial_{ij} = \partial^2/\partial x_i \partial x_j$, together with the summation convention for repeated indices.)

Lemma 1 (Maximum Principle). *Suppose that $u \in C^2(D)$ satisfies the inequalities $u \geq 0$ and $Lu \leq 0$ on D. If u vanishes at some point in D, then $u = 0$ on D.*

Proof. See [9, Chap. 2, Theorem 6]. ∎

Lemma 2 (Hopf Boundary Lemma). *Suppose that $u \in C^2(D)$ satisfies the inequalities $u > 0$ and $Lu \leq 0$ on D. Suppose, furthermore, that $P \in \partial D$ lies on the boundary of a ball B in D. If u is continuous in $D \cup P$ and $u(P) = 0$, then $\partial_\nu u(P) < 0$ for any outward direction ν.*

Proof. See [6, Section 1.3, Lemma H] and [9, Chap. 2, Theorem 8]. ∎

The derivative $\partial_\nu u$ is defined as the derivative in the direction $\nu = (\nu_1, \ldots, \nu_N)$; that is, $\partial_\nu = \nu_i \partial_i$. The vector ν is said to point outward from D if its scalar product with n, the unit normal vector in the outward direction, is positive ($\nu_i n_i > 0$).

Lemma 3 (Serrin Boundary Lemma). *Suppose that $P \in \partial D$ and that, near P, ∂D consists of two transversally intersecting C^2-hypersurfaces $\rho = 0$ and $\sigma = 0$, where $\rho < 0$ and $\sigma < 0$ in D and $(a^{ij}\partial_i\rho\partial_j\sigma)(P) \geq 0$.*

Suppose that $u \in C^2(\overline{D})$, $u > 0$ and $Lu \leq 0$ on D, and $u(P) = 0$.

(i) *If $(a^{ij}\partial_i\rho\partial_j\sigma)(P) > 0$, then $\partial_\nu u(P) < 0$ for any outward direction ν that is transverse to the hypersurfaces $\rho = 0$ and $\sigma = 0$.*

(ii) *If $(a^{ij}\partial_i\rho\partial_j\sigma)(P) = 0$, then $\partial_\nu u(P) < 0$ or $\partial_\nu^2 u(P) > 0$ for any outward direction ν that is transverse to the hypersurfaces $\rho = 0$ and $\sigma = 0$, provided a^{ij} is twice continuously differentiable in $D \cup P$ and $\partial_t(a^{ij}\partial_i\rho\partial_j\sigma)(P) \geq 0$ for any derivative in a direction t that is tangent to the submanifold $\{\rho = 0\} \cap \{\sigma = 0\}$.*

Proof. See [6, Section 1.3, Lemma S] and [10, Lemma 2]. ∎

The following lemma is a nonlinear generalization of the Hopf Boundary Lemma.

Lemma 4 *Let $f = f_1 + f_2$, where f_1 is continuous on $[0, \infty)$ and nondecreasing near 0 and f_2 is Lipschitz continuous on $[0, \infty)$, with $f(0) \geq 0$. Suppose that $u \in C^2(D)$ satisfies the inequalities $u > 0$ and $\Delta u + f(u) \leq 0$ on D. Suppose, furthermore, that $P \in \partial D$ lies on the boundary of a ball B in D. If u is continuous in $D \cup P$ and $u = 0$ on ∂D near P, then $\partial_\nu u(P) < 0$ for any outward direction ν.*

Proof. As f_1 is nondecreasing near 0, we have $f_1(u) \geq f_1(0)$ on D near P. Furthermore, f_2 is Lipschitz continuous on $[0, \infty)$, so there exists a bounded function c on D such that

$f_2(u) - f_2(0) = cu$ on D. Therefore, u satisfies the differential inequality $\Delta u + cu \leq -f(0)$ or, since $f(0) \geq 0$, $\Delta u + cu \leq 0$ on D. The assertion of the lemma follows from the Hopf Boundary Lemma (Lemma 2). ∎

The final lemma covers reaction-diffusion equations where $f(0)$ is negative. It requires more smoothness of the solution u; cf. [6, Section 2.2, Lemma 2.1].

Lemma 5 *Let f be continuous on $[0, \infty)$, with $f(0) < 0$. Suppose that $u \in C^2(D)$ satisfies the inequality $u > 0$ on D and that $\Delta u + f(u) = 0$ on D. Suppose, furthermore, that $P \in \partial D$ lies on the boundary of a ball B in D. If u is twice continuously differentiable in $D \cup P$ and $u = 0$ on ∂D near P, then $\partial_\nu u(P) < 0$ or $\partial_\nu^2 u(P) > 0$ for any outward direction ν.*

Proof. Because u vanishes identically on ∂D near P, any directional derivative in the tangent plane vanishes at P. Furthermore, as $u > 0$ on D, it must be the case that $\partial_n u(P) \leq 0$.

If $\partial_n u(P) < 0$, then $\partial_\nu u(P) < 0$ for any outward direction ν.

If $\partial_n u(P) = 0$, then all directional derivatives at P are zero. Using a local coordinate system with the origin at P, one coordinate along the outward normal vector, and the remaining $N-1$ coordinates in the tangent plane, we readily verify that $\Delta u(P) = \partial_n^2 u(P)$. Since u is twice continuously differentiable on $D \cup P$ and $u(P) = 0$, we have $\Delta u(P) = -f(0) > 0$; hence, $\partial_n^2 u(P) > 0$. Then also $\partial_\nu^2 u(P) > 0$ for any outward direction ν. ∎

3 Proof of Theorem 1

Suppose (Ω, u) is a classical solution of (1.3), where f satisfies the conditions of Theorem 1.

According to Lemma 4, we necessarily have $\partial_n u < 0$ on $\partial\Omega$ if $f(0) \geq 0$, so it must be the case that $f(0) < 0$.

We use the moving-plane method for the proof of the theorem. This method, due to Alexandroff (see [11, Section VII.1]) was first applied by Serrin [10] to prove a symmetry result for the constant mean-curvature equation $\Delta u = -c$, $c > 0$, and has been used since then by several authors.

Let T_λ be the hyperplane

$$T_\lambda = \{x = (x_1, \ldots, x_N) \in \mathbf{R}^N : x_1 = \lambda\}, \tag{3.1}$$

and let S_λ be the reflection operator about T_λ,

$$S_\lambda : \quad x = (x_1, x_2, \ldots, x_N) \mapsto S_\lambda x = (2\lambda - x_1, x_2, \cdots, x_N), \quad x \in \mathbf{R}^N. \tag{3.2}$$

As Ω is bounded, T_λ and Ω are disjoint for all sufficiently large λ. Let λ_1 be defined by

$$\lambda_1 = \inf\{\lambda : T_\lambda \cap \Omega = \emptyset\}. \tag{3.3}$$

Without loss of generality, we may assume that the set $\partial\Omega \cap T_{\lambda_1}$ consists of a single point; if necessary, we rotate the coordinate system.

For $\lambda < \lambda_1$, λ sufficiently close to λ_1, T_λ cuts a cap Σ_λ off Ω,

$$\Sigma_\lambda = \{x \in \Omega : x_1 > \lambda\}, \quad \lambda < \lambda_1. \tag{3.4}$$

The reflection of Σ_λ about T_λ is $S_\lambda(\Sigma_\lambda)$,

$$S_\lambda(\Sigma_\lambda) = \{S_\lambda x : x \in \Sigma_\lambda\}. \tag{3.5}$$

For all λ sufficiently close to λ_1, $S_\lambda(\Sigma_\lambda)$ is a proper subset of Ω. Let λ_0 be defined by

$$\lambda_0 = \inf\{\lambda : S_\lambda(\Sigma_\lambda) \subset \Omega\}. \tag{3.6}$$

As Ω is bounded, λ_0 is finite. Obviously, $\lambda_0 < \lambda_1$. For $\lambda = \lambda_0$, either the boundary of $S_{\lambda_0}(\Sigma_{\lambda_0})$ is internally tangent to the boundary of Ω at some point P not on T_{λ_0}, or T_{λ_0} is orthogonal to the boundary of Ω at some point P on $\partial\Omega$.

For each $\lambda \in (\lambda_0, \lambda_1)$, let u_λ be defined in terms of u by the expression

$$u_\lambda(x) = u(S_\lambda x), \quad x \in \Sigma_\lambda. \tag{3.7}$$

The following lemma plays a crucial role.

Lemma 6 *If $u_\lambda \geq u$ on Σ_λ, then either $u_\lambda > u$ on Σ_λ or $u_\lambda = u$ on Σ_λ. In the former case, $\partial_1 u < 0$ on $\Omega \cap T_\lambda$.*

Proof. Because u is a solution of (1.3), we have the identity $\Delta(u_\lambda - u) + f_1(u_\lambda) - f_1(u) + f_2(u_\lambda) - f_2(u) = 0$ on Σ_λ. As f_1 is nondecreasing, $f_1(u_\lambda) \geq f_1(u)$. Furthermore, f_2 is Lipschitz on $[0, \infty)$, so there exists a bounded function c_λ on Σ_λ, such that $f_2(u_\lambda) - f_2(u) = c_\lambda(u_\lambda - u)$. Hence, $u_\lambda - u \geq 0$ and $\Delta(u_\lambda - u) + c_\lambda(u_\lambda - u) \leq 0$ on Σ_λ. The first part of the lemma follows from the Maximum Principle (Lemma 1).

Obviously, $u_\lambda - u = 0$ on $\Omega \cap T_\lambda$, so the Hopf Boundary Lemma (Lemma 2) implies that $-\partial_1(u_\lambda - u) < 0$ on $\Omega \cap T_\lambda$. Since $-\partial_1 u_\lambda = \partial_1 u$ on $\Omega \cap T_\lambda$, the second part of the lemma follows. ∎

Let Λ be the set

$$\Lambda = \{\lambda \in (\lambda_0, \lambda_1) : \; u_\lambda(x) > u(x), x \in \Sigma_\lambda\}. \tag{3.8}$$

We claim that $\Lambda = (\lambda_0, \lambda_1)$. The proof is given in two steps; in the first step, we prove that Λ is nonempty and contains all λ sufficiently close to λ_1; in the second step, we prove that Λ actually contains the entire interval (λ_0, λ_1). The proof requires that u be twice continuously differentiable in the closure of Ω.

Lemma 7 *If $u \in C^2(\overline{\Omega})$, then there exists an $\varepsilon > 0$ such that $(\lambda_1 - \varepsilon, \lambda_1) \subset \Lambda$.*

Proof. Let P be the point where $\partial\Omega$ intersects the hyperplane T_{λ_1}. At P, the outward normal vector is oriented in the positive x_1-direction.

Because u vanishes on $\partial\Omega$ near P and $\partial_1 u(P) = 0$, it follows from Lemma 5 that $\partial_1^2 u(P) > 0$. This inequality extends by continuity to a small arc $\Gamma_\varepsilon = \partial\Omega \cap B_\varepsilon(P)$. Also, $\partial_1 u = 0$ on Γ_ε, so integrating over x_1 from Γ_ε into Ω and reducing ε if necessary, we find that $\partial_1 u < 0$ in a small domain $\Omega_\varepsilon = \Omega \cap B_\varepsilon(P)$ near P. By choosing λ in (λ_0, λ_1) sufficiently close to λ_1 we can ensure that Σ_λ and its reflection $S_\lambda(\Sigma_\lambda)$ are entirely contained in Ω_ε. For such λ we certainly have $u_\lambda > u$ on Σ_λ, so $\lambda \in \Lambda$. ∎

Lemma 8 *If $u \in C^2(\overline{\Omega})$, then $\Lambda = (\lambda_0, \lambda_1)$.*

Proof. Suppose that there exists a monotonically decreasing sequence $\{\lambda^i\}$ of values $\lambda^i \in (\lambda_0, \lambda_1)$ that belong to Λ and converge to some $\lambda \in (\lambda_0, \lambda_1)$. As $\lambda^i \in \Lambda$, we have $u_{\lambda^i} > u$ on Σ_{λ^i}, so $u_\lambda \geq u$ on Σ_λ, by continuity. Since $\lambda \in (\lambda_0, \lambda_1)$, we can apply Lemma 6 and conclude that $u_\lambda > u$ on Σ_λ. This result shows that the set Λ is left-closed in (λ_0, λ_1).

Next we prove that the set Λ is left-open in (λ_0, λ_1). The proof is by contradiction, where we assume that there exists a $\lambda \in (\lambda_0, \lambda_1)$ that belongs to Λ and a monotonically decreasing sequence $\{\lambda^i\}$ of values $\lambda^i \in (\lambda, \lambda_1)$ that do not belong to Λ, with $\lambda^i \to \lambda$ as $i \to \infty$.

As $\lambda^i \notin \Lambda$, we can find a point $x^i \in \Sigma_{\lambda^i}$, such that $u_{\lambda^i}(x^i) \leq u(x^i)$. Because Ω is bounded, we can extract a convergent subsequence of $\{x^i\}$, say $\{x^i\}$ itself, whose limit point, P say, is in the closure of Σ_λ. At P, we have $u_\lambda(P) \leq u(P)$. We claim that no such point P exists.

Since $\lambda \in \Lambda$, we know that $u_\lambda > u$ on Σ_λ, so $P \notin \Sigma_\lambda$.

Suppose that $P \in \partial\Sigma_\lambda \backslash (\overline{\Omega} \cap T_\lambda)$. Then $P \in \partial\Omega$, so $u(P) = 0$. On the other hand, $u_\lambda(P) = u(S_\lambda P) > 0$, because $\lambda_0 < \lambda < \lambda_1$ and, therefore, $S_\lambda P \in \Omega$. Thus we find that $u_\lambda(P) > u(P)$. But we have already shown that $u_\lambda(P) \leq u(P)$, so we must conclude that $P \notin \partial\Sigma_\lambda \backslash (\overline{\Omega} \cap T_\lambda)$.

Suppose that $P \in \overline{\Omega} \cap T_\lambda$. The line segment ℓ_i joining x^i and $S_{\lambda^i} x^i$ lies entirely in Ω. Therefore, the inequality $u_{\lambda^i}(x^i) \leq u(x^i)$ implies that $\partial_1 u(y^i) \geq 0$ at some point $y^i \in \ell_i$. Now, $x^i \to P$ and $S_{\lambda^i} x^i \to S_\lambda P$ as $i \to \infty$, and if $P \in \overline{\Omega} \cap T_\lambda$, as supposed, then $S_\lambda P = P$. Thus, ℓ_i shrinks to the single point P and we find that $\partial_1 u(P) \geq 0$. According to Lemma 6, $\partial_1 u < 0$ on $\Omega \cap T_\lambda$, so we must conclude that $P \notin \Omega \cap T_\lambda$.

The only remaining possibility is $P \in \partial\Omega \cap T_\lambda$. At such a point P, we have $\partial_1^2 u(P) > 0$, according to Lemma 5. This inequality extends by continuity to a small domain inside Ω near P, so $\partial_1 u$ is a (strictly) increasing function of x_1 there. But then it follows that $\partial_1 u < 0$ on a small domain inside Ω near P, since $\partial_1 u = 0$ on $\partial\Omega$. However, this conclusion contradicts the assumption that there exists a sequence $\{(\lambda^i, x^i)\}$ with $\lambda^i \in (\lambda, \lambda_1)$, $x^i \in \Sigma_{\lambda^i} \subset \Sigma_\lambda$, and converging to (λ, P), where $u_{\lambda^i}(x^i) = u(S_{\lambda^i} x^i) \le u(x^i)$. We must therefore conclude that $P \notin \partial\Omega \cap T_\lambda$.

Thus, we have exhausted all the possibilities and conclude that no such point P exists. The assumption that there exists a $\lambda \in (\lambda_0, \lambda_1)$ that belongs to Λ and a monotonically decreasing sequence $\{\lambda^i\}$ of values $\lambda^i \in (\lambda, \lambda_1)$ that do not belong to Λ, with $\lambda^i \to \lambda$ as $i \to \infty$, is untenable. Hence, Λ is left-open in (λ_0, λ_1).

According to Lemma 7, the set Λ contains an interval $(\lambda_1 - \varepsilon, \lambda_1)$ with $\varepsilon > 0$. Since Λ is both left-open and left-closed in (λ_0, λ_1), it must be the case that $\Lambda = (\lambda_0, \lambda_1)$. ∎

Next, we investigate what happens at $\lambda = \lambda_0$. We recall that, at $\lambda = \lambda_0$, either the boundary of $S_{\lambda_0}(\Sigma_{\lambda_0})$ is internally tangent to $\partial\Omega$ at some point not on T_{λ_0}, or T_{λ_0} is orthogonal to $\partial\Omega$.

Lemma 9 *If $u \in C^2(\overline{\Omega})$, then Ω is symmetric about T_{λ_0} and $u_{\lambda_0} = u$ on Σ_{λ_0}.*

Proof. By Lemma 8, $\Lambda = (\lambda_0, \lambda_1)$, so $u_\lambda > u$ on Σ_λ for all $\lambda \in (\lambda_0, \lambda_1)$. Then $u_{\lambda_0} \ge u$ on Σ_{λ_0}, by continuity. The same arguments as in Lemma 6 yield the inequality $\Delta(u_{\lambda_0} - u) + c_{\lambda_0}(u_{\lambda_0} - u) \le 0$ on Σ_{λ_0}, where c_{λ_0} is bounded, so it follows from the Maximum Principle (Lemma 1) that either $u_{\lambda_0} > u$ on Σ_{λ_0} or $u_{\lambda_0} = u$ on Σ_{λ_0}. We claim that the former case is impossible.

If the boundary of $S_{\lambda_0}(\Sigma_{\lambda_0})$ is internally tangent to $\partial\Omega$ at some point $S_{\lambda_0}P$, then $u_{\lambda_0}(P) = u(P)$. The boundary of Σ_{λ_0} is smooth near P, so if $u_{\lambda_0} > u$ on Σ_{λ_0}, we can apply the Hopf Boundary Lemma (Lemma 2) and conclude that $\partial_1(u_{\lambda_0} - u)(P) < 0$. But $\partial_1 u_{\lambda_0}(P)$ and $\partial_1 u(P)$ are both zero, because the normal derivative of u vanishes on $\partial\Omega$, so we have a contradiction.

On the other hand, if T_{λ_0} is orthogonal to $\partial\Omega$ and P is a point of $\partial\Omega \cap T_{\lambda_0}$, we can apply the Serrin Boundary Lemma (Lemma 3) at P. Near such a point P, $\partial\Sigma_{\lambda_0}$ consists of two orthogonally intersecting hypersurfaces $\rho(x) = \lambda_0 - x_1 = 0$ and $\sigma(x) = 0$. A simple computation shows that we are in case (ii) of Lemma 3. Because u and u_{λ_0} coincide on T_{λ_0}, any directional derivative of $u_{\lambda_0} - u$ in the hyperplane T_{λ_0} is zero. Moreover, $u = 0$ on $\partial\Omega$, so any directional derivative in the tangent plane to $\partial\Omega$ at P is also zero. Consequently, $\partial_\nu(u_{\lambda_0} - u)(P) = 0$ for any outward direction ν. If $u_{\lambda_0} > u$ on Σ_{λ_0}, then it follows from the Serrin Boundary Lemma (Lemma 3) that $\partial_\nu^2(u_{\lambda_0} - u)(P) > 0$ for any outward direction ν that is transverse to $\partial\Omega$ and T_{λ_0}. In particular, taking $\nu = (-1/\sqrt{2}, 0, \ldots, 0, 1/\sqrt{2})$, we find $(-\partial_1 + \partial_N)^2(u_{\lambda_0} - u)(P) > 0$. But $\partial_1^2 u_{\lambda_0}(P) = \partial_1^2 u(P)$ and $\partial_N^2 u_{\lambda_0}(P) = \partial_N^2 u(P)$, while $\partial_1\partial_N u_{\lambda_0}(P) = -\partial_1\partial_N u(P)$, so the inequality reduces to $\partial_1\partial_N u(P) > 0$. However, since both u and $\partial_n u$ are identically zero on $\partial\Omega$, the only second-order derivative that does not vanish at P is $\partial_n^2 u$; in fact, $\partial_n^2 u(P) = -f(0)$. At P, the normal vector is in the linear manifold spanned by $x_2, \ldots, x_N$, so it must be the case that $\partial_1\partial_N u(P) = 0$. Again, we have arrived at a contradiction.

We therefore conclude that $u_{\lambda_0} = u$ on Σ_{λ_0}.

Now, if the union of Σ_{λ_0}, $S_\lambda(\Sigma_{\lambda_0})$, and $\Omega \cap T_{\lambda_0}$ were a proper subset of Ω, then a part of $\partial S_{\lambda_0}(\Sigma_{\lambda_0}) \backslash (\Omega \cap T_{\lambda_0})$ would be in Ω. But then we would have a contradiction, as u vanishes on $\partial S_{\lambda_0}(\Sigma_{\lambda_0}) \backslash (\Omega \cap T_{\lambda_0})$. Hence, Ω is the union of Σ_{λ_0}, $S_\lambda(\Sigma_{\lambda_0})$, and $\Omega \cap T_{\lambda_0}$. ∎

Conclusion of the Proof. If (Ω, u) is a classical solution of (1.3) and $u \in C^2(\overline{\Omega})$, then Lemma 9 applies, so Ω is symmetric about the hyperplane T_{λ_0} and $u_{\lambda_0} = u$ on Σ_{λ_0}. Moreover, $u_\lambda > u$ and $\partial_1 u < 0$ on Σ_λ for all $\lambda \in (\lambda_0, \lambda_1)$.

The free boundary problem (1.3) is rotationally invariant, so the positive x_1-direction is not privileged in any way. Therefore, Ω is symmetric in every direction—that is, Ω is a ball, because it is connected—and the gradient of u in any radial direction is negative, as claimed. ∎

4 Proof of Theorem 2

Throughout this section Ω denotes the ball of radius R (R fixed) centered at the origin,

$$\Omega = B_R(0), \quad R > 0. \tag{4.1}$$

The proof of Theorem 2 is also based on the moving-plane method; in the notation of Eqs. (3.3) and (3.6), we have

$$\lambda_0 = 0, \quad \lambda_1 = R. \tag{4.2}$$

Suppose u is a classical solution of the Dirichlet problem (1.8), where f satisfies the conditions of Theorem 2. For each $\lambda \in (\lambda_0, \lambda_1)$, u defines a function u_λ on Σ_λ as in (3.7),

$$u_\lambda(x) = u(S_\lambda x), \quad x \in \Sigma_\lambda. \tag{4.3}$$

Lemma 10 *If $u_\lambda \geq u$ on Σ_λ, then either $u_\lambda > u$ on Σ_λ or $u_\lambda = u$ on Σ_λ. In the former case, $\partial_1 u < 0$ on $\Omega \cap T_\lambda$.*

Proof. The proof is similar to the proof of Lemma 6. Because u is a solution of (1.8), we have the identity $\Delta(u_\lambda - u) + f(u_\lambda) - f(u) = 0$ on Σ_λ. As $u > 0$ on Σ_λ and f is locally Lipschitz on $(0, \infty)$, there exists a locally bounded function c_λ on Σ_λ, such that $f(u_\lambda) - f(u) = c_\lambda(u_\lambda - u)$. Hence, $u_\lambda - u \geq 0$ and $\Delta(u_\lambda - u) + c_\lambda(u_\lambda - u) = 0$ on Σ_λ. The first part of the lemma follows from the Maximum Principle (Lemma 1). The second part of the lemma follows from the Hopf Boundary Lemma (Lemma 2). ∎

We define the set Λ as in (3.8),

$$\Lambda = \{\lambda \in (\lambda_0, \lambda_1) : \ u_\lambda(x) > u(x), x \in \Sigma_\lambda\}. \tag{4.4}$$

As before, our objective is to show that $\Lambda = (\lambda_0, \lambda_1)$.

Lemma 11 *There exists an $\varepsilon > 0$ such that $(\lambda_1 - \varepsilon, \lambda_1) \subset \Lambda$.*

Proof. We prove the lemma by contradiction, assuming that in every domain Σ_λ there is a point y where $u_\lambda(y) \leq u(y)$, no matter how close λ is to λ_1.

Let P be the point where $\partial\Omega$ intersects the hyperplane T_{λ_1}. Let $c = \sup\{|f_2(u) - f_2(v)|/|u-v|\} : u, v \in [0,\infty)\}$. We choose $\eta > 0$ sufficiently small that the first (positive) eigenvalue $\mu_1(\Omega_\eta)$ of the Dirichlet problem for $-\Delta$ on $\Omega_\eta = \Omega \cap B_\eta(P)$ satisfies the inequality $\mu_1(\Omega_\eta) > c$. Such a choice is always possible, as μ_1 is a decreasing function of $|\Omega_\eta|$ and $\mu_1(\Omega_\eta) \to \infty$ as $|\Omega_\eta| \to 0$. We restrict the discussion to values of λ sufficiently close to λ_1 that Σ_λ and its reflection $S_\lambda(\Sigma_\lambda)$ are both entirely contained in Ω_η.

First, we assume that in every domain Σ_λ there is a point y where $u_\lambda(y) < u(y)$. Because $u = 0$ on $\partial\Omega$ and $u > 0$ on Ω, it is certainly true that $u_\lambda \geq u$ on $\partial\Sigma_\lambda \backslash (\Omega_\eta \cap T_\lambda)$. Furthermore, $u_\lambda = u$ on $\Omega_\eta \cap T_\lambda$, so $u_\lambda \geq u$ on $\partial\Sigma_\lambda$. But then there exists a neighborhood N of y, which is entirely contained in Σ_λ, such that $u_\lambda < u$ on N and $u_\lambda = u$ on ∂N.

Now, f_1 is nonincreasing near 0 and u vanishes on $\partial\Omega$, so for λ sufficiently close to λ_1 we certainly have $f_1(u_\lambda) \geq f_1(u)$ on N. Furthermore, $f_2(u) - f_2(u_\lambda) \leq c(u - u_\lambda)$ on N, so $\Delta(u - u_\lambda) + c(u - u_\lambda) \geq 0$ on N. Multiplying both sides of this inequality by $u - u_\lambda$, which is positive on N, integrating over N, and using the inequality $(\Delta v, v) \leq -\mu_1(N)(v,v)$, where $\mu_1(N)$ is the first (positive) eigenvalue for the Dirichlet problem for $-\Delta$ on N, we find that $\mu_1(N) \leq c$. But now we have a contradiction, as $\mu_1(N) > \mu_1(\Omega_\eta)$ and η was chosen in such a way as to ensure that $\mu_1(\Omega_\eta) > c$.

On Σ_λ, both u and u_λ are positive. Since f is locally Lipschitz continuous on $(0,\infty)$, we can invoke the Maximum Principle (Lemma 1) to rule out the possibility that $u_\lambda(y) = u(y)$. Thus we conclude that $u_\lambda > u$ on Σ_λ for all λ sufficiently close to λ_1. ∎

Lemma 12 $\Lambda = (\lambda_0, \lambda_1)$.

Proof. As in the proof of Lemma 8, we show that Λ is left-closed in (λ_0, λ_1). The next task is to show that Λ is left-open in (λ_0, λ_1). The proof is similar to the proof in Lemma 8 and is based on an argument by contradiction, but it differs in a critical detail. The proof of Lemma 8 depends on Lemma 5, which holds only if $f(0) < 0$, and in the current situation the existence of a classical solution of (1.8) does not imply anything about the sign of $f(0)$.

To prove by contradiction that Λ is left-open in (λ_0, λ_1), we assume that there exists a point $\lambda \in (\lambda_0, \lambda_1)$ belonging to Λ and a monotonically decreasing sequence $\{\lambda^i\}$ of points λ^i not belonging to Λ, such that $\lambda^i \to \lambda$ as $i \to \infty$. The assumption implies that there exists a sequence of points $\{x^i\}$, $x^i \in \Sigma_{\lambda^i}$, where $u_{\lambda^i}(x^i) \leq u(x^i)$. The sequence $\{x^i\}$ converges to a point P in the closure of Σ_λ, where $u_\lambda(P) \leq u(P)$.

The same arguments as in the proof of Lemma 8 can be used to show that P does not belong to Σ_λ or to $\partial\Sigma_\lambda \backslash (\overline{\Omega} \cap T_\lambda)$ or to $\Omega \cap T_\lambda$. To rule out the remaining possibility, namely, that P belongs to $\partial\Omega \cap T_\lambda$, we need a different argument, as the proof of Lemma 8 relies on Lemma 5.

The argument is partially similar to the argument used in the proof of Lemma 11. First, we restrict the discussion to a small subdomain D_η of Σ_λ near P. Specifically, we take D_η to be the intersection of Σ_λ with a cylinder C_η, which extends to the right of T_λ (i.e., in the half-space $x_1 \geq \lambda$), whose axis passes through P and runs parallel to the x_1-axis, and whose radius η is fixed such that the first (positive) eigenvalue of the Dirichlet problem for $-\Delta$ on D_η satisfies the inequality $\mu_1(D_\eta) > c$.

Suppose that, in every neighborhood of P inside Σ_λ, there is a point y where $u_\lambda(y) < u(y)$. Then there certainly is such a point y in D_η. Now, consider the values of u_λ and u on the boundary ∂D_η of D_η.

The boundary ∂D_η consists of three hypersurfaces: one (labeled S_1) on the boundary $\partial\Omega$ of Ω, one (labeled S_2) on the surface of the circular cylinder C_η, and one (labeled S_3) on the hyperplane T_λ. Because $u = 0$ on $\partial\Omega$ and $u > 0$ on Ω, it is certainly the case that $u_\lambda \geq u$ on S_1. Also, because $\partial_1 u < 0$ on $\Omega \cap T_\lambda$, we have $u_\lambda > u$ on S_2; if necessary, we reduce η, so C_η intersects $\partial\Omega$ sufficiently close to T_λ. Finally, $u_\lambda = u$ on T_λ and therefore on S_3. Thus, we see that $u_\lambda \geq u$ on ∂D_η.

But then there is a neighborhood N of y, which is entirely contained in D_η, such that $u_\lambda < u$ on N and $u_\lambda = u$ on ∂N.

Now, we complete the proof as in Lemma 11 by showing that $\mu_1(N) \leq c$, and thus arrive at a contradiction. Using the Maximum Principle (Lemma 1), we then rule out the possibility that $u_\lambda(y) = u(y)$. Hence, we conclude that $u_\lambda > u$ on D_η.

However, this conclusion contradicts the assumption that there exists a sequence $\{(\lambda^i, x^i)\}$ with $\lambda^i \in (\lambda, \lambda_1)$, $x^i \in \Sigma_{\lambda^i} \subset \Sigma_\lambda$, and converging to (λ, P), where $u_{\lambda^i}(x^i) \leq u(x^i)$. We must therefore conclude that $P \notin \partial\Omega \cap T_\lambda$.

Again, we have exhausted all the possibilities and conclude that no point P with the stated properties exists. Therefore, Λ must be left-open in (λ_0, λ_1).

Being simultaneously nonempty near λ_1, left-open and left-closed in (λ_0, λ_1), Λ must be the entire interval (λ_0, λ_1). ∎

Conclusion of the Proof. According to Lemma 12, $\Lambda = (\lambda_0, \lambda_1)$. That is, $u_\lambda > u$ and $\partial_1 u < 0$ on Σ_λ for every $\lambda \in (\lambda_0, \lambda_1)$. Letting λ tend to λ_0, we find $u(S_{\lambda_0} x) \geq u(x)$ for all $x \in \Sigma_{\lambda_0}$ and $\partial_1 u \leq 0$ on $\Omega \cap T_{\lambda_0}$.

But the boundary value problem (1.8) is rotationally invariant, so the positive x_1-direction is not privileged in any way. Therefore, u is radially symmetric and $\partial u / \partial r < 0$ for $\lambda_0 < r < \lambda_1$. ∎

Remark. At this point, it is clear why, in Theorem 2, we needed to assume from the start that Ω is a ball. The difficulty is in the proof of the symmetry of Ω about T_{λ_0}. Here, the proof of Lemma 9 relies on the fact that f is Lipschitz continuous at 0. In the case of Theorem 2, an identity of the type $f(u_\lambda) - f(u) = c_\lambda(u_\lambda - u)$ ceases to hold when both u_λ and u vanish. Consequently, we cannot apply the Hopf Boundary Lemma and thus rule out the possibility that the boundary of $S_{\lambda_0}(\Sigma_{\lambda_0})$ is internally tangent to $\partial\Omega$. This problem is, in fact, related to the lack of uniqueness of the solution of an initial value problem of the type $u' = f(u)$, $u(0) = 0$, where the forcing term f is not Lipschitz at 0.

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