DOMAIN DECOMPOSITION: A BLOWUP PROBLEM AND THE GINZBURG–LANDAU EQUATIONS

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ABSTRACT. In two of our recent projects, the technique of domain decomposition plays a crucial role in the numerical integration of the partial differential equations. In our problems, the local behavior varies because of the nonlinear and singular nature of the equations. Attempts to introduce different length scales did not lead to improvement. We found that different forms of the equations have to be used over different subdomains. The first project arises in an effort to determine the profile at the moment of blowup or quenching of a nonlinear heat equation. The second project arises in the investigation of the Ginzburg-Landau equation in the theory of superconductivity.

1. Blowup and Quenching

1.1. PROBLEM FORMULATION

Let Ω be a domain in R^n with boundary $\partial\Omega$, and let $p > 1$ and $\lambda > 0$ be constants. We are interested in blowup phenomena for the nonlinear heat problem

$$u_t(x,t) = \Delta u(x,t) + \lambda u^p(x,t), \quad x \in \Omega, t > 0, \tag{1}$$

$$u(x,t) = \alpha, \quad x \in \partial\Omega, t > 0, \tag{2}$$

and the related problem

$$u_t(x,t) = \Delta u(x,t) + \lambda e^{u(x,t)}, \quad x \in \Omega, t > 0, \tag{3}$$

$$u(x,t) = \beta, x \in \Omega, t > 0, \lambda > 0. \tag{4}$$

With a suitable scaling, we can reduce α to 1 in (2) and β to 0 in (4). In an appropriate sense, (3) is the limit form of (1) as $p \to \infty$.

It is well known that if the initial profile $u(x,0)$ and/or λ is sufficiently large, the solutions of the initial-boundary value problems will blow up at

some finite time. More specifically, there exists a finite $T > 0$, such that the solution $u(x,t)$ exists for all $0 < t < T$, but

$$\lim_{t \to T} \max \{u(x,t) : x \in \Omega\} = \infty. \tag{5}$$

The phenomenon of blowup has been studied extensively. Criteria on the nonlinear source function and initial profile that lead to blowup are known. Estimates on the blowup time have been derived, and many properties of the set of blowup points have been established. There are so many papers on the subject that it is impossible to cite them all; a substantial number of references can be found in the monograph by Bebernes and Eberly [2]. The survey paper by Friedman and McLeod [13] discusses blowup phenomena for higher-order parabolic equations and hyperbolic equations. Blowup has also been observed in solutions of the nonlinear Schrödinger equation; see, for example, Landman et al. [21].

A related phenomenon is exhibited by the solutions of the equation

$$u_t(x,t) = \Delta u(x,t) + \frac{\lambda}{(1 - u(x,t))^p}, \quad x \in \Omega, t > 0, 0 < p < 1, \tag{6}$$

subject to the same boundary conditions (2). In this case, if the initial profile and/or λ is sufficiently large, then there exists a finite time T such that that solution $u(x,t)$ exists for all $0 < t < T$ but

$$\lim_{t \to T} \max \{u(x,t) : x \in \Omega\} = 1. \tag{7}$$

Here, the solution $u(x,t)$ remains bounded, but a singularity in the derivatives of $u(x,t)$ occurs at T, as the second term on the righthand side then becomes infinite. This phenomenon is called quenching. A survey of the subject together with a useful reference list has been given by Levine [22].

1.2. BEHAVIOR NEAR BLOWUP

There is a great deal of interest in the asymptotic behavior of the solution of (1) and (3) near the blowup time, stimulated by the classical work of Giga and Kohn [15]. Papers by Bressan [5, 6], Filippas and Kohn [11], and Herrero and Velázquez [16] are only a small sample of the more recent work on the subject. It is interesting to note that similar work on the quenching problem is lacking. Below, we shall describe a way to unify the two subjects through a single equation (9). This suggests a means to extend known results on the phenomenon of blowup to cover the phenomenon of quenching.

1.3. THE REGULARIZING TRANSFORM

The motivation for our recent work stems from the search for a reliable numerical method to compute the heat profile near the blowup/quenching time T, as well as the limiting profile at time T, the so-called blowup or quenching profile. Obviously, the unboundedness of $u(x,t)$ or its derivatives renders any regular numerical scheme for solving parabolic equations useless once t is near T. Numerical estimations for the blowup or quenching time have been carried out, usually by solving the parabolic equation until the solution or its derivatives become larger than some arbitrary bound. The only systematic numerical study of the profile in the vicinity of the blowup time that we are aware of is the rescaling algorithm proposed by Berger and Kohn [4], based on the asymptotic results obtained in Giga and Kohn [15]. No similar work has been done on the quenching problem yet.

Recently, Bellout [3] proposed a new approach based on a change of variable, which is inspired by known asymptotic results for the solutions. The method avoids the costly rescaling. Following Bellout, we use the transformation

$$v = u^{1-p} \quad (v = e^{-u}) \tag{8}$$

to change (1) ((3)) to the equation

$$v_t = \Delta v - \lambda - \mu \frac{|\nabla v|^2}{v}, \tag{9}$$

where $\mu = p/(p-1) > 1$ for (1) and $\mu = 1$ for (3). One can easily normalize the equation with a scaling in the space variable to reduce λ to 1. In the one-dimensional experiments we describe below, we retain λ in the equation and, instead, choose the scaling to reduce the domain to the unit interval.

We discovered that by using the change of variable

$$v = (1-u)^{p+1}, \tag{10}$$

(6) was transformed into an equation of the form (9), where $\mu \in (0,1)$. We have thus found a unified approach to treat all three equations (1), (3), and (6).

Blowup/quenching in $u(x,t)$ now corresponds to the vanishing of $v(x,t)$. The new variable $v(x,t)$ remains bounded up to the blowup/quenching time. However, the previous difficulty of having to deal with unbounded functions is replaced by having to deal with the *potential* singularity of the term $|\nabla v|^2/v$ as $v \to 0$. Bellout believes that this new difficulty is only apparent and conjectures that this term remains bounded uniformly up to the blowup time. The idea is that near where v is a minimum, ∇v is also small, so the ratio $|\nabla v|^2/v$ remains bounded. It can be shown that the conjecture is equivalent to the assertion that Δv remains uniformly bounded. If the

conjecture is true, then (9) is a regular parabolic equation, and classical methods of integration can be applied with no difficulty up to the blowup time.

In an attempt towards settling the conjecture, Bellout proved in [3] a useful regularity result in the one-dimensional case for (1) with $p > 5$, which implies that v_x is bounded for $p > 5$. Using a refined argument of Bellout's proof, we are able to extend his regularity result to all $p > 1$ as well as to cover the case of quenching. Furthermore, we are able to show that $v_x^2/(v|\ln v|^\gamma)$ is bounded for some $\gamma > 0$. The proofs will be given elsewhere. We have also established a rather long list of assertions that are equivalent to the conjecture. The conjecture is, however, still open.

The partial regularity results are nevertheless important because they imply that the differential equation satisfied by v^θ for some suitable power $\theta > 1$ has regular coefficients. Thus, v^θ can be solved numerically as a regular parabolic problem with classical algorithms, and the challenge to determine the blowup profile can essentially be considered answered. In practice, however, it is much more convenient to solve the differential equation for v than the differential equation for v^θ. It is also an intellectual challenge to be able to resolve the conjecture.

1.4. NUMERICAL SIMULATION

We have performed many numerical experiments, and the results always supported the conjecture that the term $|\nabla v\lambda|^2/v$ is uniformly bounded up to the blowup time. Most of the experiments were done using MATLAB, version 4.0. An interesting problem arises when μ is increased beyond 4. That is when domain decomposition comes in.

We have treated both one-dimensional and two-dimensional problems. For the sake of simplicity, we confine our discussion below to the one-dimensional case, with $\Omega = (0, 1)$. Instead of dealing with Equation (9) directly, we solve for $U(x, t) = 1-v(x, t)$. This means that we have to deal only with the simpler homogeneous boundary conditions $U(0, t) = U(1, t) = 0$. Furthermore, the evolution of $U(x, t)$ now resembles more that of the original variable $u(x, t)$; $U(x, t)$ increases towards 1 as $u(x, t)$ blows up, whereas $v(x, t)$ *decreases*.

We use the method of lines, discretizing only the space variable x, to obtain a matrix ordinary differential equation in t,

$$\frac{dU}{dt} = LU + \lambda + \mu\frac{U_x^2}{1-U}, \tag{11}$$

where U is a column vector representing the values of $U(x, t)$ at the chosen node points, L is the usual finite-difference tridiagonal matrix representing the second derivative, and U_x is some suitable finite-difference approximation

of the first derivative. The fraction in the last term is to be understood in the componentwise sense. The experiments show that U develops some sort of a boundary layer in which U_x assumes very large values. To improve accuracy, we use a five-point numerical differentiation formula to approximate U_x.

In our initial experiments, we used a uniform grid partitioning $[0,1]$ into 49 or 99 grid points. We integrate the system of Equations (11) over time intervals of size $k = t_{n+1} - t_n$. As is well known, the system is stiff and requires special techniques such as the Crank-Nicholson and other implicit methods. In our situation, we exploit the fact that the coefficient matrix L is a constant. Over each time interval $[t_n, t_{n+1}]$, we approximate the nonhomogeneous part of (11), $\lambda + \mu U_x^2/(1-U)$, by a linear function in t, namely, $A + Bt$, where A and B are vectors given by

$$A = \lambda + \mu \left. \frac{U_x^2}{1-U} \right|_{t=t_n}, \quad A + Bk = \lambda + \mu \left. \frac{U_x^2}{1-U} \right|_{t=t_{n+1}}. \tag{12}$$

The solution $U(t_{n+1})$ at the new time step is estimated by solving the approximated form of (11) exactly:

$$U(t_{n+1}) = U(t_n) + K_1(LU(t_n) + A) + K_2 B, \tag{13}$$

where K_1 and K_2 are constant matrices given by

$$K_1 = L^{-1}(L^{Lk} - I), \tag{14}$$

$$K_2 = L^{-2}(e^{Lk} - I - Lk), \tag{15}$$

where I denotes the identity matrix. This formula, however, cannot be used directly since only A, not B, is determinable from $U(t_n)$ by using (12). We use the usual predictor-corrector technique: we take an initial guess $B = 0$ and use (13) to predict the value $U(t_{n+1})$, which is then substituted into the second formula in (12) to give a corrected value for B. This step may be iterated to give successively better estimates of B, but in practice we find that one cycle is usually sufficient. Note that the matrices K_1 and K_2 can be computed easily once the spectral decomposition of L is known. Moreover, they need to be computed only once for each choice of k.

Our method of integration is unconditionally stable for any value of k, and thus the choice of k is restricted only by the error introduced in the approximation of the nonhomogeneous part of the equation. The criterion we used to determine a good choice of k is to compare the difference in the results obtained by using k and $2k$, respectively. A smaller k is usually chosen to repeat the last few time steps before the blowup time if higher accuracy is desired near the final moment.

Our numerical scheme works efficiently and accurately for a wide range of the parameter λ and for $\mu < 3$. For illustration we include the graphs from three typical runs. We used 99 grid points and the time step $k = 10^{-4}$.

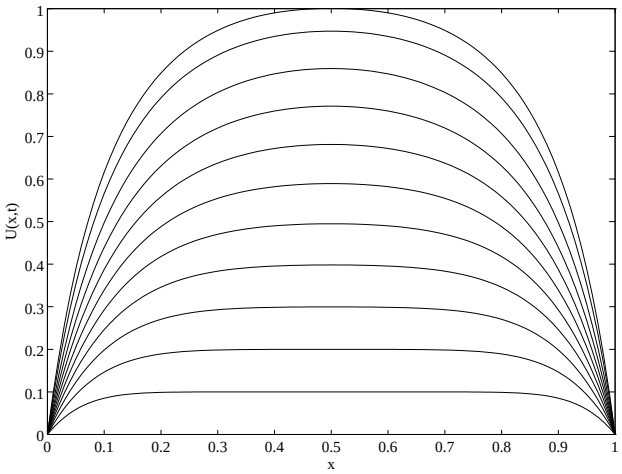

Fig. 1. $\lambda = 20, \mu = 1, u(x,0) \equiv 0$

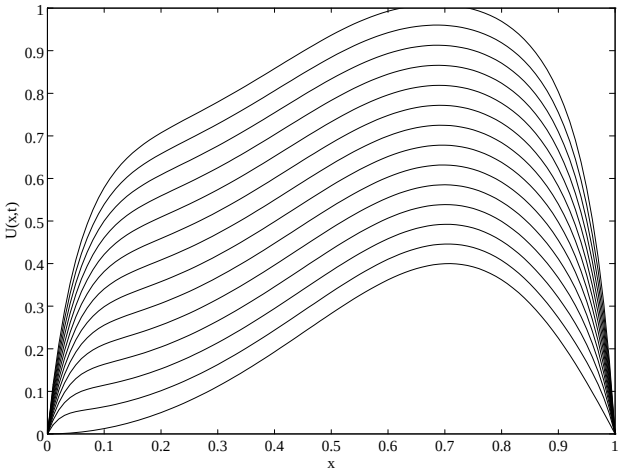

Fig. 2. $\lambda = 100, \mu = 1, u(x,0) \equiv 0.4 \sin(\pi x^2)$

The figures show the profiles $U(x, t_m)$ at selected time t_m that are uniformly apart except for the last one which is some moment before blowup. Figure 1 traces the evolution from a trivial initial profile. Figure 2 is an ex-

ample of a nonsymmetric initial profile. Figure 3 is an example of a solution blowing up at more than one point.

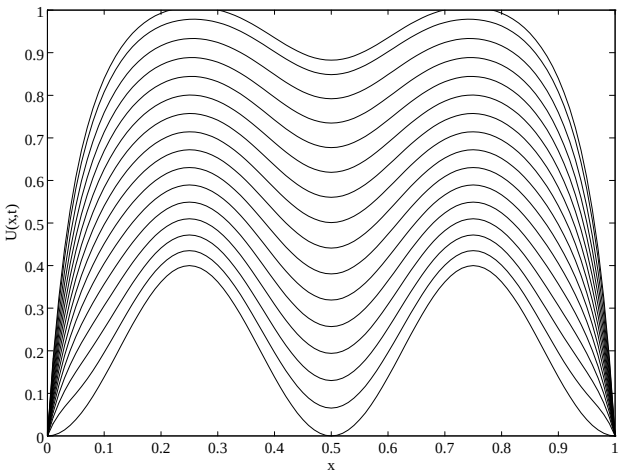

Fig. 3. $\lambda = 100, \mu = 1, u(x,0) \equiv 0.4 \sin^2(2\pi x)$

The method of rescaling is a never-ending task; a new rescaling is needed once the rescaled solution grows sufficiently large. Our method, on the other hand, allows one to accurately compute the blowup time by extrapolating $\max\{U(x,t_n), x \in (0,1)\}$ for various t_n, to find out when it will reach $U = 1$. The blowup profile can then be accurately computed by integrating (11) using $T - t_n$ as the final time step size.

1.5. DOMAIN DECOMPOSITION TECHNIQUE FOR LARGE μ

The numerical method described in Section 1.4 fails when μ is large and the technique of domain decomposition has been used to overcome the difficulty.

When the parameter μ is increased, the profile becomes flatter near the center of the interval. This produces, moments before blowup, two "boundary layers" near the endpoints, in which the solution shoots up sharply from 0 to almost 1 within a short distance. For $\mu = 4$, near the blowup time, $U(x,t)$ developed two "horns" as shown in Fig. 4. In the next time step, the solution blew up at two points. The same phenomenon prevails for larger values of μ. If one had blindly trusted computers, one could have declared the discovery of a new kind of blowup behavior. However, it has been rigorously proved (see, for example, Friedman and McLeod [13]) that if the initial profile is symmetrical and has only one local maximum, at the midpoint of the interval, then all subsequent profiles behave in the same way. In

particular, it is impossible for two "horns" to develop and the solution can blow up only at a single point, namely, the midpoint of the interval.

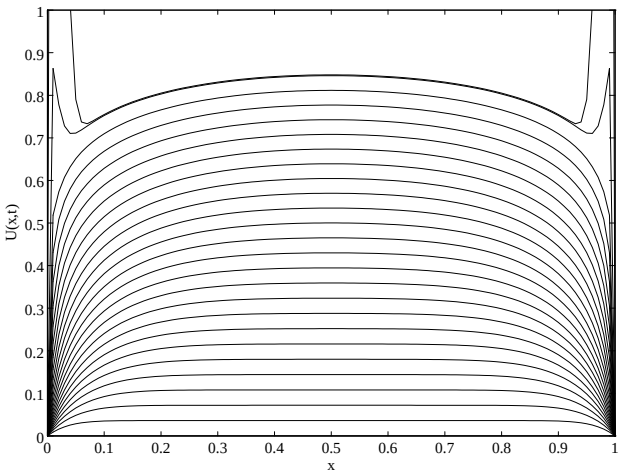

Fig. 4. **Instability sets in:** $\lambda = 100, \mu = 5, u(x,0) \equiv 0$

Our first reaction was to repeat the experiments with a significant reduction in the time step. We soon found that the plan did not work. Our next attempt was to use a higher-order approximation for the nonhomogeneous term. Not only did that lead to a much more complicated computation, but it did nothing to prevent the formation of the "horns." Our next strategy was to try to scale down the boundary layer by multiplying $U(x,t)$ by a factor such as $x(1-x)$, hoping that the final profiles would take less of the shape of a plateau. The transformed differential equation is complicated, and that discouraged us from attempting to modify the code to handle the new variable.

For a while, multiple-length scale seemed to be a reasonable explanation of the difficulty. The rapid increase of $U(x,t)$ within a short distance must mean that we need many more grid points within the boundary layer in order to represent the variation adequately. We restructured our programs to handle nonuniform grids. The first attempt was to have three subintervals, the two boundary layers and the middle section, each having a uniform subgrid. In one example we used 40 grid points between 0 and 0.1, 40 grid points between 0.9 and 1, but only 20 between 0.1 and 0.9. The results were encouraging. The appearance of the "horns" was delayed by a few time steps, and their locations were pushed outward towards the endpoints. Yet all attempts to completely eliminate the "horns" failed. We then overhauled the programs to deal with very general nonuniform grids. We used grids

that became progressively finer towards the endpoints: we started with 20 uniformly spaced grid points in $[0, 1]$, subdivided the first and last intervals into 10 subintervals, and repeatedly subdivided the resulting first and last intervals. Our program includes a subroutine to compute the coefficients for a nonuniform four-point finite difference formula. To our dismay, no matter how finely we subdivide the first and last intervals, "horns" always occur at the second or third grid point next to the endpoints.

We are puzzled by the persistence of the instability. We believe that the explanation lies in the fact that the term v_x^2/v is of the same order of magnitude as the leading term v_{xx}. The stability of the numerical scheme is upset by the presence of the nonlinear term in the same way that a large time step in the Euler method for solving parabolic equations can lead to instability. It would be interesting to see a rigorous proof.

We finally succeeded in circumventing the instability problem by using the technique of domain decomposition. We discovered that within the boundary layers the original form of the equation, namely, (1), could be solved without any instability problem. Theory guarantees that if we stay away from the blowup points (there are only a finite number of them), the original solution $u(x, t)$ remains uniformly bounded up to T, even though it may be large.

Our algorithm is as follows. Identify two suitable subdomains that contain the boundary layers. In our experiments, we used $[0, 0.1]$ and $[0.9,1]$. Based on the value of U at t_n, compute the solution in $[0.1, 0.9]$ at the new time step t_{n+1} by integrating (11) over the entire domain $[0, 1]$ using the method described above and discarding the part in the boundary layers. Solve the original equation (1) over each of the boundary layers, requiring that the solution be continuous across the common boundary point of the subdomains. The programs have been tested for values of μ as high as 20.

In theory, the requirement for the matching of the solution at a single common boundary point between two subintervals is not sufficient to guarantee a smooth matching, for there is the possibility of a jump in the first derivative. The proper algorithm should use an iterative scheme to match the solution in the common boundary points of overlapping subintervals. In all our experiments, however, the simple matching procedure suffices.

2. The Ginzburg-Landau Equations

2.1. PROBLEM FORMULATION

Our second example concerns the Ginzburg-Landau equations in the theory of superconductivity. Part of the work reported here is done in collaboration with H. G. Kaper. More details will appear in a forthcoming joint paper [18]. For the physical background, we refer to the many excellent references

cited in our earlier work [18, 19, 20] and also in the papers by Du et al. [8, 9].

We are interested in finding a complex scalar ψ and a three-dimension real-valued vector $\mathbf{A}$, both functions of position in a given domain $\Omega \in R^3$, that minimize the Helmholtz free-energy functional:

$$G(\psi, \mathbf{A}) = \int_\Omega \left(-|\psi|^2 + \frac{1}{2}|\psi|^4 + |(\nabla - i\mathbf{A})\,\psi|^2 + \kappa^2 \,|\nabla \times \mathbf{A}|^2 \right) d\Omega, \quad (16)$$

where $i = \sqrt{-1}$ and κ is a characteristic constant of the superconducting material. The Euler-Lagrange equations for this variational problem are called the Ginzburg-Landau equations.

$$(\nabla - i\mathbf{A})^2\psi + \left(1 - |\psi|^2\right)\psi = 0, \quad (\kappa^2 \nabla \times -|\psi|^2)\mathbf{A} + \mathrm{Im}(\psi^* \nabla \psi) = 0. (17)$$

Of particular interest is the quantity $|\psi|$ which represents the amount of superconducting activity going on at the pertinent location. For type-II superconductors, characterized by the fact that $\kappa > 1/\sqrt{2}$, Abrikosov [1] showed analytically, by using singular perturbation techniques, the existence of solutions that exhibited a lattice pattern.

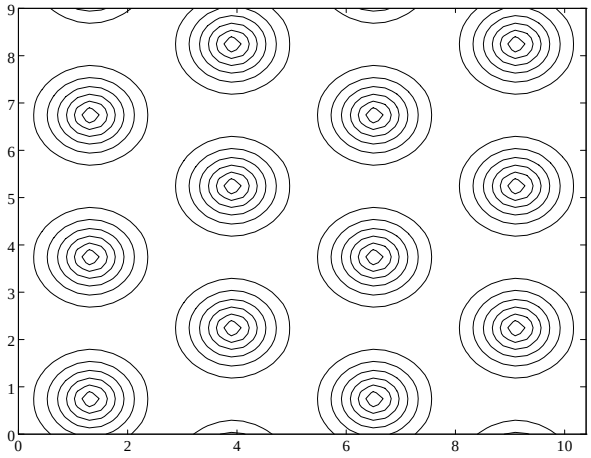

Fig. 5. **Vortex lattice**

Figure 5 shows the contour plot of $|\psi|$ for one such solution in a two-dimensional domain that models a thin-film superconductor as a normal magnetic field. The plot resembles vortices, and indeed superconducting currents are flowing along the contour lines in circular paths.

The first attempt to solve the Ginzburg-Landau equations numerically was by Doria, Gubernatis, and Rainer [10], who used the method of simulated annealing on a discrete version of the Helmholltz energy. Recently, our

colleagues at Argonne applied more sophisticated optimization and linear algebra techniques to tackle three-dimensional versions of the Ginzburg-Landau equations that model layered superconductors [14]. Du et al. [8, 9] used finite element approaches in their work.

In our recent work [17, 19, 20] we proposed a new method, the sweeping algorithm, to solve the five-point stencils that results from discretizing the two-dimensional Ginzburg-Landau equations, thus leading to an alternative numerical method for solving the Ginzburg-Landau problem. The sweeping algorithm is of interest in itself, as it can be used to invert banded matrices and general five-point stencil arising from other partial differential equations. The method offers great potential in the exploitation of parallelism and the technique of domain decomposition.

In our numerical study of the Ginzburg-Landau model, the questions we have investigated include the effect of the aspect ratio of the rectangular region Ω on the free energy, the relation between the free energy and the average magnetic field, and the simulation of vortex pinning by impurities.

2.2. THE GAUGE TRANSFORM

The boundary conditions imposed on the problem are not the usual periodic type. In fact, the vector potential cannot be periodic if the external magnetic field is nonzero. The situation is further complicated by the fact that the Helmholtz energy functional, and hence also the Ginzburg-Landau equations, are invariant under a class of transformations called gauge transforms. More precisely, given ψ and $\mathbf{A}$, one can pick any real-valued function of position $\chi(x)$, called a gauge, and form a new pair

$$\overline{\psi} = \psi e^{i\chi(x)}, \quad \overline{\mathbf{A}} = \mathbf{A} + \nabla\chi(x). \tag{18}$$

One can easily verify that the new pair gives the same Helmholtz energy as the original pair. If the original pair is a solution of the Ginzburg-Landau equations, so is the new pair. Other physical quantities derivable from each pair, such as current and magnetic field, are also the same. The two pairs indeed are different representations of the same physical state. Periodicity in physical states demands only that the functions on opposite sides of the domain differ by a gauge. This leads to the so-called modified boundary conditions.

We confine our discussions to two-dimensional simulations. In this case, one can arrange to have strict periodicity in one direction and modified periodicity in the other. One can also make the simplification that χ is a linear function of the coordinate along the appropriate edge. For instance, for a rectangular domain, $[0, L_x] \times [0, L_y]$ that contains n vortices, the boundary conditions takes the form

$$\psi(0, y) = \psi(L_x, y), \quad \psi(x, 0) = \psi(x, L_y)e^{igx}, \tag{19}$$

$$A_1(0,y) = A_1(L_x,y), \quad A_1(x,0) = A_1(x,L_y) + g, \tag{20}$$

$$A_2(0,y) = A_2(L_x,y), \quad A_2(x,0) = A_2(x,L_y), \tag{21}$$

where $g = 2n\pi/L_x$, and A_1, A_2 are the two components of $\mathbf{A}$.

2.3. DOMAIN DECOMPOSITION TECHNIQUE

The value of g in the boundary conditions (20) and (21) depends on the number of vortices inside the domain, and is related to the total magnetic flux through the region. For a region that contains many vortices, g can be very large. This poses the following problem. Suppose that $\psi(x,0)$ is a smooth, slowly varying, but nontrivial function. Its counterpart on the opposite edge must be a wildly oscillating function, as a result of the phase factor e^{igx}. Likewise, if ψ is slowly varying on the righthand edge, its value on the left-hand edge must oscillate wildly. It is impossible to find a gauge transform that gives slowly oscillating boundary values on both edges.

We illustrate this situation with the plots of the real and imaginary parts of ψ for a region that contains two of the vortices depicted in Fig. 5. The numerical simulation was performed using $\kappa = 5$. For the region shown, $L_x = 3\sqrt{3}$ and $L_y = 3$, and the number of grid points used is 24×24.

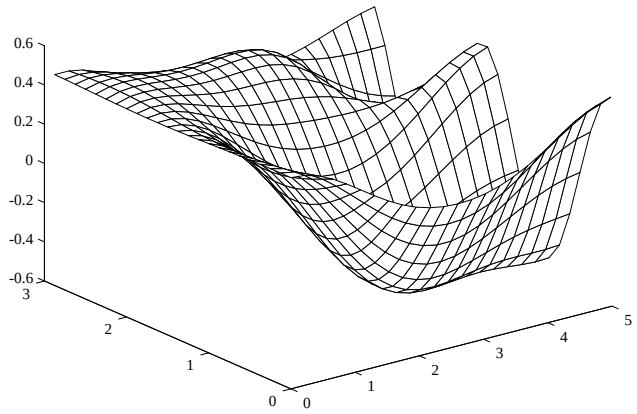

Fig. 6. **Real part of** ψ

As shown in the plots in Figs. 6 and 7, both the real and imaginary parts of ψ oscillate more and more as one approaches the righthand boundary. For a larger rectangular region that involves more vortices, ψ will oscillate even more as we move to the right. For every horizontal distance equal to

$3\sqrt{3}$, the number of oscillations of ψ increases by 2, so that at the right-hand boundary, ψ has 20 maxima and 20 minima. It is no longer possible to approximate ψ adequately using only 24 grid points in the y direction.

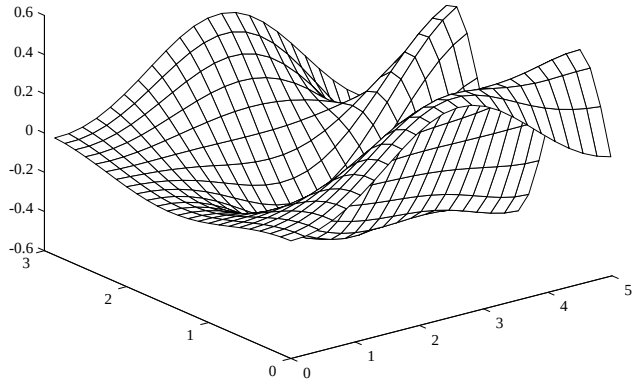

Fig. 7. **Imaginary part of ψ**

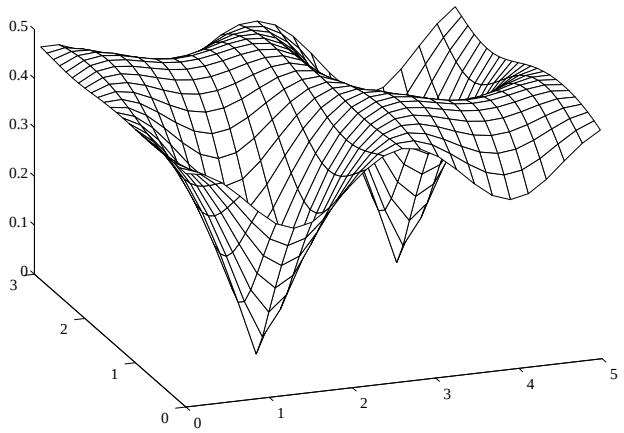

Fig. 8. **Absolute value of ψ**

Even though ψ oscillates wildly, the physical quantity we are interested in is represented by $|\psi|$, a plot of which is shown is Fig. 8. Unlike the real and imaginary parts of ψ, the oscillation of $|\psi|$ remains steady as one moves

along the x direction. The plot shown in Fig. 8 is simply duplicated when a larger rectangular region is used.

The method we used to overcome the difficulty posed by the oscillation of ψ is to divide the domain into smaller subdomains. The Helmholtz energy is iteratively minimized over each subdomain while holding ψ and $\mathbf{A}$ fixed in the other subdomains (this process can be performed in parallel). The idea of partial minimization is simply a block Gauss-Seidel procedure. The important new idea is that we are not solving directly for ψ and $\mathbf{A}$ in each subdomain, but rather for an equivalent pair $\overline{\psi}$ and $\overline{\mathbf{A}}$ derived from a suitable gauge that can smooth out the oscillations in ψ on both edges of the subdomain (this approach is now possible because the subdomain is smaller). The gauge is associated with each subdomain and may vary from one subdomain to the next. The original ψ and $\mathbf{A}$ for the entire domain can be recovered by an inverse gauge transformation over each subdomain.

After each iterative cycle, the Ginzburg-Landau equations are satisfied in the interior of all the subdomains, but not on the common boundaries. At times the errors at these boundary points can be alarmingly huge, but one must remember that a better measure of the error is some suitable average error over the entire domain. The errors on the common boundaries decrease with the iterations. A usual technique in the method of domain decomposition to speed the improvement is to find an appropriate linking operator that can give a correction term to the solution found in a subdomain based on the solution value computed in the surrounding subdomains. Instead, we use a new approach involving two overlapping decompositions.

As an example, we can use, as the first decomposition, N vertical strips of uniform width cut off starting from the left edge of the domain. The second decomposition makes the cuts exactly midway between the edges of the subdomains in the first decomposition. The two "half-strips" at the left and right ends of the domain are joined to form one subdomain. Another example is to use N vertical strips in the first decomposition and N horizontal strips in the second (the familiar alternating direction technique).

The two decompositions are used alternatively in successive cycles of the partial minimization procedure. The rationale behind the proposed method is that the errors incurred in the previous cycle are now in the interior of the subdomains of the current decomposition, far away from the common boundaries, and so will have less effect in disturbing the solution values at these points. Numerical results show that the use of overlapping decompositions greatly improves the rate of convergence over the use of a fixed decomposition (without the help of a linking operator).

3. Conclusion

Domain decomposition is an excellent device to introduce and exploit parallelism in designing numerical algorithms for solving partial differential equations. It has also been realized that domain decomposition is sometimes called for because the problem has rich local structures; see, for example, the remarks by Chin and Hedstrom in [7]. Problems involving multiple scales belong to this category.

We have given in this article two new examples in which the technique of domain decomposition forces its way into the numerical treatment of the problems. Not only does domain decomposition provide an efficient means of computation, but it appears to be the only way to resolve the difficulties we encountered in the investigation.

Traditionally, domain decomposition methods are well suited for elliptic problems. Our first example involves a semilinear parabolic problem; the second example has application in the theory of superconductivity. In both examples, the main cause of the difficulties is not in the difference in length scales but in the singular and nonlinear nature of the underlying equations. Another characteristic of our examples is that different forms of the equations must be solved in different subdomains in order to avoid instability or singularity of the solutions.

Using sets of overlapping domain decompositions in alternative iterative cycles can effectively speed up convergence. Whether further improvements can be achieved by incorporating other traditional techniques such as conjugate gradient will the the subject of future investigations.

Although asymptotics does not seem to have played an explicit role in our discussion, it is instrumental in guiding us to choose the correct transformation needed in our first example and in the analytical work that Abrikosov did in [1]. Without the asymptotic results, no one would have ever guessed the lattice structure of vortices and the modified periodic form of the boundary conditions.

Future work will include three-dimensional domains, more general nonlinear terms, and more realistic refinements of the various models.

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