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Gauges for the Ginzburg-Landau Equations of Superconductivity

This note discusses some gauge choices for the time-dependent Ginzburg-Landau *equations of superconductivity. The equations model the state of a superconducting sample in a magnetic field near the critical temperature. Any two solutions related through a "gauge transformation" describe the same state and are physically indistinguishable. This "gauge invariance" can be exploited for analytical and numerical purposes.*

1. The Ginzburg-Landau Model of Superconductivity

In the Ginzburg-Landau theory of phase transitions [1], the state of a superconducting material near the critical temperature is described by a complex-valued *order parameter* ψ, a real-valued *vector potential* $\mathbf{A}$, and, when the state changes with time, a real-valued *scalar potential* ϕ. The role of ϕ differs from that of ψ and $\mathbf{A}$: the latter are predictive variables, whose evolution is governed by diffferential equations; the former is more like a Lagrange multiplier. After suitable nondimensionalization, the equations and boundary conditions satisfied by ψ and $\mathbf{A}$ are

$$\eta\left(\frac{\partial}{\partial t}+i\kappa\phi\right)\psi = -\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)^2\psi+\left(1-|\psi|^2\right)\psi \quad \text{in } \Omega\times(0,\infty), \tag{1}$$

$$\frac{\partial\mathbf{A}}{\partial t}+\nabla\phi = -\nabla\times\nabla\times\mathbf{A}+\mathbf{J}_s+\nabla\times\mathbf{H} \quad \text{in } \Omega\times(0,\infty), \tag{2}$$

$$\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\cdot\mathbf{n} = -\frac{i}{\kappa}\gamma\psi \quad \text{on } \partial\Omega\times(0,\infty), \tag{3}$$

$$(\nabla\times\mathbf{A}-\mathbf{H})\times\mathbf{n} = \mathbf{0} \quad \text{on } \partial\Omega\times(0,\infty). \tag{4}$$

Here,

$$\mathbf{J}_s=\frac{1}{2i\kappa}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)-|\psi|^2\mathbf{A}=-\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right]. \tag{5}$$

The domain Ω corresponds to the region occupied by the superconducting material. We assume that it is bounded, $\Omega\subset\mathbb{R}^D$ with $D=2$ or $D=3$, and that its boundary $\partial\Omega$ is sufficiently regular; $\mathbf{n}$ denotes the outer unit normal to $\partial\Omega$. As usual, $\nabla\equiv$ grad, $\nabla\times\equiv$ curl, $\nabla\cdot\equiv$ div, and $\nabla^2=\nabla\cdot\nabla\equiv\Delta$. Furthermore, i is the imaginary unit, and a superscript * denotes complex conjugation.

The parameters of the model are η, a (dimensionless) friction coefficient; κ, the (dimensionless) Ginzburg-Landau parameter; and γ, a nonnegative parameter, which is zero if the superconducting material is surrounded by vacuum. The vector field $\mathbf{H}$ is a given *applied magnetic field*; in practice, $\mathbf{H}$ is either time-independent or time-periodic.

The quantity $\mathbf{J}_s$ is the so-called *supercurrent* or, more correctly, supercurrent density. The supercurrent is a phenomenological quantity, which is thought of as a flux of moving "superelectrons." The superelectrons (or Cooper pairs), whose density is $n_s=|\psi|^2$, are responsible for the superconducting properties of the material. For example, the supercurrent prevents a magnetic field from penetrating a superconducting region.

Note that $\mathbf{E}=-\partial\mathbf{A}/\partial t-\nabla\phi$ is the *electric field* and $\mathbf{B}=\nabla\times\mathbf{A}$ the *magnetic induction*. Therefore, Eq. (2) may be viewed as Ampère's law, $\nabla\times\mathbf{B}=\mathbf{J}$, where the total current $\mathbf{J}$ is the sum of a "normal" current $\mathbf{J}_n=\mathbf{E}$, the supercurrent $\mathbf{J}_s$, and the transport current $\mathbf{J}_t=\nabla\times\mathbf{H}$. The normal current obeys Ohm's law; the "normal conductivity" coefficient is equal to one in the chosen system of units.

The system of Eqs. (1)–(5), with appropriate initial conditions, constitutes the *time-dependent* Ginzburg-Landau (TDGL) *model of superconductivity*. It was first proposed by Schmid [2] and subsequently obtained as an asymptotic limit of the microscopic Bardeen-Cooper-Schrieffer (BCS) model of superconductivity by Gor'kov and Eliashberg [3]. Details can be found in the physics literature; standard references are Abrikosov [4], De-Gennes [5], and Tinkham [6]. The time-independent trivial solution ($\psi=0$, $\nabla\times\mathbf{A}=\mathbf{H}$, ϕ constant) represents the superconductor in the *normal state*, where the magnetic field penetrates the sample uniformly and the material has lost all superconducting properties.

2. Gauge Invariance

Equations (1) and (2) require initial conditions for the order parameter and the vector potential. Here, the concept of *gauge invariance* enters. Because the physical state of the system at $t = 0$ is completely determined by the magnetic induction $\mathbf{B}$, the superelectron density n_s, and the supercurrent $\mathbf{J}_s$, there is a significant degree of freedom in the choice of initial data for ψ and $\mathbf{A}$. In fact, if the pair $(\psi_0, \mathbf{A}_0)$ properly specifies the physical initial state, then so does any other pair $(\psi_0', \mathbf{A}_0')$ that is related to $(\psi_0, \mathbf{A}_0)$ by a transformation

$$\mathcal{G}_{\chi_0} : (\psi_0, \mathbf{A}_0) \mapsto (\psi_0', \mathbf{A}_0') = \left(\psi_0 e^{i\kappa\chi_0}, \mathbf{A}_0 + \nabla\chi_0\right). \tag{6}$$

Here, χ_0 can be any (sufficiently smooth) real-valued function of position. Equation (6) is the *gauge transformation* for the stationary GINZBURG-LANDAU model.

There is a similar, though more complicated, gauge transformation for the time-dependent GINZBURG-LANDAU model,

$$\mathcal{G}_{\chi} : (\psi, \mathbf{A}, \phi) \mapsto (\psi', \mathbf{A}', \phi') = \left(\psi e^{i\kappa\chi}, \mathbf{A} + \nabla\chi, \phi - \frac{\partial\chi}{\partial t}\right). \tag{7}$$

Here, χ can be any (sufficiently smooth) real-valued function of position and time. Mathematically, gauge invariance reflects a lack of uniqueness. The TDGL model defines only an equivalence class of solutions, and by choosing a particular gauge χ we select a representative from this class. The physical relevance of gauge invariance is the following. At each instant, the macroscopic state of the superconductor is entirely specified in terms of the electromagnetic variables $\mathbf{E}$ and $\mathbf{B}$, the superelectron density n_s, and the supercurrent $\mathbf{J}_s$. These quantities are invariant under the gauge transformation (7), so the states $(\psi, \mathbf{A}, \phi)$ and $(\psi', \mathbf{A}', \phi')$ are macroscopically indistinguishable. The choice of a particular gauge χ does not affect the specification of the physical state of the system.

3. Gauge Choices

The simplest gauge is the "zero-electric potential gauge," where the scalar potential is eliminated altogether. (The scalar ϕ is also known as the *electric potential.*) Given the solution $(\psi, \mathbf{A}, \phi)$ of the TDGL model at any time $t > 0$, one obtains the appropriate gauge χ at t by integrating the initial-value problem

$$\frac{\partial\chi}{\partial t} = \phi \ \text{ in } \Omega \times (0, \infty), \quad \chi|_{t=0} = \chi_0 \ \text{ in } \Omega, \tag{8}$$

from 0 to t, starting from arbitrary initial data χ_0. Then one changes to the image $(\psi', \mathbf{A}', \phi')$ of $(\psi, \mathbf{A}, \phi)$ under the gauge transformation $\mathcal{G}_\chi$. As a result, $\phi' = 0$. In the zero-electric potential gauge, Eqs. (1) and (2) reduce to

$$\eta\frac{\partial\psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi + \left(1 - |\psi|^2\right)\psi \quad \text{in } \Omega \times (0, \infty), \tag{9}$$

$$\frac{\partial\mathbf{A}}{\partial t} = -\nabla \times \nabla \times \mathbf{A} + \mathbf{J}_s + \nabla \times \mathbf{H} \quad \text{in } \Omega \times (0, \infty), \tag{10}$$

The boundary conditions (3) and (4) remain the same. The zero-electric potential gauge was used by DU [7] to prove the existence and uniqueness of strong solutions of the TDGL model for bounded initial data. It is also the gauge adopted in the kernel of the TDGL code used at ARGONNE [8, 9] for the numerical simulation of vortex dynamics in type-II superconductors.

Another gauge, popular for analytical purposes, is the "COULOMB gauge," where the vector potential is divergence-free at all times. If $(\psi_0, \mathbf{A}_0)$ are the initial data given with Eqs. (1) and (2), one determines the initial gauge χ_0 by solving the boundary value problem

$$\Delta\chi_0 = -\nabla \cdot \mathbf{A}_0 \ \text{ in } \Omega, \quad \nabla\chi_0 \cdot \mathbf{n} = -\mathbf{A}_0 \cdot \mathbf{n} \ \text{ on } \partial\Omega, \tag{11}$$

and changes to the image $(\psi_0', \mathbf{A}_0')$ of $(\psi_0, \mathbf{A}_0)$ under the gauge transformation $\mathcal{G}_{\chi_0}$. At any subsequent time $t > 0$, one takes the solution $(\psi, \mathbf{A}, \phi)$ of the TDGL model, solves the boundary value problem

$$\Delta\chi = -\nabla \cdot \mathbf{A} \ \text{ in } \Omega, \quad \nabla\chi \cdot \mathbf{n} = -\mathbf{A} \cdot \mathbf{n} \ \text{ on } \partial\Omega, \tag{12}$$

and changes to the image $(\psi', \mathbf{A}', \phi')$ of $(\psi, \mathbf{A}, \phi)$ under the gauge transformation $\mathcal{G}_\chi$. As a result, $\mathbf{A}_0'$ and $\mathbf{A}'$ are divergence-free. The procedure amounts effectively to integrating Eqs. (1) and (2), together with the boundary value

problem

$$\Delta\phi = \nabla\cdot\mathbf{J}_s \text{ in } \Omega\times(0,\infty), \quad \nabla\phi\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\times(0,\infty), \tag{13}$$

always subject to the constraint $\nabla\cdot\mathbf{A} = 0$ in Ω and the boundary conditions

$$\nabla\psi\cdot\mathbf{n} = -\gamma\psi, \quad \mathbf{A}\cdot\mathbf{n} = 0, \quad (\nabla\times\mathbf{A}-\mathbf{H})\times\mathbf{n} = \mathbf{0}, \tag{14}$$

on $\partial\Omega$. Of course, the initial data $(\psi_0, \mathbf{A}_0)$ must satisfy the constraint $\nabla\cdot\mathbf{A}_0 = 0$.

The equations of the TDGL model in the Coulomb gauge are formally similar to the Navier-Stokes equations for an incompressible fluid. This similarity was exploited in recent work by Tang and Wang [10], who proved the existence of strong solutions in two- and three-dimensional domains ($D = 2, 3$) and weak solutions in two-dimensional domains ($D = 2$), as well as the existence of a global attractor. Because the Coulomb gauge requires the solution of the elliptic boundary value problem (13) for the scalar potential at each time step, it is less suitable for numerical purposes.

It is not possible to have both the zero-electric potential gauge and the Coulomb gauge satisfied at all times. Even when the initial data are divergence-free, the solution obtained in the zero-electric potential gauge does not remain on the divergence-free manifold. To see this, take the divergence of Eq. (10): if div $\mathbf{A}$ is zero, it must be the case that div $\mathbf{J}_s = 0$. But from Eq. (5) we obtain the expression div $\mathbf{J}_s = (2i\kappa)^{-1}(\psi^*\partial_t\psi - \psi\partial_t\psi^*)$, which is zero if and only if $\psi = 0$ or the phase of ψ is constant in time.

It is, however, possible to couple ϕ to div $\mathbf{A}$, in such a way that, if one vanishes, then so does the other. The standard gauge for this purpose is the "$\phi = -$div $\mathbf{A}$ gauge," which maintains the identity $\phi + \text{div}\,\mathbf{A} = 0$ at all times. More generally, one can choose a gauge in which a linear identity of the type $\phi + \omega\,\text{div}\,\mathbf{A} = 0$ is maintained for some suitable constant ω, not necessarily equal to one. Such a gauge is obtained by taking for χ the solution of the boundary value problem

$$\frac{\partial\chi}{\partial t} - \omega\,\Delta\chi = \phi + \omega\,\text{div}\,\mathbf{A} \;\text{ in } \Omega\times(0,\infty), \quad \nabla\chi\cdot\mathbf{n} = -\mathbf{A}\cdot\mathbf{n} \;\text{ on } \partial\Omega\times(0,\infty). \tag{15}$$

The initial condition $\chi|_{t=0} = \chi_0$ can be chosen arbitrarily. Usually, χ_0 is taken so the initial data are divergence-free (cf. Du [7]), but this choice is in no way necessary. In fact, there is a distinct advantage in leaving χ_0 undetermined. The extra degree of freedom can be used, for example, to ensure that the vector potential becomes divergence free in the limit as $t\to\infty$.

In the $\phi = -\omega\,\text{div}\,\mathbf{A}$ gauge, Eqs. (1) and (2) reduce to

$$\eta\frac{\partial\psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2\psi + i\omega\eta\kappa\,(\text{div}\,\mathbf{A})\psi + \left(1-|\psi|^2\right)\psi \quad \text{in } \Omega\times(0,\infty), \tag{16}$$

$$\frac{\partial\mathbf{A}}{\partial t} = -\nabla\times\nabla\times\mathbf{A} + \omega\,\nabla(\text{div}\,\mathbf{A}) + \mathbf{J}_s + \nabla\times\mathbf{H} \quad \text{in } \Omega\times(0,\infty). \tag{17}$$

The boundary conditions are the same as in the Coulomb gauge, cf. Eq. (14). The choice of ω affects the way the divergence of $\mathbf{A}$ is accounted for. The effect is more readily appreciated if one writes Eqs. (16) and (17) in a slightly different form. Because of the identities

$$-\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 = \frac{1}{\kappa^2}\Delta - \frac{2i}{\kappa}\mathbf{A}\cdot\nabla - \frac{i}{\kappa}(\text{div}\,\mathbf{A}) - |\mathbf{A}|^2, \quad -\nabla\times\nabla\times\mathbf{A} = \Delta\mathbf{A} - \nabla(\text{div}\,\mathbf{A}), \tag{18}$$

Eqs. (16) and (17) are the same as

$$\eta\frac{\partial\psi}{\partial t} - \frac{1}{\kappa^2}\Delta\psi = -\frac{2i}{\kappa}\mathbf{A}\cdot\nabla\psi + \frac{i}{\kappa}(\omega\eta\kappa^2 - 1)(\text{div}\,\mathbf{A})\psi - |\mathbf{A}|^2\psi + (1-|\psi|^2)\psi \quad \text{in } \Omega\times(0,\infty), \tag{19}$$

$$\frac{\partial\mathbf{A}}{\partial t} - \Delta\mathbf{A} = (\omega-1)\nabla(\text{div}\,\mathbf{A}) + \mathbf{J}_s + \nabla\times\mathbf{H} \quad \text{in } \Omega\times(0,\infty). \tag{20}$$

These equations show two obvious choices for ω: $\omega = 1$ and $\omega = 1/(\eta\kappa^2)$. In the former case, div $\mathbf{A}$ contributes only to the evolution of the order parameter; in the latter, only to the evolution of the vector potential. But as Eqs. (19) and (20) show, mixed arrangements are also possible. Takáč [11] used the standard $\phi = -$div $\mathbf{A}$ gauge ($\omega = 1$) to show that the TDGL model generates a dynamical process in a Cartesian product of fractional Sobolev spaces. As we will show in a forthcoming article [12], the same methods apply in the more general setting proposed here.

The equations (16) and (17), combined withy the boundary conditions (14), are related to the gradient flow of an

energy functional. In fact, let the GINZBURG-LANDAU free energy be defined by the integral

$$E[\psi, \mathbf{A}] = \int_\Omega \left[\left| \left(\frac{i}{\kappa}\nabla + \mathbf{A} \right) \psi \right|^2 + \frac{1}{2}(1-|\psi|^2)^2 + \omega\,(\mathrm{div}\ \mathbf{A})^2 + |\nabla \times \mathbf{A} - \mathbf{H}|^2 \right] \mathrm{d}x + \int_{\partial\Omega} \gamma \left| \frac{i}{\kappa}\psi \right|^2 \mathrm{d}\tau(x). \quad (21)$$

The first variations of E with respect to ψ^* and $\mathbf{A}$ are

$$\frac{\delta E}{\delta \psi^*} = \left(\frac{i}{\kappa}\nabla + \mathbf{A} \right)^2 \psi - \left(1 - |\psi|^2\right)\psi, \quad \frac{\delta E}{\delta \mathbf{A}} = 2\left(\nabla \times \nabla \times \mathbf{A} - \omega\,\nabla(\mathrm{div}\ \mathbf{A}) - \mathbf{J}_s - \nabla \times \mathbf{H}\right). \quad (22)$$

Furthermore, if the constraint $\mathbf{A} \cdot \mathbf{n} = 0$ is imposed on $\partial\Omega$, the natural boundary conditions associated with E are

$$\nabla\psi \cdot \mathbf{n} = -\gamma\psi, \quad (\nabla \times \mathbf{A} - \mathbf{H}) \times \mathbf{n} = \mathbf{0}. \quad (23)$$

Therefore, the TDGL model in the $\phi = -\omega\ \mathrm{div}\ \mathbf{A}$ gauge can be summarized in the following form:

$$\eta \frac{\partial \psi}{\partial t} = -\frac{\delta E}{\delta \psi^*} + i\omega\eta\kappa(\mathrm{div}\ \mathbf{A})\psi, \quad \frac{\partial \mathbf{A}}{\partial t} = -\tfrac{1}{2}\frac{\delta E}{\delta \mathbf{A}}. \quad (24)$$

A detailed study of this dynamical system will appear in [12].

Observe that the modified energy functional in [7, Eq. (3.7)] is precisely the functional E introduced in (21), with $\omega = \epsilon$. Thus, the "modification," which was introduced for regularization purposes, amounted to a change from the zero-electric potential ($\phi = 0$) gauge to the $\phi = -\epsilon\,\mathrm{div}\ \mathbf{A}$ gauge.

In conclusion, we note that the $\phi = -\mathrm{div}\ \mathbf{A}$ gauge or its generalization proposed here appears to be the most natural gauge for the TDGL model, as every stationary solution $(\psi, \mathbf{A})$ automatically satisfies the COULOMB gauge $\mathrm{div}\ \mathbf{A} = 0$.

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