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FINITE-PART INTEGRALS AND THE EULER-MACLAURIN EXPANSION*

J. N. Lyness†

This paper is dedicated to Walter Gautschi on the occasion of his 65th birthday.

Abstract

The context of this note is the discretization error made by the m-panel trapezoidal rule when the integrand has an algebraic singularity at one end, say $x = 0$, of the finite integration interval. In the absence of a singularity, this error is described by the classical Euler-Maclaurin summation formula, which is an asymptotic expansion in inverse integer powers of m. When an integrable singularity (x^α with $\alpha > -1$) is present, Navot's generalization is valid. This introduces negative fractional powers of m into the expansion. Ninham has generalized this result to noninteger α satisfying $\alpha < -1$. In this note, we extend these results to all α by providing the nontrivial extension to negative integer α. This expansion differs from the previous expansions by the introduction of a term $\log m$.

1 INTRODUCTION

The classical Euler-Maclaurin summation formula (see (2.14) below with $\beta = 1$) is an asymptotic expansion in integer powers of $1/m$ of the discretization error of the m-panel trapezoidal rule. This formula is valid when $f(x)$, together with its early derivatives, is continuous over the integration interval.

A major generalization to integrable functions of the form $f(x) = x^\alpha g(x)$, where $g(x)$ is regular and $\alpha > -1$, was established by Navot in 1961. Later, Ninham (1966) generalized the result to the cases in which $\alpha < -1$ but is not an integer. He established that the expansion was formally the same but that divergent integrals should be interpreted as Hadamard finite-part integrals. In this paper we complete the theory by covering the negative integer case. We also provide much simpler proofs for the other cases than those provided by Navot and by Ninham.

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†Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL 60439

The ultimate result in this paper is Theorem 3.2. This establishes an asymptotic expansion for the m-panel offset trapezoidal rule $R^{(m)}(\beta)f$, which is valid for all integrand functions of the form $x^\alpha g(x)$, where $g(x)$ is regular and α may take any value, whether or not the integral being approximated is convergent. This expansion may include terms in m^{-j}, $m^{-(j+\alpha)}$, and $\log m$ for both positive and negative integer j. Integral representations are given for all terms in the expansion and for the remainder term.

2 CLASSICAL BACKGROUND MATERIAL

In this section, we recall some elementary results about Hadamard's finite-part integral and about the Euler-Maclaurin asymptotic expansion. In each case, we treat the integration interval [0,1]. Background material about these is available in many references, including, for example, Davis and Rabinowitz (1984), pp. 11, 188, and 136, and Diligenti and Monegato (1994). For our purposes we treat only finite-part integrals of the form

$$If = fp\int_0^1 x^\alpha g(x)dx, \tag{2.1}$$

where $g(x)$ is regular; specifically, $g(x) \in C^{q-1}[0,1]$ with $\alpha \in R$, $q \in Z^+$, and $q > -\alpha$.

We recall that any convergent conventional integral coincides with the corresponding finite-part integral and that finite-part integrals may be combined and manipulated in much the same way as conventional integrals. An exception is the classical change of variable rule, which needs minor modification; details may be found in Monegato (1994).

We now provide a two-stage elementary definition of the finite-part integrals (2.1) above.

Definition 2.1 *For all real α, the Hadamard finite-part integral If_α of the function $f_\alpha(x) = x^\alpha$ over the unit interval $[0,1]$ is defined by*

$$\begin{aligned} If_\alpha &= \int_0^1 f_\alpha(x)dx && \alpha > -1 \\ &= -\int_1^\infty f_\alpha(x)dx && \alpha < -1 \\ &= 0 && \alpha = -1. \end{aligned} \tag{2.2}$$

Clearly, If_α coincides with $f_\alpha(1)$ when this is defined. For values of α for which the first integral in (2.2) exists, the second does not, and vice versa.

Unless α is a nonnegative integer, $f_\alpha^{(s)}(x)$ is a constant multiple of $f_{\alpha-s}(x)$. A simple result, needed in a later proof, is the following lemma.

Lemma 2.2

$$\begin{aligned} \int_1^m f_\alpha^{(s)}(x)dx &= (m^{\alpha-s+1}-1)If_\alpha^{(s)} & \alpha - s \neq -1 \\ \int_1^m f_\alpha^{(s)}(x)dx &= \frac{\log m}{\log 2}\int_1^2 f_\alpha^{(s)}(x)dx & \alpha - s = -1 \end{aligned} \tag{2.3}$$

It is trivial to verify this. The underlying reason that this result and the subsequent theory is simple is that the integrand is a homogeneous function of degree $\alpha - s$ about the origin.

Extending Definition 2.1 to functions of the form $x^\alpha g(x)$, where $g(x) \in C^\ell[0,1]$, requires only the expression of $g(x)$ as a Taylor expansion about the origin and term-by-term integration of a finite series. Specifically, we may set

$$g(x) = \sum_{j=0}^{\ell-1} g^{(j)}(0)x^j/j! + G_\ell(x) \tag{2.4}$$

and

$$f(x) = x^\alpha g(x) = \sum_{j=0}^{\ell-1} \kappa_{\alpha+j} f_{\alpha+j}(x) + x^\alpha G_\ell(x), \tag{2.5}$$

where

$$\kappa_{\alpha+j} = g^{(j)}(0)/j!. \tag{2.6}$$

When $\ell + \alpha + 1 > 0$, the remainder term

$$F_{\alpha;\ell}(x) = x^\alpha G_\ell(x) = f(x) - \sum_{j=0}^{\ell-1} \kappa_{\alpha+j} f_{\alpha+j}(x) \tag{2.7}$$

is integrable over [0,1]. Term-by-term integration of (2.5) above leads to the following definition.

Definition 2.3 *Let $f(x) = x^\alpha g(x)$ and $g(x) \in C^{q-1}[0,1]$ with $q + \alpha > 0$; then*

$$If = \sum_{j=0}^{q-1} \kappa_{\alpha+j} If_{\alpha+j} + \int_0^1 (f(x) - \sum_{j=0}^{q-1} \kappa_{\alpha+j} f_{\alpha+j}(x))dx, \tag{2.8}$$

where $\kappa_{\alpha+j} = g^{(j)}(0)/j!$ and $If_{\alpha+j}$ is defined in Definition 2.1 above.

We now turn to the Euler-Maclaurin expansion.

Definition 2.4 *The offset trapezoidal rule for the interval [0,1] is denoted by*

$$R^{(m)}(\beta)f = \frac{1}{m}\sum_{j=0}^{m-1} f\left(\frac{j+\beta}{m}\right). \tag{2.9}$$

The Euler-Maclaurin expansion involves coefficients and kernel functions

$$c_s(\beta) = B_s(\beta)/s! \qquad s \geq 0, \tag{2.10}$$
$$h_s(\beta,t) = (B_s(\beta) - \bar{B}_s(\beta - t))/s! \qquad s \geq 1, \tag{2.11}$$

where $B_s(x)$ is the Bernoulli polynomial and $\bar{B}_s(x)$ the corresponding Bernoulli function, which coincides with $B_s(x)$ in the interval (0,1) and has unit period. Note that

$$h_p(\beta,t) = h_p(\beta,t+1) \tag{2.12}$$
$$c_p(\beta) = \int_0^1 h_p(\beta,t)dt. \tag{2.13}$$

Theorem 2.5 *The Euler-Maclaurin asymptotic expansion for regular $f(x)$*

Let $f(x)$ and its first p derivatives be integrable in [0,1], and let $p \geq 1$. Then

$$R^{(m)}(\beta)f = \sum_{s=0}^{p-1} \frac{c_s(\beta)}{m^s}\int_0^1 f^{(s)}(x)dx + \frac{1}{m^p}\int_0^1 h_p(\beta,mt)f^{(p)}(t)dt. \tag{2.14}$$

Note that $c_0(\beta) = 1$, and so the first term in the summation is simply If.

3 THE EULER-MACLAURIN EXPANSION FOR x^α

In this paper, we treat functions that have an algebraic singularity at an end point of our integration interval, [0,1]. We restrict our attention to functions of the form

$$f(x) = x^\alpha g(x), \tag{3.1}$$

where $g(x)$ is $C^p[0,1]$. First we treat the monomial

$$f_\alpha(x) = x^\alpha; \tag{3.2}$$

and then, as the theory is linear, we concatenate individual terms of this type, using a Taylor expansion for $g(x)$, to obtain the corresponding expansion for $f(x)$ in (3.1).

The basic theorem in this paper is the following.

Theorem 3.1 *The Euler-Maclaurin expansion for* x^α

Let α be real, and let p be a nonnegative integer satisfying $p+\alpha+1 \geq 0$; then

$$R^{(m)}(\beta)f_\alpha = \frac{A_{\alpha+1}(\beta; f_\alpha)}{m^{\alpha+1}} + \delta_{\alpha+1,0}\log m + \sum_{s=0}^{p-1} \frac{c_s(\beta)If_\alpha^{(s)}}{m^s} + E_p^{(m)}(\beta)f_\alpha, \quad (3.3)$$

where

$$A_{\alpha+1}(\beta; f_\alpha) = f_\alpha(\beta) - \sum_{s=0}^{p-1} c_s(\beta)If_\alpha^{(s)} + \int_1^\infty h_p(\beta,t)f_\alpha^p(t)dt \quad (3.4)$$

and

$$E_p^{(m)}(\beta)f_\alpha = -\frac{1}{m^{\alpha+1}} \int_m^\infty h_p(\beta,t)f_\alpha^{(p)}(t)dt = O(m^{-p}). \quad (3.5)$$

The integrals in (3.3) are defined in (2.2) as either conventional or Hadamard finite-part integrals. Simpler expressions for (3.4) are established in Section 4.

The proof given below, valid for all α, involves only elementary algebra and calculus. It exploits the fact that the result may be considered to be a matter of scaling. Comments on previous proofs of less general versions of this theorem are made in Section 5.

Proof. As a preliminary, we apply (2.14) translated to the interval $[j, j+1]$ with $m = 1$ to the function $f_\alpha(x)$. This gives

$$\begin{aligned} f_\alpha(j+\beta) = (j+\beta)^\alpha &= \sum_{s=0}^{p-1} c_s(\beta) \int_j^{j+1} f_\alpha^{(s)}(x)dx \\ &+ \int_j^{j+1} h_p(\beta,t)f_\alpha^{(p)}(t)dt \quad j = 1, 2, \ldots, m-1. \quad (3.6) \end{aligned}$$

Note that we have used the periodicity property (2.12) of $h_p(\beta,t)$.

We now substitute (3.6) into a scaled version of Definition 2.4 as follows.

$$
\begin{aligned}
R^{(m)}(\beta)f_\alpha &= \frac{1}{m}\sum_{j=0}^{m-1} f_\alpha\left(\frac{j+\beta}{m}\right) \\
&= \frac{1}{m^{1+\alpha}}\sum_{j=0}^{m-1} f_\alpha(j+\beta) \\
&= \frac{1}{m^{1+\alpha}} f_\alpha(\beta) + \sum_{s=1}^{p-1} \frac{c_s(\beta)}{m^{1+\alpha}} \int_1^m f_\alpha^{(s)}(x)dx \\
&\quad + \frac{1}{m^{1+\alpha}} \int_1^m h_p(\beta,t) f_\alpha^{(p)}(t)dt \qquad (3.7)
\end{aligned}
$$

Here we have isolated the term with $j = 0$. We have collected together integrals over $[j, j+1]$ to form an integral over $[1, m]$. In the final term we have used once more the periodicity of the kernel function $h_p(\beta, t)$. Up to this point, no singularity is involved , and any nonnegative integer p is permitted. All manipulation has been conventional. Our final manipulative steps are to apply Lemma 2.2 to reexpress the integrals whose upper limit is m, and to split the final integral into two parts. We find, when α is not an integer,

$$
\begin{aligned}
R^{(m)}(\beta)f_\alpha &= \frac{1}{m^{1+\alpha}} f_\alpha(\beta) + \sum_{s=1}^{p-1} \frac{c_s(\beta)}{m^{1+\alpha}} (m^{\alpha-s+1} - 1) I f_\alpha^{(s)} \\
&\quad + \frac{1}{m^{1+\alpha}} \int_1^\infty h_p(\beta,t) f_\alpha^{(p)} dt \\
&\quad - \frac{1}{m^{\alpha+1}} \int_m^\infty h_p(\beta,t) f_\alpha^{(p)}(t) dt \qquad p > -1-\alpha. \qquad (3.8)
\end{aligned}
$$

Note that the condition on p is needed only to validate splitting the final integral, extracting a part of order $m^{-(1+\alpha)}$, and leaving another part, the remainder term of lower order.

When α is an integer, (3.8) may require adjustment. Specifically, in the sum over index s in (3.7), any term in which $\alpha - s = -1$ should have been replaced by the second member of (2.3) instead of the first. Since s is nonnegative, this adjustment is needed, if at all, only when $\alpha \leq -1$. In these cases, one should remove the current term with $s = \alpha + 1$ and introduce the second member of (2.3), to obtain a single additional term, namely,

$$
\frac{c_{1+\alpha}(\beta)}{m^{\alpha+1}} \frac{\log m}{\log 2} \int_1^2 f_\alpha^{\alpha+1}(x)dx.
$$

Moreover, when α is nonnegative, the integrand is zero. Thus, this additional term is nonzero only when $\alpha = -1$, when it reduces to $\log m$.

Collecting together terms of specific orders in m, we recover the expressions given in the statement of the theorem. To establish the theorem, we need to show that the term (3.5) is of the order stated and that (3.4) is not dependent on p. Since $f_\alpha(x) = x^\alpha$, we see

$$E_p^{(m)}(\beta)f_\alpha = \frac{1}{m^{\alpha+1}} \int_m^\infty h_p(\beta,t)\frac{\alpha!}{(\alpha-p)!}t^{\alpha-p}dt \qquad p > -\alpha-1.$$

An elementary calculation shows that

$$|E_p^{(m)}(\beta)f_\alpha| \leq \frac{\alpha}{(\alpha-p+1)!}\max_t |h_p(\beta,t)|m^{-p},$$

establishing the correct order.

This dependence on p in the expression (3.4) is illusory. Since $h_p(\beta,t)$ is periodic in t and

$$\int_j^{j+1} h_p(\beta,t)dt = c_p(t),$$

integration by parts yields the same expression for $A_{\alpha+1}(\beta;f_\alpha)$ with p replaced by $p+1$. (We cannot replace p in this expression by any integer less than $-\alpha-1$. Besides not being justified, the resulting (incorrect) expression contains a divergent integral.) ∎

This theorem asserts that the expansion, with this value of $A_{\alpha+1}$, holds for all finite p exceeding $-\alpha-1$.

For smaller values of p, we find directly from Theorem 2.5 that

$$R^{(m)}(\beta)f_\alpha = \sum_{s=0}^{p-1} \frac{c_s(\beta)}{m_s} If_\alpha^{(s)} + \tilde{E}_p^{(m)}(\beta)f_\alpha$$

with

$$\tilde{E}_p^{(m)}(\beta])f_\alpha = \frac{1}{m^p}\int_0^1 h_p(\beta,mt)f_\alpha^{(p)}(t)dt \qquad p < -\alpha-1.$$

It follows that, when p and $\bar{p}$ are integers satisfying $p < -\alpha-1 < \bar{p}$, the forms of remainder terms are connected by

$$\tilde{E}_p^{(m)}(\beta)f_\alpha = \sum_{s=p}^{\bar{p}} \frac{c_s(\beta)}{m^s} If_\alpha^{(s)} + \frac{A_{\alpha+1}(\beta,f_\alpha)}{m^{\alpha+1}} + E_{\bar{p}}^{(m)}(\beta)f_\alpha.$$

When α is a nonnegative integer, the expression for $A_{\alpha+1}(\beta;f_\alpha)$ reduces to zero, and $f_\alpha^{(s)} = 0$ for $s \geq \alpha+1$.The result in the statement of the theorem follows directly from Theorem 2.5.

It is a simple step from the Euler-Maclaurin expansion for $f_\alpha(x)$ in this theorem to the corresponding expansion for $f_\alpha(x)g(x)$ when $g(x)$ is regular. We simply follow our definition of the Hadamard finite-part integral by expanding $g(x)$ in a Taylor series. This gives expansion (2.5). We may apply already available versions of the Euler-Maclaurin expansion to each term in this expansion. We shall apply Theorem 2.5 to the final term. To this end we require that $g(x)$ have sufficient continuity that the final term in (2.5) differentiated p times is integrable. We treat

$$g(x) \in C^{p+q-1}[0,1] \text{ with } p+q+\alpha > 0; \quad p \geq 1; \quad q \geq 0. \tag{3.9}$$

It is convenient to differentiate (2.5) term by term to obtain

$$f^{(s)}(x) = \sum_{j=0}^{p+q-1} \kappa_{\alpha+j} f^{(s)}_{\alpha+j}(x) + F^{(s)}_{\alpha;p+q}(x) \qquad s = 0, 1, \ldots, p \tag{3.10}$$

and integrate term by term to obtain

$$If^{(s)} = \sum_{j=0}^{p+q-1} \kappa_{\alpha+j} If^{(s)}_{\alpha+j} + \int_0^1 F^{(s)}_{\alpha;p+q}(x)dx \qquad s = 0, 1, \ldots, p. \tag{3.11}$$

These formulas are helpful in establishing the following theorem.

Theorem 3.2 *Let α be real, p a positive integer, and q a nonnegative integer satisfying $p+q+\alpha > 0$. Let $f(x) = x^\alpha g(x)$, where*

$$g(x) \in C^{p+q-1}[0,1].$$

Then

$$\begin{aligned} R^{(m)}(\beta)f \quad &= \quad \sum_{j=0}^{p+q-1} \frac{A_{\alpha+1+j}(\beta; f)}{m^{\alpha+1+j}} + \kappa_{-1} \ln m + \sum_{s=0}^{p-1} \frac{c_s(\beta)}{m^s} If^{(s)} \\ &\quad + O(m^{-p}), \end{aligned} \tag{3.12}$$

where κ_{-1} is taken to be zero unless α is a negative integer and

$$A_{\alpha+1+j}(\beta; f) = \kappa_{\alpha+j} A_{\alpha+1+j}(\beta; f_{\alpha+j}). \tag{3.13}$$

Proof. When $g(x) \in C^{p+q-1}[0,1]$ with $p+q+\alpha > 0$, it follows that the p-th derivative of $F_{\alpha;p+q}(x)$ is integrable in [0,1]. Hence, we may apply Theorem 2.5 to this function. Applying Theorem 3.1 to the other terms in (2.5), we find

$$
\begin{aligned}
R^{(m)}(\beta)f &= \sum_{j=0}^{p+q-1} \kappa_{\alpha+j} R^{(m)}(\beta) f_{\alpha+j} + R^{(m)}(\beta) F_{\alpha;p+q} \\
&= \sum_{j=0}^{p+q-1} \frac{\kappa_{\alpha+j} A_{\alpha+j+1}}{m^{\alpha+j+1}} + \sum_{j=0}^{p+q-1} \kappa_{\alpha+j} \delta_{\alpha+j+1,0} \ln m \\
&\quad + \sum_{s=0}^{p-1} \frac{c_s(\beta)}{m^s} \sum_{j=0}^{p+q-1} \kappa_{\alpha+j} I f_{\alpha+j}^{(s)} + \sum_{j=0}^{p+q-1} \kappa_{\alpha+j} E_p^{(m)}(\beta) f_{\alpha+j} \\
&\quad + \sum_{s=0}^{p-1} \frac{c_s(\beta)}{m^s} \int_0^1 F_{\alpha;p+q}^{(s)}(x)dx + \frac{1}{m^p} \int_0^1 h_p(\beta; mt) F_{\alpha;p+q}^{(p)}(t)dt. \quad (3.14)
\end{aligned}
$$

In view of (3.5) and (2.14), the fourth and the sixth term here are $O(m^{-p})$ and belong in the remainder term in (3.12). Also belonging in this remainder term are those elements in the summation in the first expression for which $\alpha + j + 1 \geq p$. The third and fifth expressions here can be reduced, by using (3.11), to the sum over index s in (3.12). Finally, the second term above exists only when α is an integer and $j = -\alpha - 1$. This gives the second term in (3.12). Thus, all terms in (3.14) are accounted for; together they give rise to the right-hand side of (3.12). This establishes the theorem. ∎

4 NUMERICAL VALUES OF THE EXPANSION COEFFICIENTS

It was shown by Navot and by Ninham that, when $\alpha \neq$ integer,

$$A_{\alpha+1}(\beta; f_\alpha) = \zeta(-\alpha, \beta), \quad (4.1)$$

where ζ is the Hurwitz zeta function, defined by

$$\zeta(-\alpha, \beta) = \beta^\alpha + (\beta+1)^\alpha + \ldots \qquad \beta > 0; \quad \alpha < -1, \quad (4.2)$$

and by analytic continuation for all values of α other than -1, where this analytic function has its only singularity, a pole of order 1.

A simple proof of (4.1) valid when $\alpha > -1$ may be obtained by applying a version of the Euler-Maclaurin expansion (2.14) with $m = 1$, $f(x) = x^\alpha$ to an interval $[k, k+1]$. This procedure gives

$$f_\alpha(k+\beta) = (k+\beta)^\alpha = \sum_{s=0}^{p-1} c_s(\beta) \int_k^{k+1} f_\alpha^{(s)}(x)dx + \int_k^{k+1} h_p(\beta, mt) f_\alpha^{(p)}(t)dt. \quad (4.3)$$

Set $\alpha < 1$, sum (4.3) over all positive integer k, and add $f_\alpha(\beta)$ to both sides. The result is $\zeta(-\alpha, \beta)$ on the left and Definition (3.4) for $A_{\alpha+1}(\beta; f_\alpha)$ on the right.

For α a positive integer, one may use (3.4) and (2.14) with $p > \alpha$ and $m = 1$ to show directly

$$A_{\alpha+1}(\beta; f_\alpha) = 0 \qquad \alpha = 0, 1, 2, \dots \quad . \tag{4.4}$$

Analytic continuation of the Hurwitz zeta function establishes (4.1) above for all α except $\alpha = -1$.

The reader will have noticed that, because of the singularity, no information about $A_0(\beta; f_{-1})$ has been forthcoming. This turns out to be the most intractable of the coefficients, and, as will appear below, it is the only one for which a numerical value is required in the extrapolation application. In Theorem 4.1 below, this value is expressed in terms of the Euler-Mascheroni constant

$$\gamma = \lim_{m\to\infty} \left(\sum_{j=1}^{m} \frac{1}{j} - \ln m \right) \approx 0.5772 \tag{4.5}$$

and a finite harmonic function defined by

$$H(\beta) = \sum_{n\geq 1} \left(\frac{1}{n} - \frac{1}{n+\beta} \right). \tag{4.6}$$

This function appears in Knuth (1973) where it is not given a name; when $\beta = p/q$ and p and q are positive integers satisfying $0 < p < q$, Knuth (p. 94) shows that

$$H(\beta) = \tfrac{1}{\beta} - \tfrac{1}{2}\pi \cot \beta\pi - \ln 2q + 2 \textstyle\sum_{1\leq n<q/2} \cos 2\pi n\beta \ln \sin \tfrac{n}{q}\pi. \tag{4.7}$$

Explicit forms of these expressions are listed by Knuth (p. 616) for all p, q satisfying $0 < p < q \leq 6$. For example,

$$H(1) = 1; \qquad H(\tfrac{1}{2}) = 2 - 2\ln 2.$$

Theorem 4.1

$$A_0(\beta; f_{-1}) = \gamma + \frac{1}{\beta} - H(\beta), \tag{4.8}$$

where γ is the Euler-Mascheroni constant (4.5) above, and $H(\beta)$ is Knuth's finite harmonic function (4.6) above.

Proof. We exploit the expansion in which A_0 occurs, namely, (3.3) with $\alpha = -1$. This gives

$$
\begin{aligned}
R^{(m)}(\beta)f_{-1} &= \frac{1}{m}\sum_{j=0}^{m-1}\frac{m}{j+\beta} = A_0(\beta;f_{-1}) + \log m \\
&+ \sum_{s=0}^{p-1}\frac{c_s(\beta)If_{-1}^{(s)}}{m^s} + E_p^{(m)}(\beta)f_{-1}.
\end{aligned}
$$

Since $If_{-1} = 0$, the $s = 0$ term in the sum is zero. Taking the limit as m becomes infinite, we find

$$
A_0(\beta;f_{-1}) = \lim_{m\to\infty}\left(\sum_{j=0}^{m-1}\frac{1}{j+\beta} - \log m\right).
$$

Elementary manipulation of this, involving (4.5) and (4.6) above, leads directly to the result in the theorem. ∎

Examples include

$$
A_0(1;f_{-1}) = \gamma; \qquad A_0(\tfrac{1}{2};f_{-1}) = \gamma + 2\ln 2.
$$

5 CONCLUDING REMARKS

It is beyond the scope of this paper to discuss in any detail the ways in which this expansion may be used for extrapolation or to examine its relationship with known quadrature formulas. We content ourselves by making a few points of a general nature.

When a finite-part integral is being approximated using extrapolation and so $\alpha \leq -1$, one is extracting the coefficient of a higher-order term in the expansion instead of the principal term. This is akin to numerical differentiation, and one should expect a corresponding increase in the amplification of noise level error in the calculation.

When, in addition, α is a negative integer, say $\alpha = -N$, the expansion takes the following form.

$$
\begin{aligned}
R^{(m)}(\beta)f &= \kappa_{-N}A_{1-N}m^{N-1} + \ldots + \kappa_{-2}A_{-1}m + \kappa_{-1}\ln m \\
&+ \kappa_{-1}A_0 + If + \frac{c_1(\beta)}{m}If^{(1)} + \ldots
\end{aligned}
$$

We may extrapolate in the usual way. But when doing so, we shall be obliged to extract estimates of two coefficients, namely, κ_{-1} and $\kappa_{-1}A_0 + If$. Then, to obtain

If , we have to separate it from $\kappa_{-1}A_0$. This step requires both the extrapolated value of κ_{-1} and the numerical value of constant $A_0(\beta; f_{-1})$. For some values of β, the latter is provided by Theorem 4.1. This is an unusual situation in the practice of extrapolation. Usually only simple data, such as the value of α and data relating to the structure of the expansion, are required.

The significant result of this paper is Theorem 3.2. Various proofs for the conventional case $\alpha > -1$, have appeared. The original proof by Navot (1961) involves several pages of detailed algebraic manipulation. A subsequent proof by Lyness and Ninham (1967) is shorter but involves generalized functions in an essential way. A somewhat pedestrian proof using the generalized zeta function has been given by Lyness (1971). None of these proofs is particularly illuminating. The reader finds, at the end, that the result has been established. For the finite-part integral case with noninteger α, Ninham's proof involves a Fourier decomposition of the integrand, as well as the use of generalized functions. That paper comprises an elegant but lengthy application of a little known theory contained in a twelve-page paper devoted to this single result.

The proof of Section 3 of this paper is more general and more straightforward than any of the proofs mentioned above.

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