Asymptotics of a Free Boundary Problem*

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Abstract

As was shown by Kaper and Kwong [*Differential and Integral Equations* **3**, 353–362], there exists a unique $R > 0$, such that the differential equation

$$u'' + \frac{2\nu + 1}{r} u' + u - u^q = 0, \; r > 0,$$

$(0 \leq q < 1, \; \nu \geq 0)$ admits a classical solution u, which is positive and monotone on $(0, R)$ and which satisfies the boundary conditions

$$u'(0) = 0, \; u(R) = u'(R) = 0.$$

In this article it is shown that $u(0)$ is bounded, but R grows beyond bounds as $q \to 1$.

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1 The Problem

In [1], the reaction-diffusion equation $\Delta u + u^{1/2} - 1 = 0$ was proposed as a simple model for Tokamak equilibria with magnetic islands. The equation motivated a study of free

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boundary problems for reaction-diffusion equations in $\mathbf{R}^N$ ($N = 2, 3, \ldots$) of the general form $\Delta u + u^p - u^q = 0$, where $0 \le q < p \le 1$. In [2], we showed that there is a unique R ($R > 0$) and a unique positive-valued function u on $(0, R)$, such that u is the radial solution of the differential equation which satisfies the boundary conditions $u(R) = 0$, $u'(R) = 0$. (A radial solution depends only on the radial variable $r = |x|$.) The solution (R, u) of the free boundary problem depends on the values of the exponents p and q.

In this article we analyze the special case $p = 1$ in more detail and focus on the behavior of the solution as $q \to 1$. That is, we are interested in the behavior as $q \to 1$ ($q < 1$) of the pair (R, u), R a real number ($R > 0$), u a positive-valued function on $(0, R)$, which satisfies the boundary value problem

$$u'' + \frac{2\nu + 1}{r} u' + u - u^q = 0, \ 0 < r < R, \tag{1.1}$$

$$u'(0) = 0, \ u(R) = u'(R) = 0. \tag{1.2}$$

We consider ν as a real number, not necessarily half-integer ($\nu \ge 0$). The existence and uniqueness of such a solution follow from [2]. The function u is monotone on $(0, R)$.

2 The Result

We prove the following result.

Theorem 1 *For each $q \in [0, 1)$, there is a unique $R > 0$ such that (1.1), (1.2) admits a (classical) solution u that is positive everywhere on $(0, R)$. The function u is monotonically decreasing on $(0, R)$; $u(0)$ is bounded, but R grows beyond bounds as $q \to 1$.*

In the special case $\nu = \frac{1}{2}$ ($N = 3$) we have a lower bound on R,

$$R > \sqrt{\frac{2}{1 - q}}, \quad 0 \le q < 1. \tag{2.1}$$

However, as we do not have a comparable upper bound, we cannot conclude that $R = \mathrm{O}((1 - q)^{-1/2})$ as $q \to 1$.

3 The Proof

Using a shooting argument, we replace the boundary value problem (1.1), (1.2) by the initial value problem

$$u'' + \frac{2\nu + 1}{r} u' + u - u^q = 0, \ r > 0, \tag{3.1}$$

$$u(0) = \gamma, \ u'(0) = 0. \tag{3.2}$$

The results of [2] imply that, for any $q \in [0,1)$, there is a unique $\gamma > 1$ such that the solution of (3.1), (3.2) decreases from γ to meet the r-axis with zero slope at some value $R > 0$. Denoting this solution by $u(\cdot,\gamma)$, we have

$$u(R,\gamma) = 0, \; u'(R,\gamma) = 0. \tag{3.3}$$

The lower bound on γ can be sharpened to $(2/(1+q))^{1/(1-q)}$, but 1 suffices for our purpose. The proof consists of a detailed investigation of the behavior of $u(\cdot,\gamma)$.

3.1 Down to 1

We begin by showing that $u(r,\gamma)$ decreases monotonically from the value γ at $r = 0$ to the value 1 at some finite point r_0.

Lemma 1 *There exists a point $r_0 < j_{\nu,1}/(1-q)^{1/2}$ such that $u(\cdot,\gamma)$ is monotonically decreasing on $(0,r_0)$, with $u(r_0,\gamma) = 1$ and $u'(r_0,\gamma) < 0$. Here, $j_{\nu,1}$ is the first positive zero of J_ν—the Bessel function of the first kind of order ν.*

Proof. As long as $u > 1$, we have $u - u^q > (1-q)u$, so $u(\cdot,\gamma)$ oscillates faster than the solution v of the equation

$$v'' + \frac{2\nu+1}{r}v' + (1-q)v = 0. \tag{3.4}$$

In particular, $u(\cdot,\gamma)$ reaches the value 1 before v does. Now, $v(r)$ is a constant multiple of $r^{-\nu}J_\nu(r(1-q)^{1/2})$, where J_ν is the Bessel function of the first kind of order ν—see, for example, [3]. Hence, $v(r) = 1$ for some value $r < j_{\nu,1}/(1-q)^{1/2}$, where $j_{\nu,1}$ is the first positive zero of J_ν. We conclude that there must be a point $r_0 < j_{\nu,1}/(1-q)^{1/2}$, such that $\gamma > u(r,\gamma) > 1$ for $0 < r < r_0$ and $u(r_0,\gamma) = 1$.

Since $u'(0,\gamma) = 0$ and $u''(r,\gamma) < 0$ near 0, it must be the case that $u'(r,\gamma) < 0$ near 0.

Suppose $u(\cdot,\gamma)$ were not monotone on $(0,r_0)$. Then there exists a value $r_1 \in (0,r_0)$ where $u(r,\gamma)$ has a local minimum, with $u(r_1,\gamma) > 1$. Because $u(r,\gamma)$ reaches the value 1 at r_0, there must then exist a value $r_2 \in (r_1,r_0)$ such that $u(r_2,\gamma) = u(r_1,\gamma)$ and $u'(r_2,\gamma) \le 0$. Multiplying the differential equation (3.1) by u' and integrating over (r_1,r_2), we find that

$$\frac{1}{2}(u'(r_2,\gamma))^2 = -(2\nu+1)\int_{r_1}^{r_2} \frac{(u'(r,\gamma))^2}{r}dr. \tag{3.5}$$

But here we have a contradiction, as the two sides of this identity have opposite signs. It must therefore be the case that $u(\cdot,\gamma)$ is monotone on $(0,r_0)$.

The monotonicity of $u(\cdot,\gamma)$ on $(0,r_0)$ implies that $u'(r_0,\gamma) \le 0$. If $u'(r_0,\gamma) = 0$, then it follows from the Lipschitz continuity of the function $u - u^q$ for $u > 0$ and the consequential uniqueness of the solution of the initial value problem for (3.1) in the direction of decreasing r starting at $r = r_0$ that $u(r,\gamma) = 1$ for all $r \in (0,r_0)$. But then we have a contradiction, as $u(0,\gamma) = \gamma > 1$. We conclude that $u'(r_0,\gamma) < 0$. ∎

3.2 Beyond r_0

From Lemma 1 we know that $u(r,\gamma)$ decreases monotonically until it reaches the value 1 with a negative slope at $r = r_0$. Beyond r_0, $u(r,\gamma)$ decreases further until either it reaches the value 0 with a negative or zero slope, or it bottoms out at some finite value of r with a minimum value between 0 and 1.

Let r_1 be the point where $u(r,\gamma)$ ceases to be positive,

$$r_1 = \sup \{r > r_0 : u(s,\gamma) > 0,\ 0 < s < r\}. \tag{3.6}$$

If r_1 is finite and $u(r_1,\gamma) = 0$, we do not consider $u(\cdot,\gamma)$ beyond r_1. In this case, we can use the same argument as in the proof of Lemma 1 to show that $u(\cdot,\gamma)$ is monotonically decreasing on the entire interval $(0, r_1)$. In particular, if γ is such that not only $u(r_1,\gamma) = 0$, but also $u'(r_1,\gamma) = 0$, then $u(\cdot,\gamma)$ defines the (unique) solution u of the free boundary problem (3.1), (3.3), where $R = r_1$.

If $r_1 = \infty$, then $u(r,\gamma)$ has a positive minimum at some finite value of r, after which it oscillates with decreasing amplitude around the constant value 1.

Lemma 2 *For $0 < r < r_1$, we have $0 < u(r,\gamma) < \gamma$.*

Proof. The lemma is true for $0 < r \le r_0$ (cf. Lemma 1). Beyond r_0, we use a simple energy argument. The energy E of any solution u of (3.1), defined by the expression

$$E(r) = \frac{1}{2}(u'(r))^2 + \frac{1}{2}(u(r))^2 - \frac{1}{q+1}(u(r))^{q+1}, \tag{3.7}$$

is a monotonically decreasing function of its argument, as $E'(r) = -((2\nu+1)/r)(u'(r))^2 \le 0$ for all $r \ge 0$.

Suppose the lemma were false for $r_0 < r < r_1$. Then $u(r_2,\gamma) = \gamma$ for some $r_2 \in (r_0, r_1)$, where $E(r_2) \ge \gamma^2/2 - \gamma^{q+1}/(q+1) = E(0)$, and we have a contradiction. ∎

Let w be defined in terms of $u(\cdot,\gamma)$ by the expression

$$w(r) = \frac{ru(r,\gamma)}{\gamma}. \tag{3.8}$$

This function is a solution of the initial value problem

$$w'' + \frac{2\nu-1}{r}w' + \left(1 - \frac{1}{(u(r,\gamma))^{1-q}} - \frac{2\nu-1}{r^2}\right) w = 0,\ r > 0, \tag{3.9}$$

$$w(0) = 0,\ w'(0) = 1. \tag{3.10}$$

It vanishes when u vanishes, while its derivative vanishes when both u and u' vanish. Furthermore,

$$0 < w(r) < r,\ 0 < r < r_1. \tag{3.11}$$

The following lemma gives a lower bound for r_1.

Lemma 3 *We have* $r_1 > j_{\nu,1}/\delta$, *where*

$$\delta = \sqrt{1 - \gamma^{-(1-q)}}. \tag{3.12}$$

Proof. Because $u(r,\gamma) < \gamma$, w oscillates less than the solution v of the initial value problem

$$v'' + \frac{2\nu - 1}{r} w' + \left(\delta^2 - \frac{2\nu - 1}{r^2}\right) v = 0, \; r > 0; \;\; v(0) = 0, \; v'(0) = 1, \tag{3.13}$$

at least as long as v is positive. Since $v(r) = 2^\nu \Gamma(\nu+1)\delta^{-\nu} r^{1-\nu} J_\nu(\delta r)$, the first zero of v occurs at $j_{\nu,1}/\delta$. It must therefore be the case that $r_1 > j_{\nu,1}/\delta$. ∎

3.3 Bounds on $(0, j_{\nu,1}/\delta)$

We rewrite the equation (3.9) in the form

$$w'' + \frac{2\nu - 1}{r} w' + \left(\delta^2 - \frac{2\nu - 1}{r^2}\right) w = f(w), \tag{3.14}$$

where

$$f(w) = \frac{1}{\gamma^{1-q}} \left\{ \left(\frac{r}{w}\right)^{1-q} - 1 \right\} w. \tag{3.15}$$

Using the method of variation of parameters, we obtain the integral equation for w,

$$w(r) = rg(\delta r) + \frac{\pi}{2} \int_0^r r^{1-\nu} s^\nu \left\{ J_\nu(\delta s) Y_\nu(\delta r) - Y_\nu(\delta s) J_\nu(\delta r) \right\} f(w(s))\, ds, \tag{3.16}$$

where

$$g(\rho) = 2^\nu \Gamma(\nu+1) \rho^{-\nu} J_\nu(\rho). \tag{3.17}$$

J_ν and Y_ν are the Bessel functions of the first and second kind, respectively, of order ν. The expression (3.16) holds for all $r \in [0, r_1)$ or, if r_1 is finite, for all $r \in [0, r_1]$. We now restrict r to the interval $[0, j_{\nu,1}/\delta]$.

Lemma 4 *For* $0 < r < j_{\nu,1}/\delta$, *we have*

$$0 < rg(\delta r) < w(r) < r\left(g(\delta r) + \frac{\phi(\delta r)}{\log \gamma}\right), \tag{3.18}$$

where g *is defined in* (3.17) *and*

$$\phi(\rho) = \frac{\rho^2 (g(\rho))^{-1} \log(g(\rho))^{-1}}{4(\nu+1)}. \tag{3.19}$$

Proof. Take any $r \in (0, j_{\nu,1}/\delta)$. It follows from the Kneser-Sommerfeld expansion [3, Section 15.42] that

$$J_\nu(\delta s)Y_\nu(\delta r) - Y_\nu(\delta s)J_\nu(\delta r) = \frac{4\delta r J_\nu(\delta r)}{\pi J_\nu(\delta s)} \sum_{n=1}^{\infty} \frac{(J_\nu(j_{\nu,n}s/r))^2}{(j_{\nu,n}^2 - (\delta r)^2)j_{\nu,n}J_\nu'^2(j_{\nu,n})}, \tag{3.20}$$

for $0 \le s \le r$. All the terms in the right member are positive, so the expression in the left member is positive. Furthermore, $f(w(s))$ is positive for $0 \le s \le r$. Therefore, the integral in (3.16) is positive. Obviously, $g(\delta r)$ is positive, so

$$w(r) > rg(\delta r) > 0, \ 0 < r < j_{\nu,1}/\delta. \tag{3.21}$$

It remains to establish the upper bound on $w(r)$ in (3.18). From (3.21) and the fact that g is decreasing on $(0, j_{\nu,1})$ we deduce that

$$\frac{s}{w(s)} < \frac{1}{g(\delta s)} \le \frac{1}{g(\delta r)}, \ 0 \le s \le r. \tag{3.22}$$

Therefore,

$$f(w(s)) < \frac{(g(\delta r))^{-(1-q)} - 1}{\gamma^{1-q}} \ w(s), \ 0 \le s \le r. \tag{3.23}$$

Furthermore, $w(s) \le s$, cf. (3.11), so

$$\begin{aligned} w(r) - rg(\delta r) &= \frac{\pi}{2}\int_0^r r^{1-\nu}s^\nu \left\{J_\nu(\delta s)Y_\nu(\delta r) - Y_\nu(\delta s)J_\nu(\delta r)\right\} f(w(s)) \ ds \\ &\le \ r\frac{(g(\delta r))^{-(1-q)} - 1}{\gamma^{1-q} - 1}\left[\rho^{-\nu}\frac{\pi}{2}\int_0^\rho \left\{J_\nu(z)Y_\nu(\rho) - Y_\nu(z)J_\nu(\rho)\right\} z^{\nu+1} \ dz\right]_{\rho=\delta r}. \end{aligned} \tag{3.24}$$

The expression in square brackets can be evaluated by means of the recurrence formulae for Bessel functions [3, Section 3.2] and the resulting expression can be simplified further by means of the Wronskian [3, Section 3.63],

$$\rho^{-\nu}\frac{\pi}{2}\int_0^\rho \left\{J_\nu(z)Y_\nu(\rho) - Y_\nu(z)J_\nu(\rho)\right\} z^{\nu+1} \ dz = 1 - g(\rho). \tag{3.25}$$

We estimate this expression by substituting the series expansion for the Bessel function J_ν and truncating after the first term,

$$1 - g(\rho) = 1 - 2^\nu\Gamma(\nu+1)\rho^{-\nu}J_\nu(\rho) \le \frac{\rho^2}{4(\nu+1)}. \tag{3.26}$$

Thus,

$$\left[\rho^{-\nu}\frac{\pi}{2}\int_0^\rho \left\{J_\nu(z)Y_\nu(\rho) - Y_\nu(z)J_\nu(\rho)\right\} z^{\nu+1} \ dz\right]_{\rho=\delta r} \le \frac{(\delta r)^2}{4(\nu+1)}. \tag{3.27}$$

To estimate the factor in front of the square brackets in (3.24), we observe that $0 < g(\delta r) < 1$ on $(0, j_{\nu,1}/\delta)$ and $\gamma > 1$. Furthermore, one readily verifies that

$$\frac{1 - x^{1-q}}{\log x^{-1}} \le \frac{y^{1-q} - 1}{\log y}$$

for any pair (x, y) with $0 < x \le 1 \le y$. Therefore,

$$\begin{aligned} \frac{(g(\delta r))^{-(1-q)} - 1}{\gamma^{1-q} - 1} &= (g(\delta r))^{-(1-q)} \frac{1 - (g(\delta r))^{1-q}}{\gamma^{1-q} - 1} \\ \le \quad \frac{(g(\delta r))^{-(1-q)} \log(g(\delta r))^{-1}}{\log \gamma} &\le \frac{(g(\delta r))^{-1} \log(g(\delta r))^{-1}}{\log \gamma}. \end{aligned} \tag{3.28}$$

Using (3.27) and (3.28) in (3.24), we obtain the estimate

$$w(r) - rg(\delta r) \le r \frac{\phi(\delta r)}{\log \gamma}, \tag{3.29}$$

where ϕ is defined in (3.19). The upper bound for $w(r)$ given in (3.18) follows. ∎

In terms of u, we have the following bounds:

$$0 < \gamma g(\delta r) < u(r, \gamma) < \gamma \left[g(\delta r) + \frac{\phi(\delta r)}{\log \gamma} \right], \ 0 < r < \frac{j_{\nu,1}}{\delta}. \tag{3.30}$$

Because $\phi(\rho)$ increases beyond bounds as $g(\rho)$ decreases to 0, the upper bound in (3.18) or (3.30) increases indefinitely as r approaches the right endpoint of the interval $(0, j_{\nu,1}/\delta)$.

In the following analysis we also need an estimate of the quantity $r^{1-2\nu}(r^{2\nu-1}w)'(r)$. It is given by the expression

$$r^{1-2\nu}(r^{2\nu-1}w)'(r) = h(\delta r) + \delta \frac{\pi}{2} \int_0^r r^{1-\nu} s^\nu \left\{ J_\nu(\delta s) Y_{\nu-1}(\delta r) - Y_\nu(\delta s) J_{\nu-1}(\delta r) \right\} f(w(s)) \, ds, \tag{3.31}$$

where

$$h(\rho) = 2^\nu \Gamma(\nu + 1) \rho^{1-\nu} J_{\nu-1}(\rho). \tag{3.32}$$

Like (3.16), (3.31) holds for all $r \in [0, r_1)$ or, if r_1 is finite, for all $r \in [0, r_1]$. The following lemma gives an estimate on $(0, j_{\nu,1}/\delta)$.

Lemma 5 *For $0 < r < j_{\nu,1}/\delta$, we have*

$$\left| r^{1-2\nu}(r^{2\nu-1}w)'(r) - h(\delta r) \right| < 2(\nu + 1) \frac{\phi(\delta r)}{\log \gamma}, \tag{3.33}$$

where h is defined in (3.32) and ϕ is defined in (3.19).

Proof. The proof is similar to, although slightly more involved than, the proof of Lemma 4. Instead of (3.16), we use (3.31). The analog of (3.24) is

$$\begin{aligned} &\delta \frac{\pi}{2} \int_0^r r^{1-\nu} s^\nu \left\{ J_\nu(\delta s) Y_{\nu-1}(\delta r) - Y_\nu(\delta s) J_{\nu-1}(\delta r) \right\} f(w(s)) \, ds \\ &\le \quad \frac{(g(\delta r))^{-(1-q)} - 1}{\gamma^{1-q} - 1} \left[\rho^{1-\nu} \frac{\pi}{2} \int_0^\rho \left\{ J_\nu(z) Y_{\nu-1}(\rho) - Y_\nu(z) J_{\nu-1}(\rho) \right\} z^{\nu+1} \, dz \right]_{\rho=\delta r} \end{aligned} \tag{3.34}$$

The expression in square brackets can again be evaluated; instead of (3.25) we have

$$\rho^{1-\nu}\frac{\pi}{2}\int_0^\rho \{J_\nu(z)Y_{\nu-1}(\rho) - Y_\nu(z)J_{\nu-1}(\rho)\}\, z^{\nu+1}\; dz = 2(\nu+1) - h(\rho), \tag{3.35}$$

where

$$2(\nu+1) - h(\rho) = 2(\nu+1) - 2^\nu\Gamma(\nu+1)\rho^{1-\nu}J_{\nu-1}(\rho) \le \tfrac{1}{2}\rho^2. \tag{3.36}$$

The lemma follows from (3.31), (3.34), (3.35), (3.36), and (3.27). ∎

3.4 Estimates at r_0

We use the results of Lemmas 4 and 5 to estimate r_0 and $r^{1-2\nu}(r^{2\nu-1}w)'$ at r_0.

Lemma 6 *Let $a \in (j_{\nu-1,1}, j_{\nu,1})$ be fixed. Then there exists a constant $\gamma_1 > 1$ that does not depend on q, such that*

$$r_0^{1-2\nu}(r^{2\nu-1}w)'(r_0) < -\tfrac{1}{2}|h(a)| \tag{3.37}$$

and

$$\frac{a}{\delta} < r_0 < \left(1 + \frac{4\nu}{|h(a)|}\right)^{1/(2\nu)}\frac{a}{\delta}, \tag{3.38}$$

for all $\gamma \ge \gamma_1$.

Proof. With the choice of a indicated in the statement of the lemma, we have $g(a) > 0$ and $h(a) < 0$. These inequalities follow from the interlacing property of the zeros of Bessel functions,

$$0 < j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \ldots;$$

cf. [3, Section 15.22].

We begin by observing that w oscillates less than v, where $v(r) = rg(\delta r)$, g defined by (3.17). Therefore r_0, which is defined by the identity $w(r) = r/\gamma$, is certainly beyond the point r_2, where $g(\delta r_2) = 1/\gamma$. Therefore, if

$$\gamma_0 = 1/g(a), \tag{3.39}$$

then $g(\delta r_2) \le g(a)$ for all $\gamma \ge \gamma_0$. Now, g is monotonically decreasing between a and $j_{\nu-1,1}$, so then we also have $\delta r_2 \ge a$ for all $\gamma \ge \gamma_0$. Since $r_0 > r_2$, we have thus achieved that

$$a/\delta < r_0. \tag{3.40}$$

for all $\gamma \ge \gamma_0$.

With $r_3 = a/\delta$, it follows from (3.33) that

$$r_3^{1-2\nu}(r^{2\nu-1}w)'(r_3) < -|h(a)| + 2(\nu+1)\frac{\phi(a)}{\log\gamma}. \tag{3.41}$$

Here, $h(a)$ and $\phi(a)$ do not depend on q or γ. Therefore, if we now define γ_1,

$$\gamma_1 = \min\{\gamma_0, exp\left(2(\nu+1)\frac{\phi(a)}{|h(a)|}\right)\}, \tag{3.42}$$

then γ_1 is independent of q and

$$r_3^{1-2\nu}(r^{2\nu-1}w)'(r_3) < -\tfrac{1}{2}|h(a)| \tag{3.43}$$

for all $\gamma \geq \gamma_1$. Writing the differential equation (3.9) in the form

$$\left(r^{1-2\nu}\left(r^{2\nu-1}w\right)'\right)' = -\left(1 - u^{-(1-q)}\right), \tag{3.44}$$

we observe that the function $r^{1-2\nu}(r^{2\nu-1}w)'$ is decreasing as long as $u(r,\gamma) > 1$—that is, up to r_0. Therefore, the bound (3.43) extends to the entire interval $[r_3, r_0]$, and we have

$$r^{1-2\nu}(r^{2\nu-1}w)'(r) < -\tfrac{1}{2}|h(a)|, \; r_3 \leq r \leq r_0. \tag{3.45}$$

for all $\gamma \geq \gamma_1$. In particular, the inequality holds at r_0, as asserted in (3.37).

Multiplying both sides of the inequality (3.45) by $r^{2\nu-1}$ and integrating over the interval (r_3, r_0), we find

$$\left(\frac{w(r_3)}{r_3} + \frac{|h(a)|}{4\nu}\right) r_3^{2\nu} - \frac{|h(a)|}{4\nu} r_0^{2\nu} > \frac{w(r_0)}{r_0} r_0^{2\nu}. \tag{3.46}$$

Here, we estimate the expression in the right member from below by 0. In the left member, we estimate the ratio $w(r_3)/r_3$ from above by 1; cf. (3.11). Thus,

$$r_0^{2\nu} < \left(1 + \frac{4\nu}{|h(a)|}\right) r_3^{2\nu}. \tag{3.47}$$

The inequalities (3.38) now follow from (3.40) and (3.47). ∎

3.5 Down to 0

We are now in a position to prove that the continuation of u beyond r_0 decreases to 0 for all sufficiently large γ, independently of q.

Lemma 7 *There exists a constant γ_2 that does not depend on q ($\gamma_2 \geq \gamma_1$, where γ_1 is the constant introduced in Lemma 6), such that $r_1 < \infty$ for all $\gamma \geq \gamma_2$.*

Proof. The proof is by contradiction, where we assume that, for some $\gamma \geq \gamma_1$, the solution $u(\cdot, \gamma)$ of (3.1), (3.2) is positive for all $r \geq 0$.

Consider the function w defined by (3.8). By assumption, w is positive for all $r > 0$. Because $(r^{1-2\nu}(r^{2\nu-1}w)')' = (r/\gamma)(u^q - u)$ and $u^q - u < 1 - q$ for $u > 0$, we have

$$(r^{1-2\nu}(r^{2\nu-1}w)')'(r) < \frac{(1-q)r}{\gamma}, \; r > 0. \tag{3.48}$$

Integrating (3.48) from r_0 to any point $r > r_0$, and using the estimate (3.37) at r_0, we find

$$r^{1-2\nu}(r^{2\nu-1}w)'(r) < -\tfrac{1}{2}|h(a)| + \frac{(1-q)r^2}{2\gamma},\ r > r_0, \tag{3.49}$$

for all $\gamma \geq \gamma_1$. Because $\gamma\delta^2 = \gamma - \gamma^q > \gamma^{1-q} - 1 > (1-q)\log\gamma$, it follows that

$$r^{1-2\nu}(r^{2\nu-1}w)'(r) < -\tfrac{1}{2}|h(a)| + \frac{r^2\delta^2}{2\log\gamma},\ r > r_0, \tag{3.50}$$

for all $\gamma \geq \gamma_1$.

Now we restrict r to a compact interval $[r_0, r_2]$, where

$$r_2 = b/\delta, \tag{3.51}$$

and $b > a$ is a suitably chosen constant. Defining the constant γ_2 by

$$\gamma_2 = \min\{\gamma_1, \mathrm{e}^{2b^2/|h(a)|}\}, \tag{3.52}$$

we then have

$$\frac{r^2\delta^2}{2\log\gamma} \leq \frac{1}{4}|h(a)|,\ r_0 \leq r \leq r_2, \tag{3.53}$$

for all $\gamma \geq \gamma_2$, so (3.50) reduces to

$$r^{1-2\nu}(r^{2\nu-1}w)'(r) < -\frac{1}{4}|h(a)|,\ r_0 \leq r \leq r_2, \tag{3.54}$$

for all $\gamma \geq \gamma_2$. Hence,

$$w(r_2) < \left[(a/b)^{2\nu}\frac{w(r_0)}{r_0} - (1-(a/b)^{2\nu})\frac{|h(a)|}{8\nu}\right] r_2. \tag{3.55}$$

Using (3.38) to estimate $w(r_0)/r_0$ and writing the inequality in terms of u, we thus find that

$$u(r_2,\gamma) < (a/b)^{2\nu}\left(1 + \frac{4\nu}{|h(a)|}\right)^{1/(2\nu)} - \gamma\left(1-(a/b)^{2\nu}\right)\frac{|h(a)|}{8\nu}, \tag{3.56}$$

for all $\gamma \geq \gamma_2$.

But now we have a contradiction, as the expression in the right member of this inequality certainly becomes negative for sufficiently large values of γ. We conclude therefore that $u(\cdot,\gamma)$ reaches the value 0 at some finite point r_1, as claimed. ∎

3.6 Completion of the Proof

According to Lemma 7, $u(\cdot,\gamma)$ ceases to be positive at a finite point r_1 for all $\gamma \geq \gamma_2$, where γ_2 is a constant that does not depend on q. Obviously, r_1 depends on the value of γ; in fact, it decreases as γ increases. Let

$$\Gamma = \inf\{\gamma > 1 : r_1 < \infty\}. \tag{3.57}$$

If $\gamma = u(0) = \Gamma$, then $u(\cdot, \gamma)$ reaches the r-axis with a horizontal slope, so $u(\cdot, \Gamma)$ defines the unique solution u of the free boundary problem (1.1), (1.2), where

$$R = r_1(\Gamma). \tag{3.58}$$

Obviously, Γ depends on q. However, it follows from Lemma 7 that $1 < \Gamma \le \gamma_2$, so $u(0)$ is bounded as $q \to 1$ ($q < 1$).

It remains to investigate the behavior of R as $q \to 1$ ($q < 1$). Because Γ is bounded, $\lim_{q \to 1} \Gamma^{1-q} = 1$. Then it follows from (3.12) that $\lim_{q\to 1} \delta = 0$, and therefore, by Lemma 3, $\lim_{q\to 1} R = \infty$. Thus, the proof of the theorem is complete. ∎

3.7 Special Case: $N = 3$

In the special case $N = 3$ ($\nu = \frac{1}{2}$), it is actually possible to find a lower bound for R that shows that R grows beyond bounds as $q \to 1$.

A simple energy argument gives the inequality

$$0 = E(R) < E(0) = \frac{\Gamma^2}{2} - \frac{\Gamma^{1+q}}{1+q}; \tag{3.59}$$

cf. (3.7). Hence,

$$\Gamma^{1-q} > \frac{2}{1+q}. \tag{3.60}$$

Next, we use an energy argument for (3.9). If $\nu = \frac{1}{2}$, this equation reduces to

$$w'' + w - \Gamma^{-(1-q)} r^{1-q} w^q = 0. \tag{3.61}$$

Hence,

$$\left(w'^2 + w^2 - \frac{2}{1+q} \Gamma^{-(1-q)} r^{1-q} w^{1+q} \right)' = -2 \frac{1-q}{1+q} \Gamma^{-(1-q)} r^{-q} w^{1+q}. \tag{3.62}$$

Upon integration over $(0, R)$, the left member yields -1; in the right member we use the inequality $w(r) < r$ to obtain the estimate

$$\int_0^R r^{-q} w^{1+q} \, dr < \tfrac{1}{2} R^2. \tag{3.63}$$

Thus, using (3.60), we find that

$$R > \sqrt{\frac{2}{1-q}}. \tag{3.64}$$

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