On the One-Dimensional Ginzburg-Landau BVPs*

Man Kam Kwong

Mathematics and Computer Science Division
Argonne National Laboratory
Argonne, IL 60439-4844
E-mail: kwong@mcs.anl.gov

Abstract

We study the one-dimensional system of Ginzburg-Landau equations that models a thin film of superconductor subjected to a tangential magnetic field. We prove that the bifurcation curve for the symmetric problem is the graph of a continuous function of the supremum of the order parameter. We also prove the existence of a critical magnetic field. In general, there is more than one positive solution to the symmetric boundary value problem. Our numerical experiments have shown cases with three solutions. It is still an open question whether only one of these corresponds to the physical solution that minimizes the Gibbs free energy. We establish uniqueness for a related boundary value problem.

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1 Introduction

The name Ginzburg-Landau has been associated with more than one system of differential equations in more than one area of applied mathematics, including the theory of hydrodynamics, liquid crystals, superconductivity, and harmonic maps. Although the various systems have some similarities in appearance, the actual structure of the equations and their theory are sufficiently different to warrant independent studies. In this paper, we investigate the Ginzburg-Landau system that models a superconducting thin film. Many good monographs exist on the theory of superconductivity [1], [2], [15], and several recent survey articles [4], [7], [8] have been written with mathematicians in mind; these should be consulted for further details on the physics, experimental observations, alternative models, and additional references. We describe here only those concepts that are necessary to make our mathematical problem understandable.

Superconductivity was discovered in 1911 by Kammerlingh Onnes, who observed that when certain material was cooled below some critical temperature T_c (characteristic of the material), the material abruptly lost its electric resistivity and could conduct electric current without any loss of energy. Physicists have studied the behavior of a superconductor when it is subjected to a magnetic field (referred to as the external magnetic field to distinguish it from the internal field measured at a point inside the material). One of the early findings is that when the magnetic field is sufficiently high, a superconductor will lose its superconductivity and revert back to a normal conductor. The least magnetic field for which this occurs depends on the temperature at which the experiment is conducted, and is called the critical field $H_c(T)$. Measurements show that the graph of $H_c(T)$ is close to a parabola with vertex on $T = 0$ and horizontal intersect T_c.

Physicists explain the loss of resistivity below T_c as a phase transition, conceptually similar to the change of a liquid into a solid upon cooling, although the transition is at a subatomic level and no alteration in outward appearance is discernible. It was Ginzburg and Landau's ingenious idea to apply the theory of phase transition to explain the onset of superconductivity. A crucial step is in postulating the form of the Gibbs free energy to be minimized. For our purpose, it suffices to say that the electro-magnetic properties of the superconductor are completely described by two quantities, the order parameter $\phi(\mathbf{x})$ and the vector potential $\mathbf{A}(\mathbf{x})$, defined in the three-dimensional region Ω occupied by the material. The former is a complex-valued scalar function $\phi : \Omega \to \mathbf{C}$ (analogous to the wave function in quantum theory), and the latter is a real-valued three-dimensional vector $\mathbf{A} : \Omega \to \mathbf{R}^3$. The Ginzburg-Landau theory hypothesizes that the pair $(\phi, \mathbf{A})$ seeks to minimize the Gibbs energy functional

$$\mathcal{G} = \int_\Omega \left(-|\phi|^2 + \frac{1}{2}|\phi|^4 + \left| \left(\frac{\nabla}{\kappa^2} - i\mathbf{A} \right) \phi \right|^2 + |\nabla \times \mathbf{A} - \mathbf{H}|^2 \right) d\Omega, \qquad (1.1)$$

where κ is a characteristic constant of the material called its Ginzburg-Landau parameter, $i = \sqrt{-1}$, and $\mathbf{H}$ is the external magnetic field. It appears as if the temperature T does not play a role in the energy functional. This is not true. The formula for the Gibbs free energy in terms of known physical quantities is rather complicated. It is only after some suitable scaling (with scaling constants depending on T) of the quantities ϕ and $\mathbf{A}$ that the formula reduces to the simpler form (1.1). Thus, after the ϕ and $\mathbf{A}$ are solved by minimizing (1.1), they should be scaled back to give the actual physical values representing the system. The final answers will then contain the temperature T. We also point out that the Ginzburg-Landau equations are believed to be valid only for ideal superconductors and at temperatures near T_c.

No measurable quantities actually correspond directly to ϕ and $\mathbf{A}$; rather, physical quantities are given by values derived from ϕ and $\mathbf{A}$. For instance, the density of superelectrons is $|\phi|^2$ and the superconducting current is $\nabla \times \nabla \times \mathbf{A}$. It can happen that two distinct pairs $(\phi_1, \mathbf{A}_1)$ and $(\phi_2, \mathbf{A}_1)$ give identical answers when used to compute these measurable quantities. More precisely, $(\phi_1, \mathbf{A}_1)$ and $(\phi_2, \mathbf{A}_1)$ are said to be gauge equivalent if there exists a scalar function $\chi(\mathbf{x})$ such that

$$\phi_1 = \phi_2 e^{i\chi(x)}, \qquad (1.2)$$

$$\mathbf{A}_1 = \mathbf{A}_2 + \nabla\chi(x). \qquad (1.3)$$

The correspondence that gives $(\phi_1, \mathbf{A}_1)$ from $(\phi_2, \mathbf{A}_2)$ is called a gauge transform. All gauge equivalent pairs represent the same physical state. In particular, they all give the same Gibbs free energy when substituted into (1.1).

Two- and three-dimensional Ginzburg-Landau systems have very rich structures. The existence of vortex solutions, first shown by Abrikosov, led to the discovery of Type II superconductors (we refer the reader to any of the books and articles cited above).

The special case of a superconducting thin film is idealized by taking Ω to be an unbounded slab of thickness $2l$, $\Omega = \{(x,y,z) : -l < x < l\}$. When the external magnetic field $\mathbf{H}$ is parallel (also said to be *tangential*) to the film, one assumes that only the component of $\mathbf{A}$ parallel to $\mathbf{H}$ is significant and that both ϕ and $\mathbf{A}$ are uniform in the y and z directions. In reality, for a given external magnetic field $\mathbf{H}$ and Ginzburg-Landau constant κ, these assumptions are approximately valid only when the thickness is sufficiently small. The ensuing mathematics problem, however, is well defined for all positive values of the parameters. In addition, a suitable gauge can be chosen to reduce ϕ to a real-valued function. To summarize, in the one-dimensional case, the electromagnetic state of the superconducting film is described by a pair of scalar functions $(\phi(x), \alpha(x))$ that minimizes the energy functional

$$\mathcal{G} = \int_{-l}^{l} \left(\frac{a^2\phi^2}{2} - \phi^2 + \frac{1}{2}\phi^4 + \frac{\phi'^2}{\kappa^2} + (a' - H)^2 \right) dx. \tag{1.4}$$

In the differential equation approach, the minimizer of the energy functional satisfies the following Ginzburg-Landau system of equations,

$$\begin{cases} \phi(x)'' = \kappa^2 \left(\phi(x)^2 + a(x)^2 - 1\right) \phi(x), & x \in (-l, l) \\ a''(x) = \phi^2(x) a(x), \end{cases} \tag{1.5}$$

with the natural boundary conditions

$$\phi'(\pm l) = 0, \quad a'(\pm l) = H. \tag{1.6}$$

A solution of the Ginzburg-Landau system, on the other hand, need not be a minimizer of the energy functional. A symmetric solution of (1.5) is one that satisfies $\phi(-x) = \phi(x)$ and $a(-x) = -a(x)$ or, alternatively, the boundary conditions

$$\phi'(0) = \phi'(l) = a(0) = 0, a'(l) = H. \tag{1.7}$$

The one-dimensional problem has been studied by the authors [13], [14], [5], [16], [17], [19]. Odeh [14] gave an existence proof of the minimizer based on a variational approach. More recently, Yang [18] gave proofs of the existence and regularity of the solutions. For certain ranges of the parameters, the minimizer is the trivial solution $\phi(x) = 0$, which corresponds to the normal (nonsuperconducting) state of the material. The more interesting case is, of course, when the minimizer is nontrivial. Wang and Yang [16] showed the existence of a minimizer in the class of symmetric solutions and derived some useful properties of a nontrivial symmetric minimizer, among them the fact that ϕ is positive and monotonically decreasing in $[0,1]$ and a is positive and monotonically increasing in $[0,1]$.

The question of uniqueness remains open. One approach to tackle the problem is to study the bifurcation curve. One or more of the three parameters H, l, and κ are varied and the corresponding boundary value problem (BVP) is solved. Some characteristic value of the solution, usually $\max\phi$, is then plotted against the parameter(s) to obtain the bifurcation curve. In Section 2, we show that the bifurcation curve for the symmetric solution, relating $\phi(0)$ to H, is the graph of a continuous function.

If the Ginzburg-Landau system has a unique solution, then the energy functional has a unique minimizer, which must then also be symmetric. Unfortunately, numerical experiments indicate that, in general, a solution to the system is not unique and not even necessarily symmetric. A study of the asymptotic behavior and bifurcation of the solutions of the Ginzburg-Landau system can be found in [3]. We have discovered through numerical experiments examples in which there can be three nontrivial symmetric solutions. There is strong evidence, but still unproven, that for any given κ, uniqueness prevails when l is sufficiently small.

Another interesting problem is the existence of the upper critical field. In Section 4 of [16], Wang and Yang mentioned that "it seems impossible to achieve a sharp verification of ... [the existence of] a finite critical [magnetic field]." They managed to show the weaker result that as $H \to \infty$, the corresponding $\phi \to 0$ uniformly. We give a rigorous proof of the existence of the critical field in Section 3.

We return to the uniqueness question in Section 4. Even for a scalar field equation, uniqueness can be difficult to show. A method first used by Coffman has recently been applied to resolve some long-standing conjectures involving semilinear elliptic equations; see [10], [11], [12]. In the case of systems of equations, very few uniqueness results are known, except when the energy functionals are convex, leading to equations of sublinear type. In Section 4 we use the Kolodner-Coffman method to obtain a uniqueness result for a boundary value problem of the Ginzburg-Landau system with fixed-end boundary conditions for ϕ that is decreasing.

2 The Symmetric BVP: Monotonic Shooting and the Bifurcation Curve

We scale the symmetric Ginzburg-Landau equations (1.5) to fit into the unit interval $[0, 1]$ instead of $[0, l]$, and we use the new constants

$$K = \kappa^2 l^2, \quad L = l^2, \quad h = lH, \tag{2.1}$$

to obtain the system

$$\begin{cases} \phi(x)'' = K\left(\phi(x)^2 + a(x)^2 - 1\right)\phi(x), & x \in (0,1) \\ a''(x) = L\phi^2(x)a(x), \end{cases} \tag{2.2}$$

with boundary conditions

$$\phi'(0) = \phi'(1) = a(0) = 0, \quad a'(1) = h. \tag{2.3}$$

The system always has a trivial solution, $\phi(x) = 0$ and $a(x) = hx$. As shown in [16], the physically interesting solutions (namely, the minimizer of the energy functional), if nontrivial, must be positive and monotonic in $[0, 1]$. The maximum value of ϕ is $\phi(0)$.

We assume that K and L are given, and we solve the boundary value problem for varying h. The set of pairs $(\phi(0), h)$ forms the bifurcation curve of the system. We present a monotone shooting method to solve the boundary value problem.

Let $\beta < 1$ and α be given positive numbers. We solve (2.2) as an initial value problem with the initial values

$$\phi(0) = \beta, \quad \phi'(0) = a(0) = 0, \quad a'(0) = \alpha, \tag{2.4}$$

and denote the solution as

$$\phi(x; \alpha, \beta) \quad \text{and} \quad a(x; \alpha, \beta) \tag{2.5}$$

to emphasize the dependence on the initial values. We can no longer guarantee that $\phi(x; \alpha, \beta)$ remains positive in $[0, 1]$, nor can we guarantee that ϕ and a remain finite for all $x \in [0, 1]$.

The fact that the right-hand sides of the equations in (2.2) have coefficients that are increasing functions of ϕ and a (when both are positive) yields a useful comparison result. We need the following form of the classical Sturm comparison theorem in the proof.

Sturm Comparison Theorem. *Suppose that y and Y are solutions of the second order differential equations*

$$y'' = q(x)y, \quad Y'' = Q(x)Y, \quad x \in (c, d), \tag{2.6}$$

respectively, and that the following comparison conditions hold:

$$q(x) \leq Q(x), \quad q(x) \not\equiv Q(x), \quad \frac{y'(c)}{y'(c)} \leq \frac{Y'(c)}{Y(c)}. \tag{2.7}$$

Then

$$\frac{y'(d)}{y'(d)} < \frac{Y'(d)}{Y(d)}. \tag{2.8}$$

As a consequence, y oscillates strictly more (so y bends downward faster) than Y in (c, d). If, furthermore,

$$y(c) \leq Y(c), \tag{2.9}$$

then

$$y(d) < Y(d) \quad \textit{and} \quad y'(d) < Y'(d). \tag{2.10}$$

Lemma 1 *For fixed x, the values $\phi(x; \alpha, \beta)$, $\phi'(x; \alpha, \beta)$, $a(x; \alpha, \beta)$, and $a'(x; \alpha, \beta)$ are strictly increasing in α and β, as long as the values remain positive and finite.*

Proof. Suppose that $\alpha < \overline{\alpha}$ and $\beta < \overline{\beta}$. For simplicity, we write $\phi(x) = \phi(x; \alpha, \beta)$ and $\overline{\phi}(x) = \phi(x; \overline{\alpha}, \overline{\beta})$, with similar notations for $a(x)$. Then

$$\phi(x) < \overline{\phi}(x) \quad \text{and} \quad a(x) < \overline{a}(x) \tag{2.11}$$

for $x > 0$ and x sufficiently near 0. If the inequalities remain true for all x, then we are basically done. Let us suppose the contrary, namely, that there is some $d < 1$, such that (2.11) holds for all $x \in (0, d)$ but that either $\phi(d) = \overline{\phi}(d)$ or $a(d) = \overline{a}(d)$. In $(0, d)$, the right-hand sides of (2.2) for ϕ and a have smaller coefficients than those for $\overline{\phi}$ and $\overline{a}$. By the Sturm comparison theorem, ϕ and a oscillate strictly more than $\overline{\phi}$ and $\overline{a}$, respectively. Then

$\phi(d) < \overline{\phi}(d)$ and $a(d) < \overline{a}(d)$, contradicting our assumption. Hence (2.11) must hold for all x. That the same inequalities hold between the derivative of the solutions is the last assertion in the conclusion of the Sturm comparison theorem. ∎

Lemma 2 *For any $0 < \beta < 1$, there exists one and only one $\alpha = \alpha(\beta)$ such that $(\phi(x; \alpha(\beta), \beta), a(x, \alpha(\beta), \beta))$ is a solution to the Ginzburg-Landau boundary value problem (2.2)–(2.3).*

Proof. Uniqueness is a consequence of Lemma 1. Existence can be proved by a shooting argument. If we choose $\alpha = 0$, then $\phi(x; 0, \beta)$ is decreasing in x, so either ϕ crosses the x axis before reaching 1 or $\phi(1; 0, \beta) < 0$. If we choose α sufficiently large, then $\phi(x; \alpha, \beta)$ will increase very rapidly after an initial dip; ϕ will either blow up at a finite point or $\phi'(1; \alpha, \beta)$ will be greater than 0. A continuity argument then gives an intermediate α such that $\phi'(1; \alpha, \beta) = 0$, and we have a solution to the boundary value problem. ∎

Lemma 3 *The correspondence $\alpha(\beta) : \beta \mapsto \alpha$ asserted in Lemma 2 is a continuous decreasing function.*

Proof. Monotonicity is a consequence of Lemma 1. Continuity follows if we can show that the range of the function is onto an interval. To this end, let $\alpha_1 = \alpha(\beta_1) < \alpha_2 = \alpha(\beta_2)$ be two given images in the range, and let $\alpha_1 < \alpha_0 < \alpha_2$. We have to show that there exists a β_0 such that $\alpha(\beta_0) = \alpha_0$. By Lemma 1,

$$\phi'(1; \beta_1, \alpha_0) > \phi'(1; \beta_1, \alpha_1) = 0 \tag{2.12}$$

and

$$\phi'(1; \beta_2, \alpha_0) < \phi'(1; \beta_2, \alpha_1) = 0. \tag{2.13}$$

By continuity, some intermediate β_0 exists such that $\phi'(1; \beta_0, \alpha_0) = 0$. ∎

Theorem 1 *The bifurcation curve of our boundary value problem is the union of the positive h axis (which represents the trivial solution) and the graph of the continuous composite function*

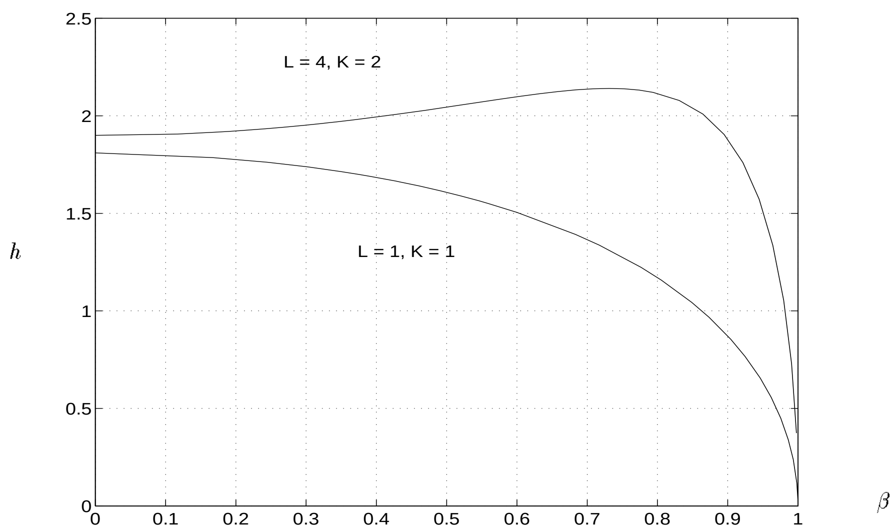
$$\beta \longmapsto \alpha(\beta) \longmapsto h(\beta) = a'(1; \alpha(\beta), \beta). \tag{2.14}$$

All curves start from the h axis with zero slope and end at $\beta = 1$ on the β axis. The initial height of the curve at the h axis is the value λ for which the boundary value problem

$$u'' = K(h^2x^2 - 1)u, \quad u'(0) = u'(1) = 0 \tag{2.15}$$

has a positive solution.

Proof. The behavior of the bifurcation curve near $\phi = 0$ can be examined using classical asymptotic analysis; we omit the details. We merely point out the heuristic arguments that as $\phi \to 0$, the second equation in (2.2) degenerates to $a'' = 0$. The limiting solution a is thus a linear function of x, and the first equation in (2.2) degenerates to (2.15). The boundary conditions in (2.15) are derived from the boundary conditions on ϕ. ∎



Figire 1. Typical bifurcation curves

Even though $\alpha(\beta)$ is monotone, $h(\beta)$ need not be so. Indeed, the boundary value problem has a unique solution if and only if $h(\beta)$ is a strictly decreasing function of β. We implemented the monotone shooting method in MATLAB. In our program we use α as the shooting parameter and adjust β to satisfy the boundary conditions. Two typical bifurcation curves are shown in Figure 1. The differential equations are solved by using a fourth-fifth order Runge-Kutta method with error bound estimation. We used the error bound of 10^{-10} for most of our experiments and even smaller bounds if there is a need for higher precision.

The vertical axis is h, and the horizontal axis is β. The lower curve is typical for L small. It is monotone, and uniqueness for the boundary value problem prevails. The upper curve is representative for L large. Whenever a horizontal line intersects the curve at more than one point, the boundary value problem has multiple solutions. For a while, we conjectured that all cases of nonuniqueness occur with a unimodal bifurcation curve. More extensive experiments turned up the example $L = 4, K = 3.6$, in which the bifurcation curve first decreases and then increases to a global maximum before it decreases again to the point $\beta = 1, h = 0$. Table 1 lists the results of our numerical computation. The first column gives β and the second column the corresponding h. Similar behavior was observed by varying K in the range 3.4 to 4.1, while keeping $L = 4$.

Table 1. Results with $L = 4$, $K = 3.4 - 4.1$

β	h
0.86920073040461	2.02892378533588
0.83936182769457	2.06803427304492
0.80640879787522	2.08883938928138
0.77018919995251	2.09700047341807
0.73046894383778	2.09721411626507
0.68689914935239	2.09322143483159
0.63895895150244	2.08788602769435
0.58585181632874	2.08330544961489
0.52630220502554	2.08093445344534
0.45810552909473	2.08170706592965
0.37692312413923	2.08614998905450
0.27174054997177	2.09448313906868
0.07322708691641	2.10670529625534

3 Existence of a Critical Magnetic Field

The critical magnetic field of (2.2) is defined to be the smallest value h_c such that the only solution of the boundary value problem is the trivial solution.

In the case of the symmetric boundary value problem, the existence of the critical field is a simple consequence of the continuity of h as a function of β. In fact, h_c is the supremum of $h(\beta)$ over $[0,1]$, since for $h > h_c$, the only point on the bifurcation curve with height h lies on the h axis and corresponds to the trivial solution. This proof, however, is not applicable to the full-range Ginzburg-Landau boundary value problem, since the continuity of h on ϕ has not yet been proved. We give below a different proof of the existence of h_c that is applicable in general. The idea of the proof is to show that when h is sufficiently large, then a must be sufficiently large outside of a neighborhood of $x = 0$. Thus the coefficient of the right-hand side of the first equation must be very large, outside of a neighborhood of $x = 0$. The Sturm comparison theorem can then be used to conclude that ϕ cannot oscillate fast enough to satisfy the endpoint conditions at $x = 1$.

Theorem 2 *For given K and L, there exists a critical magnetic filed h_c such that for all $h > h_c$, the Ginzburg-Landau boundary value problem has only the trivial solution.*

Proof. We give the proof only for the symmetric problem. It can easily be generalized to the full-range problem. Since $\phi(x) \leq 1$, the second equation in (2.2) gives the differential inequality

$$a''(x) \leq La(x). \tag{3.1}$$

Solving this inequality with the given boundary conditions on a gives

$$a(x) \geq \frac{h \sinh(\sqrt{L}x)}{\sinh(\sqrt{L})}. \tag{3.2}$$

When h is sufficiently large, the above lower bound of $a(x)$ can be made arbitrarily large in $[1/2, 1]$. We thus have

$$a(x) \geq \begin{cases} 0, & \text{in } [0, 1/2] \\ k, & \text{in } [1/2, 1] \end{cases}, \tag{3.3}$$

with $k \to \infty$ as $h \to \infty$. The coefficient on the right-hand side of the first equation in (2.2) satisfies the inequality

$$(a^2 + \phi^2 - 1) \geq q(x) = \begin{cases} -1, & \text{in } [0, 1/2] \\ k^2 - 1, & \text{in } [1/2, 1] \end{cases}. \tag{3.4}$$

Using the Sturm comparison theorem, we conclude that ϕ oscillates less than the solution of the differential equation

$$u''(x) = q(x)u(x), \quad u(0) = \phi(0), u'(0) = 0. \tag{3.5}$$

In particular, if ϕ is nontrivial, $\phi'(1) > u'(1)$. Direct computation shows that $u'(1) > 0$ for k sufficiently large. Hence, ϕ cannot satisfy the boundary condition $\phi'(1) = 0$. We have thus proved that ϕ cannot be nontrivial. ∎

4 Uniqueness of a Related BVP

The uniqueness of the minimizer of the Gibbs energy functional for both the full-range and symmetric Ginzburg-Landau system proves to be an elusive conjecture. The uniqueness of the positive solution of the corresponding Ginzburg-Landau equations is not even true in general. It is thus surprising to be able to obtain uniqueness for a related problem. As far as we know, our result is the first application of the Kolodner-Coffman method to a system of equations.

We consider the Ginzburg-Landau equations (2.2) subject to the boundary conditions

$$\phi(0) = \beta_0,\ \phi(1) = \beta_1 \text{ are given}, \quad a(0) = 0, a'(1) = h \tag{4.1}$$

and the condition (we confine ourselves to positive decreasing ϕ)

$$\phi(x) > 0, \quad \phi'(x) < 0. \tag{4.2}$$

Theorem 3 *The Ginzburg-Landau boundary value problem (2.2) subject to (4.1) and (4.2) has at most one solution.*

One can solve this boundary value problem with a monotone shooting method similar to the one described in Section 2 for the symmetric problem. Instead of using the initial height of ϕ as the shooting parameter, we use the initial slope $\phi'(0) = \gamma$. One can easily prove a comparison result similar to Lemma 1, using α and γ instead of α and β. To solve the boundary value problem, one shoots out a solution ϕ with initial height β_0 and some chosen initial slope γ. The initial slope α of a is then adjusted so that $\phi(1)$ hits the target height β_1. This defines a function that maps γ to α and then to $h(\gamma) = a'(1; \alpha, \gamma)$. Uniqueness will hold if the correspondence $\gamma \mapsto h$ is a monotonic function.

Suppose we already have a solution to our boundary value problem. Following the Kolodner-Coffman method, we define

$$v(x) = \frac{\partial \phi(x)}{\partial \gamma}, \quad w(x) = \frac{\partial a(x)}{\partial \gamma}. \tag{4.3}$$

Uniqueness follows if we can show that

$$w'(1) < 0. \tag{4.4}$$

To this end, we investigate the differential equations satisfied by v and w, obtained by differentiating (2.2) with respect to γ:

$$\begin{cases} v'' = K\left(a^2 + 3\phi^2 - 1\right) v + 2Ka\phi w \\ w'' = 2La\phi v + L\phi^2 w, \end{cases} \tag{4.5}$$

$$v(0) = v(1) = 0, \quad v'(0) = 1, \quad w(0) = 0. \tag{4.6}$$

The condition $v'(0) = 1$ implies that $v(x)$ is positive in some neighborhood of $x = 0$.

We regard ϕ and a as known functions; then (4.5) is a system of linear equations in v and w. The system (4.5), in fact, has another solution. It is easy to verify that

$$\overline{v}(x) = \phi'(x), \quad \overline{w}(x) = a'(x) \tag{4.7}$$

satisfy (4.5). Note the following properties of $\overline{v}$ and $\overline{w}$.

Lemma 4

$$\overline{v}(x) < 0, \quad \overline{w}(x) > 0, \quad x \in [0, 1], \tag{4.8}$$

$$\overline{w}'(1) > 0. \tag{4.9}$$

For any positive constant c, the functions $\hat{v} = \overline{v} + cv$ and $\hat{v} = \overline{w} + cw$ are solutions of (4.5).

Now suppose that (4.4) is not true.

Lemma 5 *If $w'(1) \geq 0$, then there exists a c greater than 0 such that*

$$\hat{v}(x) \leq 0, \quad \hat{w}(x) \geq 0, \quad x \in [0,1], \tag{4.10}$$

and one of the two functions $\hat{v}$ and $\hat{w}$ touches the x axis tangentially at an interior point in $(0,1)$.

Proof. Since v and $\overline{v}$ have opposite signs for x near 0, $\hat{v}$ cannot remain negative for all positive c. Let c_1 be the critical value after which $\overline{v}$ is no longer always negative. If $\hat{w}(x) = \overline{w}(x) + c_1 w(x) \geq 0$ for all x, then c_1 is the choice of c required in the lemma. Otherwise, let c_2 be the critical value after which $\overline{w}$ is no longer always positive. Then c_2 is the choice of c if we can show that $\hat{w}(x) = \overline{w}(x) + c_2 w(x)$ does not cross the x axis at $x = 1$. This follows from the fact that $\hat{w}'(1) = \overline{w}'(1) + c_2 w'(1) > 0$. ∎

We can now derive a contradiction, to complete the proof of Theorem 3. First we see that $\hat{v}$ and $\hat{w}$ cannot be tangential to the x axis at the same point, because, by the uniqueness theorem for linear systems of equations, the only solution for which both functions are tangential to the x axis at the same point is the trivial solution. Suppose $\hat{v}$ is tangential to the x axis at $x = \sigma$. Then $\hat{w}(\sigma) > 0$. The point σ is a local maximum of $\hat{v}$, so $\hat{v}''(\sigma) \leq 0$. However, the first equation in (4.5) gives $\hat{v}''(\sigma) = 2Ka(\sigma)\phi(\sigma)\hat{w}(\sigma) > 0$, a contradiction. A similar contradiction can be obtained for the case when $\hat{w}$ touches the x axis at a point σ, by using the second equation in (4.5) and the fact that $\hat{w}$ has a local minimum at σ.

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