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THE PROBLEM OF REASONING BY CASE ANALYSIS*

LARRY WOS
Argonne National Laboratory, 9700 South Cass Ave., Argonne, IL 60439, U.S.A.

Abstract. This article is the twenty-ninth of a series of articles discussing various open research problems in automated reasoning. The problem proposed for research asks one to find criteria that an automated reasoning program can apply to decide when to conduct a case analysis argument and to decide which cases are appropriate. When the choice of employing a case analysis argument is wise and the cases are well chosen, the likelihood that the reasoning program—or, for that matter, a person—will find an answer to the given question is sharply increased.

Key words. Automated reasoning, case analysis, unsolved research problem.

Question: *When considering some given question, what criteria should be used to decide to employ a case analysis argument, and, for such an argument, what criteria should be used to choose the cases to consider?*

(This question is the twenty-eighth of 33 problems proposed for research in [3] and will be referred to as Research Problem 28 throughout this article. All references to sections, chapters, test problems, and such also refer to [3].)

One of the better known approaches to attacking some given question is to employ case analysis. The basic idea is to split the argument into appropriately chosen cases, each of which is then considered independently. When debugging a computer program, for example, the cases might consist of $a < b$, $a = b$, and $a > b$, for a specific a and b. By keeping (where possible) the various subarguments independent of each other, the intention is to simplify the overall argument leading to the desired information. When the choice of employing a case analysis argument is wise and the cases are well chosen, the likelihood that the reasoning program—or, for that matter, a person—will find an answer to the given question is sharply increased.

Research Problem 28 asks for criteria for correctly choosing when to instruct an automated reasoning program to employ case analysis. This research problem also asks for criteria for choosing well the specific cases to consider.

The evidence that a proposed solution to Research Problem 28 in fact accurately dictates when it is wise to employ a case analysis can be demonstrated with statistics and comparisons of the type suggested in Research Problems 19, 22, and 24. That the solution chooses the specific cases well can be tested on problems such as the index 2 theorem of group theory (Test Problem 1, Section 6.1.1), the problem known as Wang 3 [2, 1], various problems in program verification, and different puzzles. We suggest, in addition, the following two theorems from group theory for testing a proposed solution.

The first theorem asks one to prove that any group of order 7 is commutative (Test Problem 6, Section 6.1.6). One could, of course, attempt to prove this theorem by using various fundamental

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lemmas in group theory, such as the lemma that asserts that the order of a subgroup divides the order of a group. Even further, one might in another context consider proving the well-known theorem that all groups of prime order are commutative—for example, one might attempt to prove this theorem using Gödel's axiomatization of set theory (see Section 6.6)—but not for testing a proposed solution to Research Problem 28. Instead, we are suggesting that one's solution be tested by attempting to produce a brute-force proof by case analysis, a proof that focuses on the clause

EQUAL(x,a1) | EQUAL(x,a2) | EQUAL(x,a3) | EQUAL(x,a4) |
EQUAL(x,a5) | EQUAL(x,a6) | EQUAL(x,e)

where *e* is the identity of the group.

The second theorem focuses on proving that there are exactly two groups of order 4 (Test Problem 7, Section 6.1.7). Rather than suggesting that one attempt to prove this theorem in its entirety, we suggest first proving a weaker result. In particular, the test problem asks one to prove that if G is a group, then G satisfies one of the following four sets of relations, where the elements of the group G are *a*, *b*, *c*, and *e* (the identity of the group).

1. The square of every element is the identity *e*.
2. The square of *a* is *b*, the cube of *a* is *c*, and the fourth power of *a* is *e*.
3. The square of *b* is *c*, the cube of *b* is *a*, and the fourth power of *b* is *e*.
4. The square of *c* is *a*, the cube of *c* is *b*, and the fourth power of *c* is *e*.

We, of course, are suggesting that the proof of this intermediate result be based on case analysis. If one wonders, for example, what happened to the set of relations in which the square of *c* is *b*, that possibility easily reduces to the second set of relations. A proof of the weaker result can be loosely thought of as establishing that there are at most four groups of order 4 (see Section 6.1.7 for an appropriate set of clauses). One could then apply an additional case analysis to show that three of the possibly different groups are in fact isomorphic—the group in which the first set of relations holds is not isomorphic to any of the other three—and one would then have a proof of the second theorem we suggest as a test problem. An excellent program for conducting experiment is William McCune's OTTER; see [5] for examples of its use and a diskette of the program, and see [4] for additional examples.

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