A REMARK ON A WEIGHTED LANDAU INEQUALITY OF KWONG AND ZETTL

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Abstract. In this note we extend a Theorem of Kwong and Zettl concerning the inequality

$$\int_0^\infty t^\beta |u'|^p \le K \left(\int_0^\infty t^\gamma |u|^p\right)^{1/2} \left(\int_0^\infty t^\alpha |u''|^p\right)^{1/2}$$

to all α, β, γ such that $\beta = (\alpha+\gamma)/2$ except for the triple: $\alpha = p-1$, $\beta = -1$, $\gamma = -1-p$. In this case the inequality is false; however u satisfies the inequality

$$\int_0^\infty t^\beta |u'|^p \le K_1 \left\{ \left(\int_0^\infty t^\gamma |u|^p\right)^{1/2} \left(\int_0^\infty t^\alpha |u''|^p\right)^{1/2} + \int_0^\infty t^\gamma |u|^p \right\}.$$

1. Notation. Let $I = (a,b)$, $-\infty \le a < b \le \infty$, and "$AC_{loc}(I)$" denote the class of locally absolutely continuous functions on I. If α, γ are real numbers define

$$\mathcal{D}_{\alpha\gamma}(I) =: \left\{ u : u' \in AC_{loc}(I) : \int_I t^\gamma |u|^p,\ \int_I t^\alpha |u''|^p < \infty \right\},$$

$$\mathcal{D}_L(I) =: \{u \in \mathcal{D}_{\alpha\gamma}(I)) : \lim_{t\to a^+} u^{(i)} = 0,\ i = 0,1\},$$

$$\mathcal{D}_R(I) =: \{u \in \mathcal{D}_{\alpha\gamma}(I) : \lim_{t\to b^-} u^{(i)} = 0,\ i = 0,1\}.$$

Additionally let K denote a constant of interest whose value may change from line to line; if required different constants will be denoted by K_1, K_2, etc.

2. A weighted multiplicative inequality. In [4, Theorem 9] Kwong and Zettl proved the following result:

Theorem 1. *Suppose $1 \le p < \infty$, β, γ and α are real numbers such that*

$$\beta = \frac{\alpha+\gamma}{2}. \tag{1}$$

Then there is a constant K independent of u such that the inequality

$$\int_0^\infty t^\beta |u'|^p \le K \left(\int_0^\infty t^\gamma |u|^p\right)^{1/2} \left(\int_0^\infty t^\alpha |u''|^p\right)^{1/2} \tag{2}$$

holds for $u \in \mathcal{D}_{\alpha\gamma}((0,\infty))$ if $\beta > -1$ and $\gamma > -1-p$.

We are going to extend this Theorem by proving:

Theorem 2. *Let $u \in \mathcal{D}_{\alpha\gamma}((0,\infty))$, $1 \le p < \infty$, then the inequality (2) holds if and only if the pair $\{\alpha,\gamma\} \ne \{p-1, -1-p\}$ and β satisfies (1). Also in the exceptional case $\{\alpha,\gamma\} = \{p-1, -1-p\}$ the inequality*

$$\int_0^\infty t^{-1}|u'|^p \le K_1 \left\{ \left(\int_0^\infty t^{-1-p}|u|^p\right)^{1/2} \left(\int_0^\infty t^{p-1}|u''|\right)^{1/2} + \int_0^\infty t^{-1-p}|u|^p \right\} \tag{3}$$

is valid.

Proof. That (2) implies (1) is a statement of "dimensional balance" and follows if we introduce the change of variables $t = \lambda s$ in (2). Next suppose that $\beta \ne \alpha - p$. If $\beta > \alpha - p$, we get that $(\alpha-\gamma)/2p < 1$, and if $\beta < \alpha - p$ then $(\alpha-\gamma)/2p > 1$; thus in either case $(\alpha-\gamma)/2p \ne 1$. *Case (i):* Let $\beta \le -1$ and $\beta > \alpha - p$. Now

$$u'(t) = u'(s) - \int_t^s u''.$$

Since $\alpha < p-1$, $\alpha p'/p < 1$, so that $t^{-\alpha p'/p}$ is integrable if $p > 1$ ($t^{|\alpha|}$ is bounded if $p = 1$) on right neighborhoods of 0. This fact and Hölder's inequality implies that $\lim_{t\to 0+} \int_t^s u''$ is finite; consequently $\lim_{t\to 0+} u'$ exists. Since t^β fails to be integrable on right neighborhoods of 0, the limit must be 0. Because $\beta > \alpha - p$ a form of Hardy's inequality for $\mathcal{D}_L((0,1])$ (see [5, Example 6.8(i)) gives

$$\int_0^1 t^\beta|u'|^p \le K\int_0^1 t^{\beta+p}|u''|^p < K\int_0^1 t^\alpha|u''|^p$$

when $\beta < -1$, and

$$\int_0^1 t^\beta|u'|^p < K\int_0^1 t^{\alpha-p}|u''|^p \le K\int_0^1 t^\alpha|u''|^p$$

when $\beta = -1$. The sum inequality on $\mathcal{D}_{\alpha\gamma}((0,1))$

$$\int_0^1 t^\beta|u'|^p < K\left\{\int_0^1 t^\gamma|u|^p + \int_0^1 t^\alpha|u''|^p\right\} \tag{4}$$

follows trivially. By existing theory (take $I = [1,\infty)$, $\delta := (\alpha-\gamma)/2p < 1$, and $\epsilon = 1$ in [1, Example 1]) we obtain the sum inequality

$$\int_1^\infty t^\beta|u'|^p \le K\left\{\int_1^\infty t^\gamma|u|^p + \int_1^\infty t^\alpha|u''|^p\right\} \tag{5}$$

on $\mathcal{D}_{\gamma\alpha}([1,\infty))$. Addition of (4) and (5) gives the sum inequality on the entire interval. Set $t = \lambda s$. Then $u_\lambda := u(\lambda s)$ is in $\mathcal{D}_{\alpha\gamma}((0,\infty))$ so that

$$\int_0^\infty s^\beta|u_\lambda'(s)|^p\,ds < K\left\{\int_0^\infty s^\gamma|u_\lambda(s)|^p\,ds + \int_0^\infty s^\alpha|u_\lambda''(s)|^p\,ds\right\},$$

which is equivalent to the inequality

$$\int_0^\infty t^\beta |u'(t)|^p \, dt < K \left\{ \lambda^\phi \int_0^\infty t^\gamma |u(t)|^p \, dt + \lambda^{-\phi} \int_0^\infty t^\alpha |u''(t)|^p \right\} dt \tag{6}$$

where $\phi = (\alpha - \gamma)/2 - p$. (2) follows by minimizing the right side of (6) with respect to λ (the minimization is possible since $(\alpha - \gamma)/2p \neq -1$).

The other possibilities concerning β follow a similar logic. *Case (ii):* Assume $\beta > \{-1, \alpha - p\}$. Then Hardy's inequality for $\mathcal{D}_R((0,1]$ (see [5, Example 6.8(ii)]), Minkowski's inequality, and the integrability of t^β on $(0,1]$ gives

$$\int_0^1 t^\beta |u'|^p \leq K \left\{ \int_0^1 t^\alpha |u''|^p + |u'(1)|^p \right\} . \tag{7}$$

Since [1, Lemma 2.1]

$$|u'(1)| \leq K \left\{ \int_1^2 |u| + \int_1^2 |u''| \right\} , \tag{8}$$

a standard Hölder's inequality argument applied to (8) in conjunction with (7) yields that

$$\int_0^1 t^\beta |u'|^p < K \left\{ \int_0^\infty t^\gamma |u|^p + \int_0^\infty t^\alpha |u''|^p \right\} . \tag{9}$$

Since (5) remains valid, addition of (5) and (9) gives the sum inequality on $(0,\infty)$ and the the same scaling argument as in the previous case may be applied. *Case (iii):* If $\beta < \alpha - p$ and $\beta < -1$, Hardy's inequality for $\mathcal{D}_L([1,\infty))$ (see [5, Example 6.9(i)]), Minkowski's inequality, Lemma 2.1 of [1], etc., give as in *Case (ii)* the sum inequality (5). On the other hand since $(\alpha - \gamma)/2p \geq 1$, existing theory (see [1, Example 2]) gives

$$\int_0^1 t^\beta |u'|^p \leq K \left\{ \int_0^1 t^\gamma |u|^p + \int_0^1 t^\alpha |u''|^p \right\} ; \tag{10}$$

we then add and scale as before. *Case (iv):* If $\beta < \alpha - p$ and $\beta \geq -1$, we can show that $\lim_{t\to\infty} u'(t) = 0$ by an argument similar to *Case (i)*. Hardy's inequality for $\mathcal{D}_R([1,\infty))$ (see [5, Example 6.9(ii)]) then leads trivially to (5). Adding this to (10) (the argument of *Case (iii)* continues to apply) and finishing the argument as before completes the proof.

Now suppose that $\beta = \alpha - p$. To handle this situation we modify an argument previously given in the proof of [2, Theorem 2.1]: Let ϕ be a C_0^∞ function with support on $[-3/4, 1]$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ on $[1/2, 1]$. Define a bi-infinite partition $\{t_i\}_{-\infty}^{\infty}$ of $(0,\infty)$ by letting $t_0 = 1$ and $t_i = 2^i$. For $m \in \mathbb{Z}$ and $u \in \mathcal{D}_{\alpha\gamma}((0,\infty))$ set

$$y_m(t) = u(t)\phi((t - t_m)/t_m) ; \tag{11}$$

thus y_m has support on $[t_{m-2}, t_{m+1}]$ and $y_m = u$ on $[t_{m-1}, t_m]$. It is not difficult to show applying Leibniz's rule of differentiation that there is a constant C independent of u and m such that

$$|y''_m(t)| \leq C \sum_{i=0}^{2} |u^{(i)}|/t_m^{2-i} , \tag{12}$$

a.e. Next we recall that if $\alpha = \beta = \gamma = 0$ (2) is a special case of a far more general and well known Gabushin) inequality (cf. [3]). (Also note that if $p > 1$ the unweighted inequality follows from *Case (ii)* above.) Substituting (11) into this inequality and using (12) gives

$$\begin{aligned}\left(\int_{t_{m-1}}^{t_m} |u'|^p\right) &\le \left(\int_{t_{m-2}}^{t_{m+1}} |y_m'|^p\right)\\ &\le K\left(\int_{t_{m-2}}^{t_{m+1}} |y_m|^p\right)^{1/2}\left(\int_{t_{m-2}}^{t_{m+1}} |y_m''|^p\right)^{1/2}\\ &\le KC^{1/2}\left(\int_{t_{m-2}}^{t_{m+1}} |u|^p\right)^{1/2}\left(\sum_{i=0}^{2} |u^{(i)}|^p/(t_m^{2-i})^p\right)^{1/2}. \qquad (13)\end{aligned}$$

We multiply the last line of (13) by t^β, noting both that β satisfies (1) and that if $t \in [t_{m-2}, t_m]$, then $1/4 \le t/t_m \le 2$ because of the nature of the partition. This gives

$$\int_{t_{m-1}}^{t_m} t^\beta |u'|^p \le K_1\left(\int_{t_{m-2}}^{t_{m+1}} t^\gamma |u|^p\right)^{1/2}\left(\sum_{i=0}^{2} t_m^{\alpha-(2-i)p}|u^{(i)}|^p\right)^{1/2} \qquad (14)$$

for a constant K_1 independent of u. Summing (14) over m and using the discrete sum form of the Cauchy-Schwartz inequality yields that

$$\int_0^\infty t^\beta |u'|^p \le K_1\left(\sum_{m=1}^{\infty}\int_{t_{m-2}}^{t_{m+1}} t^\gamma |u|^p\right)^{1/2}\left(\sum_{m=0}^{\infty}\int_{t_{m-2}}^{t_{m+1}}\left(\sum_{i=0}^{2} t_m^{\alpha-(2-i)p}|u^{(i)}|^p\right)\right)^{1/2}$$

Because each t belongs in at most three intervals $[t_{m-2}, t_{m+1}]$ and by Minkowski's inequality applied to the last integral in (15), it follows that

$$\int_0^\infty t^\beta |u'|^p \le K_2\left(\int_0^\infty t^\gamma |u|^p\right)^{1/2}\left[\sum_{i=0}^{2}\left(\int_0^\infty t_m^{\alpha-(2-i)p}|u^{(i)}|^p\right)\right]^{1/2}. \qquad (16)$$

Assume now that $\beta < -1$ so that $\gamma < -1$. Because $\alpha < p-1$ and $\beta < -1$ an argument given in *Case (i)* shows that $\lim_{t\to 0^+} u'(t) = 0$. Similarly the fact that

$$\frac{-\beta p'}{p} > \frac{1}{p-1} > 0$$

together with Hölder's inequality applied to the integral in the identity

$$u(t) = u(s) - \int_t^s u'$$

demonstrates that $\lim_{t\to 0^+} u(t)$ exists. Since $\gamma < -1$ the limit is 0. This shows that $\mathcal{D}_{\alpha\gamma}((0,\infty)) = \mathcal{D}_L((0,\infty))$. Since $\alpha - p < -1$, iteration of Hardy's inequality for $\mathcal{D}_L((0,\infty))$ (cf. [5, Example 6.7]) gives

$$\int_0^\infty t^{\alpha-(2-i)p}|u^{(i)}|^p \le K_3 \int_0^\infty t^\alpha |u''|^p. \qquad (17)$$

Substitution of (17) into (16) yields (2). The case $\beta > -1$, $\gamma > -1$ is covered by Theorem 1. However an argument similar to the previous case shows that this alternative implies that $\lim_{t\to\infty} u^{(i)}(t) = 0$ for $i = 0, 1$; so that $\mathcal{D}_{\alpha\gamma}((0,\infty)) = \mathcal{D}_R((0,\infty))$. Since (17) holds on $\mathcal{D}_R((0,\infty))$ if $\alpha - p > -1$ we can substitute it into (16) as before to complete the proof. Summarizing, (2) holds for all choices of α, β, and γ satisfying (1) except possibly for

$$\begin{aligned} \alpha &= p-1\,, \\ \beta &= -1\,, \\ \gamma &= -1-p\,, \end{aligned} \tag{18}$$

which was to be proved.

We next show by a counterexample that (2) cannot hold in the exceptional case (18), Let

$$u_\delta(t) := \begin{cases} u_{1,\delta}(t) \equiv t^{1+\delta} & \text{for } t \in [0,1] \\ u_{2,\delta} \equiv ((1+\delta)t^{1-\delta} - 2\delta)/(1-\delta) & \text{for } t \in (1,\infty) \end{cases}$$

where $\delta > 0$ is a parameter. Since u_δ and u'_δ are continuous at 1, $u_\delta \in \mathcal{D}_{\alpha\gamma}((0,\infty))$. To prove that (2) cannot hold for this family of functions it is sufficient to show that if

$$Q(u_\delta) := \frac{\left(\displaystyle\int_0^\infty t^{-1}|u'_\delta|^p\right)^2}{\left(\displaystyle\int_0^\infty t^{-1-p}|u_\delta|^p\right)\left(\displaystyle\int_0^\infty t^{p-1}|u''_\delta|\right)}\,,$$

then

$$\lim_{\delta\to 0} Q(u_\delta) = \infty\,. \tag{19}$$

A calculation yields that

$$\int_0^\infty t^{-1}|u'_\delta|^p = \frac{2(1+\delta)^p}{p\delta} \tag{20}$$

$$\int_0^\infty t^{p-1}|u''_\delta|^p = \frac{2\delta^p(\delta+1)^p}{p\delta}\,. \tag{21}$$

Moreover

$$\begin{aligned} \int_1^\infty t^{-1-p}|u_2|^p &< \frac{\displaystyle\int_1^\infty (1+\delta)t^{1-\delta}}{1-\delta} \\ &< \frac{(1+\delta)^p}{p\delta(1-\delta)^p}\,, \end{aligned} \tag{22}$$

so that

$$\begin{aligned}\int_0^\infty t^{-1-p}|u_\delta|^p &= \frac{1}{p\delta} + \int_1^\infty t^{-1-p}|u_2|^p \\ &< \frac{1}{p\delta} + \frac{(1+\delta)^p}{p\delta(1-\delta)^p}\,. \end{aligned} \tag{23}$$

Combining (22) and (23) and substituting them together with the estimates (20) and (21) into (19) gives

$$\begin{aligned}\lim_{\delta\to 0} Q(u_\delta) &\ge \lim_{\delta\to 0} \frac{2(1+\delta)^p}{\delta^p\left(1+\frac{(1+\delta)^p}{(1-\delta)^p}\right)} \\ &= \infty\,.\end{aligned}$$

It remains to prove (3) in the exceptional case: Let $f(t)=t$, $N=t^{-1}$, $W=t^{-1-p}$, and $P=t^{p-1}$ in condition (C_3) of [1]. Then a calculation shows that

$$\begin{aligned} S_1 &= (\epsilon t)^p\left((\epsilon t)^{-1}\int_t^{t(1+\epsilon)} s^{-1}\,ds\right)\left((\epsilon t)^{-1}\int_t^{t(1+\epsilon)} s^{-(p-1)p'/p}\,ds\right)^{p-1} \\ &\le \epsilon \\ S_2 &= (\epsilon t)^{-p}\left((\epsilon t)^{-1}\int_t^{t(1+\epsilon)} s^{-1}\,ds\right)\left((\epsilon t)^{-1}\int_t^{t(1+\epsilon)} s^{(p+1)p'/p}\,ds\right)^{p-1} \\ &\le \epsilon^{-p}(1+\epsilon)^{p+1}\,. \end{aligned}$$

This yields by [1, Theorem 2.1 (iv)] the sum inequality

$$\int_0^\infty t^{-1}|u|^p \le K\left\{\epsilon^{-p}(1+\epsilon)^{p+1}\int_0^\infty t^{-1-p}|u|^p + \epsilon^p\int_0^\infty t^{p-1}|u''|^p\right\}\,. \tag{24}$$

We substitute the elementary inequality

$$(1+\epsilon)^{p+1} \le 2^p\left(1+\epsilon^{p+1}\right)$$

into (24) and then minimize

$$2^p\epsilon^{-p}\int_0^\infty t^{-1-p}|u|^p + \epsilon^p\int_0^\infty t^{p-1}|u''|^p$$

on the right-hand side. The result is (3) with $K_1=\max\{2K,\epsilon_0^p\}$ where ϵ_0 is minimizing value of ϵ . The proof is complete.

References

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