DYNAMICS OF THE GINZBURG-LANDAU EQUATIONS OF SUPERCONDUCTIVITY

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Abstract. This article is concerned with the dynamical properties of solutions of the time-dependent Ginzburg-Landau (TDGL) equations of superconductivity. It is shown that the TDGL equations define a dynamical process when the applied magnetic field varies with time and a dynamical system when the applied magnetic field is stationary. In the latter case, every solution of the TDGL equations is attracted to a set of stationary solutions, which are divergence free. These results are obtained in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge with $\omega > 0$, which generalizes the standard "$\phi = -\nabla \cdot \boldsymbol{A}$" gauge. The implications for the limiting case $\omega = 0$ (zero-electric potential gauge) are discussed.

Keywords. Ginzburg-Landau equations, superconductivity, gauge, weak solutions, global existence, uniqueness, dynamical process, global attractor.

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1 Introduction

In this article, we are concerned with the dynamical properties of solutions of the time-dependent Ginzburg-Landau (TDGL) equations of superconductivity. While the emphasis is on the formal mathematical aspects of the equations, we make every effort to comply with the physical nature of the problem. We make no simplifications for the convenience of mathematics, and our rigorous treatment is motivated by known facts from physics. We show that the TDGL equations define a dynamical process when the applied magnetic field varies with time and a dynamical system when the applied magnetic field is stationary. We work consistently in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge introduced in [1] and [2] and deduce by logical arguments the ramifications for the zero-electric potential gauge ($\phi = 0$). The "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge enables us to rigorously establish the large-time asymptotic behavior and make the connection with solutions of the time-independent GL equations of superconductivity.

1.1 Ginzburg-Landau Model of Superconductivity

In the Ginzburg-Landau theory of phase transitions [3], the state of a superconducting material near the critical temperature is described by a complex-valued *order parameter* ψ, a real vector-valued *vector potential* $\boldsymbol{A}$, and, when the system changes with time, a real-valued *scalar potential* ϕ. The latter is a diagnostic variable; ψ and $\boldsymbol{A}$ are prognostic variables, whose evolution is governed by a system of coupled differential equations,

$$\eta \left(\frac{\partial}{\partial t} + i\kappa\phi \right) \psi = - \left(\frac{i}{\kappa}\nabla + \boldsymbol{A} \right)^2 \psi + \left(1 - |\psi|^2 \right) \psi, \tag{1.1}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi = -\nabla \times \nabla \times \boldsymbol{A} + \boldsymbol{J}_s + \nabla \times \boldsymbol{H}. \tag{1.2}$$

The *supercurrent density* $\boldsymbol{J}_s$ is a nonlinear function of ψ and $\boldsymbol{A}$,

$$\boldsymbol{J}_s \equiv \boldsymbol{J}_s(\psi, \boldsymbol{A}) = \frac{1}{2i\kappa} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right) - |\psi|^2 \boldsymbol{A} = -\operatorname{Re}\left[\psi^* \left(\frac{i}{\kappa}\nabla + \boldsymbol{A} \right) \psi \right]. \tag{1.3}$$

The system of Eqs. (1.1)–(1.3) must be satisfied everywhere in Ω, the region occupied by the superconducting material, and at all times $t > 0$. The boundary conditions associated with the differential equations are

$$\boldsymbol{n} \cdot \left(\frac{i}{\kappa}\nabla + \boldsymbol{A} \right) \psi + \frac{i}{\kappa}\gamma\psi = 0 \quad \text{and} \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A} - \boldsymbol{H}) = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\partial\Omega$ is the boundary of Ω and $\boldsymbol{n}$ the local outer unit normal to $\partial\Omega$. They must be satisfied at all times $t > 0$. Henceforth, the term "TDGL equations" refers to the system of Eqs. (1.1)–(1.4).

We assume that Ω is a bounded domain in $\mathbf{R}^n$ with a boundary $\partial\Omega$ of class $C^{1,1}$. That is, Ω is an open and connected set whose boundary $\partial\Omega$ is a compact $(n-1)$-manifold described by Lipschitz-continuously differentiable charts. We consider two- and three-dimensional problems ($n = 2$ and $n = 3$, respectively). The vector potential $\boldsymbol{A}$ takes its values in $\mathbf{R}^n$. The vector $\boldsymbol{H}$ represents the (externally) *applied magnetic field*, which is a given function of space and time; like $\boldsymbol{A}$, it takes its values in $\mathbf{R}^n$. The function γ is defined and Lipschitz continuous on $\partial\Omega$, and $\gamma(x) \geq 0$ for $x \in \partial\Omega$. The parameters in the TDGL equations are η, a (dimensionless) friction coefficient, and κ, the (dimensionless) Ginzburg-Landau parameter. The former measures the temporal rate of change, the latter the spatial rate of change of the order parameter relative to the vector potential. As usual, $\nabla \equiv$ grad, $\nabla\times \equiv$ curl, $\nabla\cdot \equiv$ div, and $\nabla^2 = \nabla \cdot \nabla \equiv \Delta$; i is the imaginary unit, and a superscript * denotes complex conjugation. Sometimes, we use the symbol ∂_t to denote the partial derivative $\partial/\partial t$.

The order parameter can be thought of as the wave function of the center-of-mass motion of the "superelectrons" (Cooper pairs), whose density is $n_s = |\psi|^2$ and whose flux is $\boldsymbol{J}_s$. The vector potential $\boldsymbol{A}$ determines the electromagnetic field; $\boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla\phi$ is the *electric field* and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ the *magnetic induction*. Equation (1.2) is essentially Ampère's law, $\nabla \times \boldsymbol{B} = \boldsymbol{J}$, where $\boldsymbol{J}$, the total current, is the sum of a "normal" current $\boldsymbol{J}_n = \boldsymbol{E}$, the supercurrent $\boldsymbol{J}_s$, and the transport current $\boldsymbol{J}_t = \nabla \times \boldsymbol{H}$. The normal current obeys Ohm's law $\boldsymbol{J}_n = \sigma_n \boldsymbol{E}$; the "normal conductivity" coefficient σ_n is equal to one in the adopted system of units. The difference $\boldsymbol{M} = \boldsymbol{B} - \boldsymbol{H}$ is known as the *magnetization*. The trivial solution ($\psi = 0$, $\boldsymbol{B} = \boldsymbol{H}$, $\boldsymbol{E} = \boldsymbol{0}$) represents the *normal state*, where all superconducting properties have been lost.

The TDGL equations generalize the original GL equations to the time-dependent case. The GL equations themselves embody in a most simple way the macroscopic quantum-mechanical nature of the superconducting state. The generalization, first proposed by Schmid [4], was analyzed by Gor'kov and Eliashberg [5] in the context of the microscopic Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity. Although the validity of the TDGL equations seems to be limited to a narrow range of temperatures near the critical temperature, T_c, the equations have been used extensively and successfully in large-scale numerical simulations to study vortex dynamics in type-II superconductors; see [6, 7, 8, 9]. We refer the reader to the physics literature [10, 11, 12] for further details.

1.2 Previous Work and Outline of Present Work

The TDGL equations have been the object of several recent mathematical studies. Elliott and Tang [13] proved the existence and uniqueness of solutions in two-dimensional domains under some complicated mathematical boundary conditions, using a time-discretization procedure. Subsequently, Tang applied the same methods to the TDGL equations with fixed total magnetic flux [14]. Du [15], using a finite-element approach, established the existence and uniqueness of weak solutions in two- and three-dimensional domains, under the assumption that the order parameter is initially bounded in $L^\infty(\Omega)$. The same results were obtained independently by Chen, Hoffmann, and Liang [16], who used the Leray-Schauder fixed-point theorem. Du adopted the zero-electric potential gauge ($\phi = 0$), Chen, Hoffmann, and Liang the "$\phi = -\nabla \cdot \boldsymbol{A}$" gauge for their analysis.

In [17], Liang and Tang considered the dynamics of the TDGL equations in bounded domains in $\mathbf{R}^3$, assuming the "$\nabla \cdot \boldsymbol{A} = 0$" London gauge at all times. They claimed to prove the existence of a dynamical system. But since they failed to verify the continuous dependence of the solution operator on the initial data, it is not evident that the solution operator actually defines a dynamical system. Moreover, the limiting relation displayed in the proof of [17, Theorem 6.1] does not follow from [18, Theorem 4.3.4], as claimed.

Recently, Tang and Wang [19] exploited the formal similarity between the TDGL equations in the London gauge and the Navier-Stokes equations for incompressible fluids. They applied the methods developed for the Navier-Stokes equations to prove the existence of strong solutions in two and three dimensions, weak solutions in two dimensions, and a global attractor for the TDGL equations.

One might think that, with Ref. [19], the issues of existence, uniqueness, and large-time asymptotic behavior for the TDGL equations had been settled. However, not only are there lacunae in the proofs, but we claim that the methods developed for the Navier-Stokes equations are most unnatural for the TDGL equations. By imposing the London gauge and forcing the TDGL equations into the framework of the Navier-Stokes equations, one turns a standard semilinear parabolic equation into something much more complicated. Although it is true, as our work will show, that the London gauge is the appropriate gauge for the time-independent GL equations, the "$\phi = -\nabla \cdot \boldsymbol{A}$" gauge is a natural gauge for the TDGL equations. As first noted by Takáč [2], the TDGL equations generate a dynamical system in this gauge, and every stationary solution satisfies the London gauge.

In this article, we use a generalization of the "$\phi = -\nabla \cdot \boldsymbol{A}$" gauge, which was introduced by FLECKINGER–PELLÉ and KAPER in [1]. The "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge, where ω is any nonnegative number, generalizes the standard "$\phi = -\nabla \cdot \boldsymbol{A}$" gauge and reduces to the zero-electric potential gauge ($\phi = 0$) in the limit $\omega = 0$. The zero-electric potential gauge, which is the preferred choice for numerical calculations, yields a form of the TDGL equations that does not fit the framework of the Navier-Stokes equations and is not covered by the analysis of Ref. [19]. Applying the methods developed by TAKÁČ in [2], we establish rigorously the existence of a dynamical process for the TDGL equations in the case where the applied magnetic field is time dependent and the existence of a dynamical system in the case where it is time independent. In the latter case, we prove that every solution of the TDGL equations is attracted to a set of stationary solutions, which are divergence free if $\omega > 0$. This result indicates how the stationary solutions of the TDGL equations can be connected to the solutions of the time-independent GL equations. The case $\omega = 0$ is degenerate and needs to be treated separately; in this case, we *cannot* conclude that the stationary solutions are divergence free.

Following is an outline of the article. Section 2 contains preliminary material. We derive some auxiliary identities from the TDGL equations (Section 2.1), introduce the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge (Section 2.2), and give various estimates that follow from an energy-type functional (Section 2.3). Section 3 gives the formulation of the TDGL equations as an abstract initial-value problem in a Hilbert space. We first introduce the notation (Section 3.1), homogenize the boundary conditions by means of the applied vector potential (Section 3.2), and define the abstract initial-value problem (Section 3.3). In Section 3.4 we prove a regularity result for an integral involving the applied vector potential, which eventually determines the regularity of a mild solution of the abstract initial-value problem. Section 4 summarizes the results of our analysis in three theorems, each with a corollary. Theorem 1 gives an existence and uniqueness result (Section 4.1), Theorem 2 a regularity result (Section 4.2). Both theorems hold when the applied magnetic field varies with time. A corollary of Theorem 2 is the existence of a dynamical process. Specializing to the case of a time-independent magnetic field, we obtain a dynamical system whose properties are given in Theorem 3 (Section 4.3). The degenerate case $\omega = 0$ is discussed in Section 4.4. The proofs of the theorems are given in Section 5.

2 Preliminaries

In this section we establish several auxiliary identities, which follow from the TDGL equations (1.1)–(1.4). We also introduce the gauge choice and define an energy-type functional for the TDGL equations.

2.1 Auxiliary Identities

The TDGL model of superconductivity is a system of semilinear parabolic equations. This is most easily seen if, in Eqs. (1.1) and (1.2), one uses the identities

$$-\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \psi = \frac{1}{\kappa^2}\Delta\psi - \frac{2i}{\kappa}(\nabla\psi)\cdot\boldsymbol{A} - \frac{i}{\kappa}\psi(\nabla\cdot\boldsymbol{A}) - \psi|\boldsymbol{A}|^2 \tag{2.1}$$

and

$$-\nabla\times\nabla\times\boldsymbol{A} = \Delta\boldsymbol{A} - \nabla(\nabla\cdot\boldsymbol{A}). \tag{2.2}$$

Many of the methods developed for such systems are indeed applicable to the TDGL equations. But, as we will see in the following analysis, the TDGL equations have several distinct features that make them mathematically interesting in their own right and different from, say, the Navier-Stokes equations.

The curl of a gradient vanishes, so the TDGL equations do not change if we replace $\boldsymbol{H}$ by $\boldsymbol{H}' = \boldsymbol{H} + \nabla\Phi$, for any (sufficiently smooth) real scalar-valued function Φ of position and time. If $\Phi = 0$ on $\partial\Omega$, we also have $\boldsymbol{n}\times\boldsymbol{H} = \boldsymbol{n}\times\boldsymbol{H}'$ on $\partial\Omega$, so the boundary conditions do not change either. In particular, if we take Φ at any time as the (unique) solution of the Dirichlet problem for Poisson's equation $\Delta\Phi = -\nabla\cdot\boldsymbol{H}$, we have $\nabla\cdot\boldsymbol{H}' = 0$ at all times. Hence, there is no loss of generality if, from now on, we assume that the applied magnetic field $\boldsymbol{H}$ is divergence free,

$$\nabla\cdot\boldsymbol{H} = 0 \quad \text{in } \Omega. \tag{2.3}$$

The quantity $n_s = |\psi|^2$ corresponds to the superelectron density. Its evolution is governed by the equation

$$\eta\frac{\partial|\psi|^2}{\partial t} = -2\ \mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2\psi\right] + 2\left(1 - |\psi|^2\right)|\psi|^2 \tag{2.4}$$

or, equivalently,

$$\eta\frac{\partial|\psi|^2}{\partial t} = \frac{1}{\kappa^2}\Delta|\psi|^2 - 2\left|\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)\psi\right|^2 + 2\left(1 - |\psi|^2\right)|\psi|^2. \tag{2.5}$$

Clearly, if the inequality $|\psi| \le 1$ is satisfied on Ω at $t = 0$, it is satisfied at all later times. Note that the scalar potential ϕ does not figure in Eq. (2.4).

The divergence of a curl vanishes, so Eq. (1.2) implies the identity

$$\nabla \cdot \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \phi \right) = \nabla \cdot \boldsymbol{J}_s \quad \text{in } \Omega. \tag{2.6}$$

An expression for $\nabla \cdot \boldsymbol{J}_s$ is easily obtained by taking the divergence of Eq. (1.3),

$$\nabla \cdot \boldsymbol{J}_s = -\kappa \ \mathrm{Im} \left[\psi^* \left(\frac{i}{\kappa} \nabla + \boldsymbol{A} \right)^2 \psi \right]. \tag{2.7}$$

From this expression and Eq. (1.1) we obtain the more interesting expression

$$\nabla \cdot \boldsymbol{J}_s = \eta \kappa^2 \left[\frac{1}{2i\kappa} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) + \phi |\psi|^2 \right]. \tag{2.8}$$

An immediate consequence of the definition (1.3) of $\boldsymbol{J}_s$ and the first boundary condition in Eq. (1.4) is that $\boldsymbol{n} \cdot \boldsymbol{J}_s = 0$ on $\partial\Omega$. By assumption, $\partial\Omega$ is locally the level surface (or curve) of a $C^{1,1}$-function $\Phi : \mathbf{R}^n \to \mathbf{R}$. Hence, the unit normal vector is given by $\boldsymbol{n} = |\nabla \Phi|^{-1} \nabla \Phi$, where $\nabla \Phi$ is nonvanishing and Lipschitz continuous near every point of $\partial\Omega$. Consequently, $\boldsymbol{n} \cdot (\nabla \times \boldsymbol{n}) = 0$ on $\partial\Omega$. According to the second boundary condition in Eq. (1.4), $\nabla \times \boldsymbol{A} - \boldsymbol{H}$ and $\boldsymbol{n}$ are colinear on $\partial\Omega$. Therefore, it must be the case that $\boldsymbol{n} \cdot \nabla \times (\nabla \times \boldsymbol{A} - \boldsymbol{H}) = 0$ on $\partial\Omega$. When we combine this identity and the identity $\boldsymbol{n} \cdot \boldsymbol{J}_s = 0$ with Eq. (1.2), we see that $\boldsymbol{n} \cdot (\partial_t \boldsymbol{A} + \nabla \phi) = 0$ on $\partial\Omega$. Therefore, any solution of the TDGL equations is such that

$$\boldsymbol{n} \cdot \left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla \phi \right) = 0 \quad \text{and} \quad \boldsymbol{n} \cdot \boldsymbol{J}_s = 0 \quad \text{on } \partial\Omega. \tag{2.9}$$

These identities express the physical fact that the electric field and the supercurrent are always tangential to the surface of the superconductor.

2.2 Gauge Choice

The TDGL equations are invariant under the gauge transformation

$$\mathcal{G}_\chi : (\psi, \boldsymbol{A}, \phi) \mapsto \left(\psi e^{i\kappa\chi}, \boldsymbol{A} + \nabla \chi, \phi - \partial_t \chi \right). \tag{2.10}$$

The *gauge* χ can be any (sufficiently smooth) real scalar-valued function of position and time. For the present investigation we adopt the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$"

gauge, where ω is a real nonnegative parameter. This gauge, introduced in [1], is determined by taking $\chi \equiv \chi_\omega(x,t)$ as the (unique) solution of the boundary-value problem

$$(\partial_t - \omega\Delta)\chi = \phi + \omega(\nabla \cdot \boldsymbol{A}) \quad \text{in } \Omega \times (0,\infty), \tag{2.11}$$

$$\omega(\boldsymbol{n} \cdot \nabla\chi) = -\omega(\boldsymbol{n} \cdot \boldsymbol{A}) \quad \text{on } \partial\Omega \times (0,\infty), \tag{2.12}$$

subject to a suitable initial condition, $\chi(\cdot\,,0) = \chi_0$ in Ω. (See the remark following Eq. (4.6).)

In the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge, we have, at all times $t \geq 0$, the identities

$$\phi + \omega(\nabla \cdot \boldsymbol{A}) = 0 \quad \text{in } \Omega, \qquad \omega(\boldsymbol{n} \cdot \boldsymbol{A}) = 0 \quad \text{on } \partial\Omega. \tag{2.13}$$

The second identity can be strengthened. If $\omega > 0$, it simplifies to $\boldsymbol{n} \cdot \boldsymbol{A} = 0$ on $\partial\Omega$. If $\omega = 0$, the first identity reduces to $\phi = 0$ in Ω; hence, $\boldsymbol{n} \cdot \nabla\phi = 0$ on $\partial\Omega$. But then it follows from the first identity in Eq. (2.9) that $\boldsymbol{n} \cdot \partial_t \boldsymbol{A} = 0$, so $\boldsymbol{n} \cdot \boldsymbol{A} = \boldsymbol{n} \cdot \boldsymbol{A}_0$ on $\partial\Omega$, where $\boldsymbol{A}_0 = \boldsymbol{A}(\cdot\,,0)$. By appropriately choosing χ_0, we can realize the identity $\boldsymbol{n} \cdot \boldsymbol{A} = 0$ on $\partial\Omega$ for all times $t \geq 0$, the same as for $\omega > 0$. Instead of (2.13), we thus have, at all times $t \geq 0$,

$$\phi + \omega(\nabla \cdot \boldsymbol{A}) = 0 \quad \text{in } \Omega, \qquad \boldsymbol{n} \cdot \boldsymbol{A} = 0 \quad \text{on } \partial\Omega. \tag{2.14}$$

In the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge, the differential equations (1.1) and (1.2) reduce to

$$\eta\frac{\partial\psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \psi + i\eta\kappa\omega\psi(\nabla \cdot \boldsymbol{A}) + \left(1 - |\psi|^2\right)\psi \quad \text{in } \Omega \times (0,\infty), \tag{2.15}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \times \nabla \times \boldsymbol{A} + \omega\nabla(\nabla \cdot \boldsymbol{A}) + \boldsymbol{J}_s + \nabla \times \boldsymbol{H} \quad \text{in } \Omega \times (0,\infty), \tag{2.16}$$

where $\boldsymbol{J}_s$ is again given by Eq. (1.3), and the boundary conditions (1.4) to

$$\boldsymbol{n}\cdot\nabla\psi + \gamma\psi = 0, \quad \boldsymbol{n}\cdot\boldsymbol{A} = 0, \quad \boldsymbol{n}\times(\nabla \times \boldsymbol{A} - \boldsymbol{H}) = \boldsymbol{0} \quad \text{on } \partial\Omega\times(0,\infty). \tag{2.17}$$

Henceforth, the term "gauged TDGL equations" refers to the TDGL equations in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge, given by the system of Eqs. (2.15)–(2.17).

The gauged TDGL equations govern the evolution of the pair $(\psi, \boldsymbol{A})$ from the initial data,

$$\psi = \psi_0 \quad \text{and } \boldsymbol{A} = \boldsymbol{A}_0 \quad \text{on } \Omega \times \{0\}, \tag{2.18}$$

where ψ_0 and $\boldsymbol{A}_0$ are given. The boundary-value problem (2.15)–(2.17) is strongly parabolic for $\omega > 0$. It becomes degenerate for $\omega = 0$.

The scalar potential ϕ does not figure in the evolution equation (2.4), so the gauge choice does not affect the observation that $|\psi| \leq 1$ on Ω at all times

$t > 0$ if the inequality is satisfied at $t = 0$. (Cf. the "maximum modulus principle" in Section 4.1, Theorem 1.)

In the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge, the auxiliary identity (2.6), the expression (2.8), and the identities (2.9) reduce to

$$(\partial_t - \omega\Delta)(\nabla \cdot \boldsymbol{A}) = \nabla \cdot \boldsymbol{J}_s \quad \text{in } \Omega, \tag{2.19}$$

$$\nabla \cdot \boldsymbol{J}_s = \eta\kappa^2 \left[\frac{1}{2i\kappa} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - \omega |\psi|^2 (\nabla \cdot \boldsymbol{A}) \right], \tag{2.20}$$

$$\omega(\boldsymbol{n} \cdot \nabla)(\nabla \cdot \boldsymbol{A}) = 0 \quad \text{and} \quad \boldsymbol{n} \cdot \boldsymbol{J}_s = 0 \quad \text{on } \partial\Omega. \tag{2.21}$$

2.3 Energy-Type Functionals

Consider the functional $E_\omega \equiv E_\omega[\psi, \boldsymbol{A}]$,

$$E_\omega[\psi, \boldsymbol{A}] = \int_\Omega \left[\left| \left(\frac{i}{\kappa} \nabla + \boldsymbol{A} \right) \psi \right|^2 + \frac{1}{2} \left(1 - |\psi|^2 \right)^2 + 2\omega(\nabla \cdot \boldsymbol{A})^2 \right.$$
$$\left. + |\nabla \times \boldsymbol{A} - \boldsymbol{H}|^2 \right] \mathrm{d}x + \int_{\partial\Omega} \gamma \left| \frac{i}{\kappa} \psi \right|^2 \mathrm{d}\sigma(x). \tag{2.22}$$

If ψ and $\boldsymbol{A}$ satisfy the gauged TDGL equations, the time derivative of E_ω is

$$\frac{\mathrm{d}E_\omega}{\mathrm{d}t} = -2 \int_\Omega \left[\eta \left| \frac{\partial \psi}{\partial t} - i\kappa\omega\psi(\nabla \cdot \boldsymbol{A}) \right|^2 + \left| \frac{\partial \boldsymbol{A}}{\partial t} \right|^2 + \omega^2 \left| \nabla(\nabla \cdot \boldsymbol{A}) \right|^2 \right] \mathrm{d}x$$
$$-2 \int_\Omega \frac{\partial \boldsymbol{H}}{\partial t} \cdot (\nabla \times \boldsymbol{A} - \boldsymbol{H})\, \mathrm{d}x. \tag{2.23}$$

If $\partial_t \boldsymbol{H} = \boldsymbol{0}$ (stationary applied magnetic field), the expression in the right member is negative semidefinite, and $E_\omega(t) \le E_\omega(0)$ for all $t \ge 0$. In general, the applied magnetic field is not stationary, and E_ω is not necessarily bounded by a constant. However, as the following lemma shows, $E_\omega(t)$ can still be estimated in terms of the quantity $P(t)$,

$$P(t) = \int_0^t \left(\int_\Omega |\partial_t \boldsymbol{H}(x, s)|^2 \, \mathrm{d}x \right)^{1/2} \mathrm{d}s. \tag{2.24}$$

Lemma 1 *If $E_\omega \equiv E_\omega(t)$ exists and is finite, and $P(T) < \infty$ for some $T > 0$, then*

$$E_\omega(t) + 2 \int_0^t \int_\Omega \left[\eta \left| \frac{\partial \psi}{\partial t} - i\kappa\omega\psi(\nabla \cdot \boldsymbol{A}) \right|^2 + \left| \frac{\partial \boldsymbol{A}}{\partial t} \right|^2 + \omega^2 \left| \nabla(\nabla \cdot \boldsymbol{A}) \right|^2 \right] \mathrm{d}x\, \mathrm{d}t'$$

$$\leq \left((E_\omega(0))^{1/2} + P(t) \right)^2, \quad t \in [0,T]. \tag{2.25}$$

Proof. It follows from Eq. (2.23) and the Cauchy-Schwarz inequality that

$$\frac{\mathrm{d}E_\omega}{\mathrm{d}t} \leq 2 \left(\int_\Omega \left| \frac{\partial \boldsymbol{H}}{\partial t} \right|^2 \mathrm{d}x \right)^{1/2} \left(\int_\Omega |\nabla \times \boldsymbol{A} - \boldsymbol{H}|^2 \, \mathrm{d}x \right)^{1/2} \leq 2 \frac{\mathrm{d}P}{\mathrm{d}t} \left(E_\omega(t) \right)^{1/2}. \tag{2.26}$$

Hence, $\mathrm{d}E_\omega^{1/2}/\mathrm{d}t \leq \mathrm{d}P/\mathrm{d}t$. Upon integration, we obtain

$$E_\omega(t) \leq \left((E_\omega(0))^{1/2} + P(t) \right)^2, \quad t \in [0,T]. \tag{2.27}$$

To obtain the inequality (2.25), we use Eq. (2.23) again, this time including the first integral, and apply the estimate (2.27),

$$\frac{\mathrm{d}E_\omega}{\mathrm{d}t} + 2 \int_\Omega \left[\eta \left| \frac{\partial \psi}{\partial t} - i\kappa\omega\psi(\nabla \cdot \boldsymbol{A}) \right|^2 + \left| \frac{\partial \boldsymbol{A}}{\partial t} \right|^2 + \omega^2 \left| \nabla(\nabla \cdot \boldsymbol{A}) \right|^2 \right] \mathrm{d}x$$

$$\leq 2 \frac{\mathrm{d}P}{\mathrm{d}t} \left(E_\omega(t) \right)^{1/2} \leq 2 \frac{\mathrm{d}P}{\mathrm{d}t} \left((E_\omega(0))^{1/2} + P(t) \right). \tag{2.28}$$

The inequality (2.25) follows upon integration. ∎

Lemma 2 *Assume that* $M = \text{ess sup}\{|\psi(x,t)| : (x,t) \in \Omega \times (0,T)\} < \infty$. *Then*

$$2 \int_0^t \int_\Omega \left[\eta \left| \frac{\partial \psi}{\partial t} \right|^2 + \left| \frac{\partial \boldsymbol{A}}{\partial t} \right|^2 + \omega^2 \left| \nabla(\nabla \cdot \boldsymbol{A}) \right|^2 \right] \mathrm{d}x \, \mathrm{d}t'$$

$$\leq (3 + \eta\kappa^2\omega M^2 t) \left((E_\omega(0))^{1/2} + P(t) \right)^2, \quad t \in [0,T], \tag{2.29}$$

whenever the terms in the inequality are well defined.

Proof. Using the elementary inequality $|a|^2 \leq 2(|a-b|^2 + |b|^2)$ and the inequality (2.25), we obtain

$$\int_0^t \int_\Omega \eta \left| \frac{\partial \psi}{\partial t} \right|^2 \mathrm{d}x \, \mathrm{d}t' \leq \left((E_\omega(0))^{1/2} + P(t) \right)^2 + \eta\kappa^2\omega M^2 \int_0^t \int_\Omega 2\omega (\nabla \cdot \boldsymbol{A})^2 \, \mathrm{d}x \, \mathrm{d}t',$$

where

$$\int_0^t \int_\Omega 2\omega(\nabla \cdot \boldsymbol{A})^2 \, \mathrm{d}x \, \mathrm{d}t' \leq \int_0^t E_\omega(t') \, \mathrm{d}t' \leq t \left((E_\omega(0))^{1/2} + P(t) \right)^2.$$

The remaining terms have already been estimated by $\Big((E_\omega(0))^{1/2} + P(t)\Big)^2$ in Lemma 1, in the inequality (2.25). ∎

The term $2\omega(\nabla\cdot\boldsymbol{A})^2$ in the functional E_ω has no basis in physics. Indeed, E_ω is *not* an energy functional unless $\omega = 0$. If $\omega = 0$, E_ω reduces to the Ginzburg-Landau free-energy functional,

$$E_0[\psi,\boldsymbol{A}] = \int_\Omega \left[\left|\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)\psi\right|^2 + \frac{1}{2}\left(1-|\psi|^2\right)^2 + |\nabla\times\boldsymbol{A}-\boldsymbol{H}|^2\right]\mathrm{d}x$$

$$+ \int_{\partial\Omega}\gamma\left|\frac{i}{\kappa}\psi\right|^2 \mathrm{d}\sigma(x). \tag{2.30}$$

The gauge restriction (2.14) reduces to $\phi = 0$ in Ω, and the Euler equations and natural boundary conditions associated with E_0 are

$$-\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2\psi + \left(1-|\psi|^2\right)\psi = 0 \quad \text{in } \Omega, \tag{2.31}$$

$$-\nabla\times\nabla\times\boldsymbol{A} + \boldsymbol{J}_s + \nabla\times\boldsymbol{H} = \boldsymbol{0} \quad \text{in } \Omega, \tag{2.32}$$

$$\boldsymbol{n}\cdot\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)\psi + \gamma\frac{i}{\kappa}\psi = 0 \quad \text{and} \quad \boldsymbol{n}\times(\nabla\times\boldsymbol{A}-\boldsymbol{H}) = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{2.33}$$

These are the time-independent Ginzburg-Landau (GL) equations of superconductivity. The relationship between stationary solutions of the TDGL equations and solutions of the time-independent GL equations is discussed in a forthcoming article [20].

3 Functional Formulation

In this section, we formulate the gauged TDGL equations as an abstract evolution equation in a Hilbert space.

3.1 Notation

The symbol C denotes a generic positive constant, not necessarily the same at different instances. All Banach spaces are real; the (real) dual of a Banach space X is denoted by X'.

$L^p(\Omega)$, for $1 \le p \le \infty$, is the usual Lebesgue space, with norm $\|\cdot\|_{L^p}$; $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$. $W^{m,2}(\Omega)$, for nonnegative integer m, is the usual Sobolev space, with norm $\|\cdot\|_{W^{m,2}}$; $W^{m,2}(\Omega)$ is a Hilbert space for the inner product $(\cdot,\cdot)_{m,2}$, given by $(u,v)_{m,2} = \sum_{|\alpha|\le m}(\partial^\alpha u, \partial^\alpha v)$ for $u, v \in W^{m,2}(\Omega)$. Fractional Sobolev spaces $W^{s,2}(\Omega)$, with noninteger s, are defined by interpolation [21, Chapter VII].

$C^\nu(\Omega)$, for $\nu \ge 0$, $\nu = m + \lambda$ with $0 \le \lambda < 1$, is the space of m times continuously differentiable functions on Ω, whose mth order derivatives satisfy a Hölder condition with exponent λ if ν is not an integer; the norm $\|\cdot\|_{C^\nu}$ is defined in the usual way.

The definitions extend to spaces of vector-valued functions in the standard way, with the *caveat* that the inner product in $[L^2(\Omega)]^n$ is defined by $(u,v) = \int_\Omega u \cdot v$, where $\cdot$ indicates the scalar product in $\mathbf{R}^n$. Complex-valued functions are interpreted as vector-valued functions with two real components.

Functions that vary in space and time, like the order parameter and the vector potential, are considered as mappings from the time domain, which is a subinterval of $[0,\infty)$, into spaces of complex- or vector-valued functions defined in Ω. Let $X = (X, \|\cdot\|_X)$ be a Banach space of functions defined in Ω. Then functions of space and time defined on $\Omega \times (0,T)$, for $T > 0$, may be considered as elements of $L^p(0,T;X)$, for $1 \le p \le \infty$, or $W^{m,2}(0,T;X)$, for nonnegative integer m, or $C^\nu(0,T;X)$, for $\nu \ge 0$, $\nu = m + \lambda$ with $0 \le \lambda < 1$. Detailed definitions can be found, for example, in [18].

Obviously, function spaces of ordered pairs $(\psi, \boldsymbol{A})$, where $\psi : \Omega \to \mathbf{R}^2$ and $\boldsymbol{A} : \Omega \to \mathbf{R}^n$ $(n = 2,3)$, play an important role in the study of the gauged TDGL equations. We therefore adopt the following special notation: $\mathcal{X} = [X(\Omega)]^2 \times [X(\Omega)]^n$ for any Banach space $X(\Omega)$ of real-valued functions defined in Ω. Here, $[X(\Omega)]^2$ and $[X(\Omega)]^n$ are the underlying Banach spaces for the order parameter ψ and the vector potential $\boldsymbol{A}$, respectively. A suitable framework for the functional analysis of the gauged TDGL equations is the Cartesian product $\mathcal{W}^{1+\alpha,2} = [W^{1+\alpha,2}(\Omega)]^2 \times [W^{1+\alpha,2}(\Omega)]^n$ with $\frac{1}{2} < \alpha < 1$. This space is continuously imbedded in $\mathcal{W}^{1,2} \cap \mathcal{L}^\infty$.

3.2 Reduction to Homogeneous Form

When $\boldsymbol{H} \not\equiv \mathbf{0}$, the boundary conditions (2.17) are inhomogeneous, and it is necessary to first reduce them to homogeneous form.

Assume $\boldsymbol{H} \not\equiv \mathbf{0}$ and $\boldsymbol{H} \in [L^2(\Omega)]^n$. Let $\boldsymbol{A}_{\mathbf{H}}$ be a minimizer of the convex quadratic form $J_\omega \equiv J_\omega[\boldsymbol{A}]$,

$$J_\omega[\boldsymbol{A}] = \int_\Omega \left[\omega(\nabla\cdot\boldsymbol{A})^2 + |\nabla\times\boldsymbol{A}-\boldsymbol{H}|^2\right]\,\mathrm{d}x, \tag{3.1}$$

on the domain

$$\mathcal{D}(J_\omega) = \{\boldsymbol{A}\in[W^{1,2}(\Omega)]^n : \boldsymbol{n}\cdot\boldsymbol{A} = 0 \text{ on } \partial\Omega\}.$$

Lemma 3 *The functional J_ω has a unique minimizer $\boldsymbol{A}_\mathbf{H}$ on $\mathcal{D}(J_\omega)$ if $\omega > 0$, and this minimizer has the property $\nabla\cdot\boldsymbol{A}_\mathbf{H} = 0$ in Ω. The functional J_0 has a unique minimizer $\boldsymbol{A}_\mathbf{H}$ on the closed linear subspace $\mathcal{D}_0(J_0) = \{\boldsymbol{A}\in\mathcal{D}(J_0) : \nabla\cdot\boldsymbol{A} = 0 \text{ in } \Omega\}$ of $\mathcal{D}(J_0)$.*

Proof. Assume $\omega > 0$. Then $J_\omega[\boldsymbol{A}]\to\infty$ as $\|\boldsymbol{A}\|_{W^{1,2}}\to\infty$; see [22, Chapter I, Eq. (5.45)]. Also, J_ω is strictly convex and weakly lower semicontinuous. Standard methods of the calculus of variations yield the existence of a unique minimizer. This minimizer, $\boldsymbol{A}_\mathbf{H}$, is necessarily divergence free; otherwise, we could replace it by $\boldsymbol{A}_\mathbf{H}+\nabla\Phi$ without changing the term $\nabla\times\boldsymbol{A}-\boldsymbol{H}$ and, by taking Φ as the solution of the Neumann problem for Poisson's equation $\Delta\Phi = -\nabla\cdot\boldsymbol{A}_\mathbf{H}$ in Ω, reduce the value of the functional to $J_\omega[\boldsymbol{A}_\mathbf{H}+\nabla\Phi] = \int_\Omega |\nabla\times\boldsymbol{A}_\mathbf{H}-\boldsymbol{H}|^2\,\mathrm{d}x$, which is strictly less than $J_\omega[\boldsymbol{A}_\mathbf{H}]$. The case $\omega = 0$ is similar. ∎

The lemma shows that the property $\nabla\cdot\boldsymbol{A}_\mathbf{H} = 0$ in Ω is a *consequence* of the fact that $\boldsymbol{A}_\mathbf{H}$ minimizes the functional J_ω if $\omega > 0$. If $\omega = 0$, we *impose* the condition $\nabla\cdot\boldsymbol{A}_\mathbf{H} = 0$. In either case, $\boldsymbol{A}_\mathbf{H}$ is the (unique) weak solution of the strongly elliptic boundary-value problem

$$\nabla\times\nabla\times\boldsymbol{A}_\mathbf{H} = \nabla\times\boldsymbol{H} \quad\text{and}\quad \nabla\cdot\boldsymbol{A}_\mathbf{H} = 0 \quad\text{in } \Omega, \tag{3.2}$$

$$\boldsymbol{n}\cdot\boldsymbol{A}_\mathbf{H} = 0 \quad\text{and}\quad \boldsymbol{n}\times(\nabla\times\boldsymbol{A}_\mathbf{H}-\boldsymbol{H}) = \mathbf{0} \quad\text{on } \partial\Omega. \tag{3.3}$$

We refer to $\boldsymbol{A}_\mathbf{H}$ as the *applied vector potential.*

Lemma 4 *If $\boldsymbol{H}\in[L^2(\Omega)]^n$, then $\boldsymbol{A}_\mathbf{H}\in\mathcal{D}(J_\omega)$. The mapping $\boldsymbol{H}\mapsto\boldsymbol{A}_\mathbf{H}$ is linear, time independent, and continuous from $[W^{\theta,2}(\Omega)]^n$ to $[W^{1+\theta,2}(\Omega)]^n$, for $0\le\theta\le 1$.*

Proof. The continuity of the mapping $\boldsymbol{H} \mapsto \boldsymbol{A}_{\mathbf{H}}$ follows from the regularity results in GEORGESCU [23]. ∎

We now introduce the *reduced vector potential* $\boldsymbol{A}'$,

$$\boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}}. \tag{3.4}$$

In terms of ψ and $\boldsymbol{A}'$, the gauged TDGL equations assume the form

$$\frac{\partial \psi}{\partial t} - \frac{1}{\eta\kappa^2}\Delta\psi = \varphi \quad \text{in } \Omega \times (0, \infty), \tag{3.5}$$

$$\frac{\partial \boldsymbol{A}'}{\partial t} + \nabla \times \nabla \times \boldsymbol{A}' - \omega\nabla(\nabla \cdot \boldsymbol{A}') = \boldsymbol{F} \quad \text{in } \Omega \times (0, \infty), \tag{3.6}$$

$$\boldsymbol{n} \cdot \nabla\psi + \gamma\psi = 0, \quad \boldsymbol{n} \cdot \boldsymbol{A}' = 0, \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A}') = \mathbf{0} \quad \text{on } \partial\Omega \times (0, \infty). \tag{3.7}$$

Here, φ and $\boldsymbol{F}$ are nonlinear functions of ψ and $\boldsymbol{A}'$,

$$\varphi \equiv \varphi(t, \psi, \boldsymbol{A}') = \frac{1}{\eta}\left[-\frac{2i}{\kappa}(\nabla\psi) \cdot (\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}})\right.$$

$$\left. -\frac{i}{\kappa}(1 - \eta\kappa^2\omega)\psi(\nabla \cdot \boldsymbol{A}') - \psi|\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}|^2 + \left(1 - |\psi|^2\right)\psi\right], \tag{3.8}$$

$$\boldsymbol{F} \equiv \boldsymbol{F}(t, \psi, \boldsymbol{A}') = \boldsymbol{J}_s' - |\psi|^2\boldsymbol{A}_{\mathbf{H}} - \partial_t\boldsymbol{A}_{\mathbf{H}}. \tag{3.9}$$

Here we have used the abbreviation $\boldsymbol{J}_s' = \boldsymbol{J}_s(\psi, \boldsymbol{A}')$, where $\boldsymbol{J}_s$ is the expression for the supercurrent density, given by Eq. (1.3). The equations are supplemented by initial data, which follow from Eqs. (2.18) and (3.4),

$$\psi = \psi_0 \quad \text{and } \boldsymbol{A}' = \boldsymbol{A}_0 - \boldsymbol{A}_{\mathbf{H}}(0) \quad \text{on } \Omega \times \{0\}. \tag{3.10}$$

In the next section we connect the evolution of the solution $(\psi, \boldsymbol{A}')$ of the system of Eqs. (3.5)–(3.7) from the initial data $(\psi_0, \boldsymbol{A}_0 - \boldsymbol{A}_{\mathbf{H}}(0))$ with the dynamics of a vector u in the Hilbert space $\mathcal{L}^2 = [L^2(\Omega)]^2 \times [L^2(\Omega)]^n$.

3.3 Gauged TDGL Equations

The following analysis is restricted to the case $\omega > 0$; we comment on the case $\omega = 0$ in Section 4.4.

Let the vector $u : [0, \infty) \to \mathcal{L}^2$ represent the pair $(\psi, \boldsymbol{A}')$,

$$u = (\psi, \boldsymbol{A}') \equiv (\psi, \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}}), \tag{3.11}$$

and let $\mathcal{A}$ be the linear selfadjoint operator in $\mathcal{L}^2$ associated with the quadratic form $Q_\omega \equiv Q_\omega[u]$,

$$Q_\omega[u] = \int_\Omega \left[\frac{1}{\eta\kappa^2} |\nabla\psi|^2 + \omega(\nabla \cdot \boldsymbol{A}')^2 + |\nabla \times \boldsymbol{A}'|^2 \right] \mathrm{d}x + \int_{\partial\Omega} \frac{\gamma}{\eta\kappa^2} |\psi|^2 \, \mathrm{d}\sigma(x), \tag{3.12}$$

on the domain

$$\mathcal{D}(Q_\omega) = \mathcal{D}(\mathcal{A}^{1/2}) = \{u = (\psi, \boldsymbol{A}') \in \mathcal{W}^{1,2} : \boldsymbol{n} \cdot \boldsymbol{A}' = 0 \text{ on } \partial\Omega\}.$$

The quadratic form Q_ω is nonnegative. Furthermore, since $\omega > 0$, $Q_\omega[\psi, \boldsymbol{A}'] + c\|\psi\|_{L^2}$ is coercive on $\mathcal{W}^{1,2}$ for any constant $c > 0$. Hence, $\mathcal{A}$ is positive definite in $\mathcal{L}^2$ [22, Chapter I, Eq. (5.45)]. If no confusion is possible, we use the same symbol $\mathcal{A}$ for the restrictions $\mathcal{A}_\psi$ and $\mathcal{A}_{\mathbf{A}}$ of $\mathcal{A}$ to the respective linear subspaces $[L^2(\Omega)]^2 \equiv [L^2(\Omega)]^2 \times \{0\}$ (for ψ) and $[L^2(\Omega)]^n \equiv \{0\} \times [L^2(\Omega)]^n$ (for $\boldsymbol{A}$) of $\mathcal{L}^2$.

Now, consider the initial-value problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u = \mathcal{F}(t, u(t)) \quad \text{for } t > 0; \quad u(0) = u_0, \tag{3.13}$$

in $\mathcal{L}^2$, where $\mathcal{F}(t,u) = (\varphi, \boldsymbol{F})$, φ and $\boldsymbol{F}$ given by Eqs. (3.8) and (3.9), and $u_0 = (\psi_0, \boldsymbol{A}_0 - \boldsymbol{A}_{\mathbf{H}}(0))$.

With $\frac{1}{2} < \alpha < 1$ and $u_0 \in \mathcal{W}^{1+\alpha,2}$, we say that u is a *mild solution* of Eq. (3.13) on the interval $[0,T]$, for some $T > 0$, if $u : [0,T] \to \mathcal{W}^{1+\alpha,2}$ is continuous and

$$u(t) = \mathrm{e}^{-\mathcal{A}t} u_0 + \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)} \mathcal{F}(s, u(s)) \, \mathrm{d}s \quad \text{for } 0 \le t \le T \tag{3.14}$$

in $\mathcal{L}^2$. A mild solution of the initial-value problem (3.13) defines a *weak solution* $(\psi, \boldsymbol{A}')$ of the boundary-value problem (3.5)–(3.7), which in turn defines a weak solution $(\psi, \boldsymbol{A})$ of the gauged TDGL equations, provided $\boldsymbol{A}_{\mathbf{H}}$ is sufficiently regular.

Given any $f = (\varphi, \boldsymbol{F}) \in \mathcal{L}^2$, the equation $\mathcal{A}u = f$ in $\mathcal{L}^2$ is equivalent with a system of uncoupled boundary-value problems,

$$-\frac{1}{\eta\kappa^2} \Delta\psi = \varphi \quad \text{in } \Omega, \quad \boldsymbol{n} \cdot \nabla\psi + \gamma\psi = 0 \quad \text{on } \partial\Omega; \tag{3.15}$$

$$\nabla \times \nabla \times \boldsymbol{A}' - \omega\nabla(\nabla \cdot \boldsymbol{A}') = \boldsymbol{F} \quad \text{in } \Omega, \quad \boldsymbol{n} \cdot \boldsymbol{A}' = 0, \ \boldsymbol{n} \times (\nabla \times \boldsymbol{A}') = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{3.16}$$

(More precisely, the system of Eqs. (3.15)–(3.16) holds in the dual space $\mathcal{D}(Q_\omega)'$ of $\mathcal{D}(Q_\omega)$ with respect to the inner product in $\mathcal{L}^2$.) Boundary-value problems of this type have been studied by Georgescu [23]. Applying his results, we see that $\mathcal{D}(\mathcal{A})$ is a closed linear subspace of $\mathcal{W}^{2,2}$. Since $\mathcal{A}$ is positive definite on $\mathcal{L}^2$, its fractional powers $\mathcal{A}^\theta$ are well defined for all $\theta \in \mathbf{R}$; they are unbounded for $\theta > 0$. Interpolation theory shows that $\mathcal{D}(\mathcal{A}^\theta)$ is a closed linear subspace of $\mathcal{W}^{2\theta,2}$ for $0 < \theta < 1$.

3.4 Smoothing of the Applied Vector Potential

The term $\partial_t \boldsymbol{A}_{\mathbf{H}}$ in Eq. (3.9) introduces an integral $\mathcal{J}_{\mathbf{H}}(t)$ in Eq. (3.14),

$$\mathcal{J}_{\mathbf{H}}(t) = \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(s)\, \mathrm{d}s, \tag{3.17}$$

where $\mathcal{J}_{\mathbf{H}}(t) \in [L^2(\Omega)]^n \equiv \{0\} \times [L^2(\Omega)]^n \subset \mathcal{L}^2$ for $t \in (0,T)$. The regularity of this integral determines the regularity of the solution u of Eq. (3.13).

Lemma 5 *If $\boldsymbol{H} \in W^{1,2}(0,T;[L^2(\Omega)]^n)$, then $\mathcal{J}_{\mathbf{H}}(t) \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ for $0 \le \alpha < 1$, for every $t \in (0,T)$, and $\mathcal{J}_{\mathbf{H}} \in C^\beta(0,T;[W^{1+\alpha,2}(\Omega)]^n)$ for $0 \le \beta < \frac{1}{2}(1-\alpha)$.*

Proof. Assume that $0 \le \alpha < 1$ and $0 \le \beta < \frac{1}{2}(1-\alpha)$. The proof of the lemma uses the inequalities

$$\|\mathcal{A}^{\alpha/2}\mathrm{e}^{-\mathcal{A}s}\|_{\mathcal{L}^2} \le C s^{-\alpha/2} \quad \text{for } 0 < s \le T, \tag{3.18}$$

$$\|(\mathrm{e}^{-\mathcal{A}s} - I)\mathcal{A}^{-\beta}\|_{\mathcal{L}^2} \le C s^\beta \quad \text{for } 0 \le s \le T, \tag{3.19}$$

where the positive constants C do not depend on s; see [18, Theorem 1.4.3].

Because $\partial_t \boldsymbol{H} \in L^2(0,T;[L^2(\Omega)]^n)$, it follows immediately from Lemma 4 that $\partial_t \boldsymbol{A}_{\mathbf{H}} \in L^2(0,T;[W^{1,2}(\Omega)]^n)$. Standard arguments lead to the continuity of $\mathcal{J}_{\mathbf{H}} : [0,T] \to [W^{1+\alpha,2}(\Omega)]^n$; cf. [18, Proof of Theorem 3.3.4]. Also, $\mathcal{A}^{1/2}\partial_t \boldsymbol{A}_{\mathbf{H}} \in L^2(0,T;[L^2(\Omega)]^n)$ and

$$\mathcal{A}^{(1+\alpha)/2}\mathcal{J}_{\mathbf{H}}(t) = \int_0^t \mathcal{A}^{\alpha/2}\mathrm{e}^{-\mathcal{A}(t-s)}\mathcal{A}^{1/2}\frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(s)\, \mathrm{d}s \quad \text{for } 0 \le t \le T \tag{3.20}$$

in $[L^2(\Omega)]^n$. Applying the estimate (3.18), we obtain

$$\|\mathcal{A}^{(1+\alpha)/2}\mathcal{J}_{\mathbf{H}}(t)\|_{L^2} \le \int_0^t \|\mathcal{A}^{\alpha/2}\mathrm{e}^{-\mathcal{A}s}\|_{\mathcal{L}^2} \left\| \mathcal{A}^{1/2}\frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(t-s)\right\|_{L^2} \mathrm{d}s$$

$$\leq C \int_0^t \left\| \mathcal{A}^{1/2} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(t-s) \right\|_{L^2} s^{-\alpha/2} \, \mathrm{d}s$$

$$\leq C \left(\int_0^t \left\| \mathcal{A}^{1/2} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(t-s) \right\|_{L^2}^2 \mathrm{d}s \right)^{1/2} \left(\int_0^t s^{-\alpha} \, \mathrm{d}s \right)^{1/2}$$

$$= \frac{C}{(1-\alpha)^{1/2}} t^{(1-\alpha)/2} \left(\int_0^t \left\| \mathcal{A}^{1/2} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(s) \right\|_{L^2}^2 \mathrm{d}s \right)^{1/2}, \tag{3.21}$$

so $\mathcal{J}_{\mathbf{H}}(t) \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$, a closed subspace of $[W^{1+\alpha,2}(\Omega)]^n$, for every $t \in [0,T]$.

To prove the Hölder continuity of $\mathcal{J}_{\mathbf{H}}$, we take $0 \leq t < t' \leq T$ and use the following identity in $[L^2(\Omega)]^n$, which follows immediately from the definition (3.17),

$$\mathcal{A}^{(1+\alpha)/2} \left(\mathcal{J}_{\mathbf{H}}(t') - \mathcal{J}_{\mathbf{H}}(t) \right)$$
$$= \mathcal{A}^{\alpha/2} \left[\int_0^{t'} \mathrm{e}^{-\mathcal{A}(t'-s)} \mathcal{A}^{1/2} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(s) \, \mathrm{d}s - \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)} \mathcal{A}^{1/2} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(s) \, \mathrm{d}s \right]$$
$$= \mathcal{J}_1(t,t') + \mathcal{J}_2(t,t'), \tag{3.22}$$

where

$$\mathcal{J}_1(t,t') = \int_0^{t'-t} \mathcal{A}^{\alpha/2} \mathrm{e}^{-\mathcal{A}s} \mathcal{A}^{1/2} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(t'-s) \, \mathrm{d}s,$$
$$\mathcal{J}_2(t,t') = \left(\mathrm{e}^{-\mathcal{A}(t'-t)} - I \right) \int_0^t \mathcal{A}^{\alpha/2} \mathrm{e}^{-\mathcal{A}(t-s)} \mathcal{A}^{1/2} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(s) \, \mathrm{d}s.$$

We estimate the $[L^2(\Omega)]^n$-norms of $\mathcal{J}_1(t,t')$ and $\mathcal{J}_2(t,t')$ as in (3.21), making use of the inequalities (3.18) and (3.19),

$$\|\mathcal{J}_1(t,t')\|_{L^2} \leq \frac{C}{(1-\alpha)^{1/2}} |t'-t|^{(1-\alpha)/2} \left(\int_t^{t'} \left\| \mathcal{A}^{1/2} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(s) \right\|_{L^2}^2 \mathrm{d}s \right)^{1/2}, \tag{3.23}$$

$$\|\mathcal{J}_2(t,t')\|_{L^2} = \left\| \left(\mathrm{e}^{-\mathcal{A}(t'-t)} - I \right) \mathcal{A}^{-\beta} \int_0^t \mathcal{A}^{\beta+(\alpha/2)} \mathrm{e}^{-\mathcal{A}(t-s)} \mathcal{A}^{1/2} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(s) \, \mathrm{d}s \right\|_{L^2}$$
$$\leq \frac{C}{(1-2\beta-\alpha)^{1/2}} t^{(1-\alpha)/2-\beta} |t'-t|^{\beta} \left(\int_0^t \left\| \mathcal{A}^{1/2} \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}(s) \right\|_{L^2}^2 \mathrm{d}s \right)^{1/2}. \tag{3.24}$$

Here, the positive constants C depend only on $\mathcal{A}$, α, and β. The statement of the lemma follows. ∎

4 Results

We present our results in the form of three theorems, each with a corollary. The proofs are deferred until Section 5. Unless indicated otherwise, we assume that the data entering the equations satisfy the following hypotheses:

(H1) $\Omega \subset \mathbf{R}^n$ ($n = 2$ or 3) is bounded, with $\partial\Omega$ of class $C^{1,1}$. (That is, $\partial\Omega$ is a compact $(n-1)$-manifold described by Lipschitz-continuously differentiable charts.)

(H2) $\gamma : \partial\Omega \to \mathbf{R}$ is Lipschitz continuous, with $\gamma(x) \geq 0$ for all $x \in \partial\Omega$.

(H3) $\omega, T, \alpha, \beta \in \mathbf{R}$ are constants such that $0 < \omega < \infty$, $0 < T < \infty$, $\frac{1}{2} < \alpha < 1$, and $0 \leq \beta < \frac{1}{2}(1-\alpha)$.

(H4) $\boldsymbol{H} \in L^\infty(0,T;[W^{\alpha,2}(\Omega)]^n) \cap W^{1,2}(0,T;[L^2(\Omega)]^n)$.

4.1 Existence and Uniqueness

Our first theorem gives the existence and uniqueness of a mild solution of the initial-value problem (3.13).

Theorem 1 *Let the initial data $(\psi_0, \boldsymbol{A}_0)$ be such that $u_0 = (\psi_0, \boldsymbol{A}'_0) \equiv (\psi_0, \boldsymbol{A}_0 - \boldsymbol{A}_{\mathbf{H}}(0))$ is in $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$. Then the initial-value problem (3.13) has a unique mild solution $u = (\psi, \boldsymbol{A}') \equiv (\psi, \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}})$ such that $u \in C(0,T;\mathcal{W}^{1+\alpha,2})$. The order parameter ψ of this solution satisfies the "maximum modulus principle,"*

$$|\psi(x,t)| \leq \max\left\{1, \|\psi_0\|_{L^\infty(\Omega)}\right\} \quad \textit{for all } (x,t) \in \overline{\Omega} \times [0,T]. \tag{4.1}$$

Also, $(\psi, \boldsymbol{A}) \in W^{1,2}(0,T;\mathcal{L}^2)$ and $\nabla \cdot \boldsymbol{A} \in L^2(0,T;[W^{1,2}(\Omega)]^n)$.

The proof of Theorem 1 is given in Section 5.1.

Observe that the theorem states that $(\psi, \boldsymbol{A}') \in C(0,T;\mathcal{W}^{1+\alpha,2})$. To obtain a comparable result for $(\psi, \boldsymbol{A})$, we need the continuity $\boldsymbol{A}_{\mathbf{H}}$ in time, which, according to Lemma 4, is controlled by the continuity of $\boldsymbol{H}$ in time. In the hypothesis **(H4)**, we have imposed a minimum condition on $\boldsymbol{H}$. If **(H4)** is strengthened to $\boldsymbol{H} \in C(0,T;[W^{\alpha,2}(\Omega)]^n)$, then $(\psi, \boldsymbol{A}) \in C(0,T;\mathcal{W}^{1+\alpha,2})$.

Theorem 1 implies the existence and uniqueness of a weak solution of the gauged TDGL equations.

Corollary 1 *The pair $(\psi, \boldsymbol{A}')$ obtained in Theorem 1 is a weak solution of the gauged TDGL equations; Eqs. (3.5) and (3.6) are satisfied in the $L^2(\Omega\times(0,T))$-sense, Eq. (3.7) in the sense of traces in $L^\infty(0,T;W^{\alpha-1/2,2}(\partial\Omega))$.*

Theorem 1 justifies the introduction of a solution map $S_0 : \mathcal{D}(\mathcal{A}^{(1+\alpha)/2}) \to C(0,T;\mathcal{W}^{1+\alpha,2})$ by the definition

$$u(t) = S_0(t)u_0, \quad u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2}),\ t \in [0,T]. \tag{4.2}$$

The properties of S_0 are considered in more detail in the following section.

4.2 Regularity

The following theorem improves the continuous dependence of the solution u on the initial data u_0. Let the map $S_{\beta'} : \mathcal{D}(\mathcal{A}^{(1+\alpha')/2}) \to C^\beta(0,T;\mathcal{W}^{1+\alpha,2})$ be defined by the identity

$$t^{\beta'}u(t) = S_{\beta'}(t)u_0, \quad u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha')/2}),\ t \in [0,T], \tag{4.3}$$

for suitable exponents α, α', β, and β'.

Theorem 2 *Assume that $\frac{1}{2} < \alpha' \leq \alpha < 1$, $0 \leq \beta < \frac{1}{2}(1-\alpha)$, and $\beta' = \beta + \frac{1}{2}(\alpha - \alpha')$. Then the mapping $S_{\beta'}$ defined in Eq. (4.3) is uniformly Lipschitz continuous on bounded subsets of $\mathcal{D}(\mathcal{A}^{(1+\alpha')/2})$.*

The proof of Theorem 2 is given in Section 5.2.

Observe that the theorem states a regularity result for $(\psi, \boldsymbol{A}')$. To obtain a comparable result for $(\psi, \boldsymbol{A})$, we need sufficient regularity of $\boldsymbol{A}_{\mathbf{H}}$. According to Lemma 4, the regularity of $\boldsymbol{A}_{\mathbf{H}}$ is controlled by the regularity of $\boldsymbol{H}$. In the hypothesis **(H4)**, we have imposed minimum regularity on $\boldsymbol{H}$. If **(H4)** is strengthened to $\boldsymbol{H} \in C^\beta(0,T;[W^{\alpha,2}(\Omega)]^n)$, then $(\psi, \boldsymbol{A}) \in C^\beta(0,T;\mathcal{W}^{1+\alpha,2})$.

Theorem 2 implies the existence of a *dynamical process* for the gauged TDGL equations with a time-dependent applied magnetic field; cf. [25, Section 3.6].

Corollary 2 *The mild solutions $u(t)$ of Eq. (3.13) obtained in Theorem 1 generate a dynamical process $U = \{U(t,s) : 0 \leq s \leq t \leq T\}$ on $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ by the definition*

$$u(t) = U(t,s)u(s) \quad \textit{for } 0 \leq s \leq t \leq T. \tag{4.4}$$

Moreover, for $0 \leq s < t \leq T$, each $U(t,s) : \mathcal{D}(\mathcal{A}^{(1+\alpha)/2}) \to \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ maps bounded sets into relatively compact sets.

4.3 Large-Time Asymptotic Behavior

Next, we investigate the asymptotic behavior of the mild solution $u(t)$ of Eq. (3.13) as $t \to \infty$. We restrict ourselves to the case of a time-independent applied magnetic field $\boldsymbol{H}$.

If $\partial_t \boldsymbol{H} = 0$, the hypothesis **(H4)** reduces to $\boldsymbol{H} \in [W^{\alpha,2}(\Omega)]^n$, the quantity P defined in Eq. (2.24) is zero, and the inequality (2.25) simplifies to

$$E_\omega(t) + 2\int_0^t \int_\Omega \left[\eta \left| \frac{\partial \psi}{\partial t} - i\kappa\omega\psi(\nabla \cdot \boldsymbol{A}) \right|^2 + \left| \frac{\partial \boldsymbol{A}}{\partial t} \right|^2 + \omega^2 \left| \nabla(\nabla \cdot \boldsymbol{A}) \right|^2 \right] \mathrm{d}x\, \mathrm{d}t'$$
$$\leq E_\omega(0), \quad t \in [0,T]. \tag{4.5}$$

The dynamical process $U = \{U(t,s) : 0 \leq s \leq t \leq T\}$ on $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ introduced in Corollary 2 is defined for every $T > 0$ (see Lemma 1) and becomes a *dynamical system* $S = \{S(t) : t \geq 0\}$ on $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ by the definition

$$S(t-s) = U(t,s) \quad \text{for } t \geq s \geq 0. \tag{4.6}$$

Note that the definition of the dynamical system S is not feasible if one imposes the condition $\nabla \cdot \boldsymbol{A}_0 = 0$ at $t = 0$ because the linear space

$$\{\boldsymbol{A} \in [W^{1,2}(\Omega)]^n : \nabla \cdot \boldsymbol{A} = 0 \text{ in } \Omega,\ \boldsymbol{n} \cdot \boldsymbol{A} = 0 \text{ on } \partial\Omega\}$$

is not invariant under the action of S. Consequently, one is not free to choose the initial data χ_0 for the gauge χ, defined in Eqs. (2.11) and (2.12), such that $\Delta\chi_0 = -\nabla\cdot\boldsymbol{A}_0$ in Ω.

The set $\{S(t)u_0 : t \geq 0\}$ is called the *(forward) orbit* of $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ under S. We denote the set of all limit points (as $t \to \infty$) of the orbit of u_0 by $\omega(u_0)$ and call it the *omega-limit set* of u_0.

The following theorem shows that the functional E_ω is a *Liapunov functional* for the dynamical system S in the following sense (cf. [24, Chapter VII, Definition 4.1]): (i) $E_\omega : \mathcal{D}(\mathcal{A}^{(1+\alpha)/2}) \to \mathbf{R}$ is continuous, (ii) for every $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$, the function $t \mapsto E_\omega[S(t)u_0]$ is nonincreasing, and (iii) if $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ is such that $E_\omega[S(t)u_0] = E_\omega[u_0]$ for some $t > 0$, then u_0 is a stationary point for S.

Theorem 3 *The dynamical system S defined in Eq. (4.6) has the following properties:*

(i) *E_ω is a Liapunov functional for S.*

(ii) *The orbit of each $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ has compact closure in $\mathcal{W}^{1+\alpha,2}$.*

(iii) *The omega-limit set of each $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ is a nonempty compact connected set of divergence-free equilibria.*

The proof of Theorem 3 is given in Section 5.3.

Property (iii) of Theorem 3 says, in effect, that every element of any omega-limit set is a solution of the time-independent GL equations (2.31)–(2.33) in the London gauge.

An *attractor* for the dynamical system S is the omega-limit set of one of its open neighborhoods. An attractor is called a *global attractor* if it attracts all its open bounded neighborhoods. The existence of a global attractor for the dynamical system S follows from Corollary 2 and Theorem 3; see [25, Theorem 3.4.8] and [19, Theorem 4.4]. The structure of the global attractor follows from Theorem 3; see [24, Chapter VII, Theorem 4.1].

Corollary 3 *The dynamical system S has a global attractor, $\mathcal{A}$. If the set*

$\mathcal{E}$ of all stationary points of S is discrete, then $\mathcal{A}$ is the union of $\mathcal{E}$ and the heteroclinic orbits between points of $\mathcal{E}$.

4.4 Zero-Electric Potential Gauge

Beginning with the functional formulation of the gauged TDGL equations in Section 3.3, we restricted the parameter ω in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge to positive values. If $\omega = 0$, the quadratic form $Q_\omega[\psi, \boldsymbol{A}'] + c\|\psi\|_{L^2}$ is no longer coercive on $\mathcal{W}^{1,2}$ for any constant $c > 0$, because $Q_0[0, \nabla\chi] = 0$ for any $\chi \in W^{2,2}(\Omega)$ satisfying $\boldsymbol{n} \cdot \nabla\chi = 0$ on $\partial\Omega$. The initial-value problem (3.13) is degenerate, and much of the regularity of its solution is lost. This loss is evident when the solution $(\psi, \boldsymbol{A}, \phi)$ of the TDGL equations, with $\phi = -\omega(\nabla \cdot \boldsymbol{A})$ and $\omega > 0$, is transformed to its gauge-equivalent form in the zero-electric potential gauge. The gauge χ that accomplishes this transformation is found by integrating the equation $\partial_t\chi = -\omega(\nabla \cdot \boldsymbol{A})$ from an initial condition $\chi = \chi_0$. The resulting expression for the vector potential is

$$\boldsymbol{A}(t) + \nabla\chi_0 - \omega \int_0^t \nabla(\nabla \cdot \boldsymbol{A})(t')\,\mathrm{d}t' \quad \text{for } t \geq 0.$$

Since $\nabla \cdot \boldsymbol{A}(t) \in [W^{1,2}(\Omega)]^n$ for every $t \geq 0$ (see Theorem 1), the time integral is only in $[L^2(\Omega)]^n$ for any fixed t.

5 Proofs

In this section we give the proofs of the theorems presented in the preceding section. We begin by recalling some general properties of the fractional powers of the operator $\mathcal{A}$ defined in Eq. (3.12); see [18, Section 1.4] for details.

The fractional powers $\mathcal{A}^\theta$ of the second-order elliptic differential operator $\mathcal{A}$ defined in Eq. (3.12) are well defined for all $\theta \in \mathbf{R}$. They are unbounded for $\theta > 0$. The domain $\mathcal{D}(\mathcal{A}^\theta)$ is a closed linear subspace of $\mathcal{W}^{2\theta,2}$ for $0 < \theta < 1$; hence, $C^\beta(0, T; \mathcal{D}(\mathcal{A}^\theta))$ is a closed linear subspace of $C^\beta(0, T; \mathcal{W}^{2\theta,2})$ for this range of values of θ. Furthermore, for $\frac{3}{2} < \theta \leq 2$ (and $n = 2$ or 3), the traces of $\nabla\psi$, $\boldsymbol{A}$, and $\nabla \times \boldsymbol{A}$ belong to the spaces $[W^{\theta-3/2,2}(\partial\Omega)]^{2n}$, $[W^{\theta-1/2,2}(\partial\Omega)]^n$, and $[W^{\theta-3/2,2}(\partial\Omega)]^n$, respectively, and satisfy the boundary conditions specified in Eqs. (3.15) and (3.16). Similarly, the applied vector potential $\boldsymbol{A}_\mathbf{H}$ and its curl $\nabla \times \boldsymbol{A}_\mathbf{H}$ satisfy the boundary conditions (3.3) if $\boldsymbol{H} \in [W^{\theta-1,2}(\Omega)]^n$.

5.1 Proof of Theorem 1

Proof. (i) *Local existence and uniqueness.* The proof is based on the contraction mapping principle applied to Eq. (3.14) in the space $C(0,T;\mathcal{W}^{1+\alpha,2})$ for T sufficiently small positive. The choice of the target space $\mathcal{W}^{1+\alpha,2}$ is justified because $\mathcal{W}^{1+\alpha,2}$ is continuously imbedded in $\mathcal{W}^{1,2}\cap\mathcal{L}^\infty$ for $\frac{1}{2}<\alpha<1$.

It suffices to prove that $\mathcal{F}(s,\cdot)$ is locally Lipschitz for each $s\in[0,T]$, where T may depend on the Lipschitz constant. Each term in $\mathcal{F}$ is estimated separately. For example, for any two elements $u_1=(\psi_1,\boldsymbol{A}_1')$ and $u_2=(\psi_2,\boldsymbol{A}_2')$ of $\mathcal{W}^{1+\alpha,2}$, we have

$$\begin{aligned}\|\psi_1^*\nabla\psi_1-\psi_2^*\nabla\psi_2\|_{L^2} &\le \|\psi_1\|_{L^\infty}\|\psi_1-\psi_2\|_{W^{1,2}}+\|\psi_2\|_{W^{1,2}}\|\psi_1-\psi_2\|_{L^\infty}\\ &\le C\|u_1-u_2\|_{\mathcal{W}^{1+\alpha,2}},\end{aligned}$$

where C is a positive constant, which depends only on the norms of u_1 and u_1 in $\mathcal{W}^{1+\alpha,2}$. Similar estimates hold for the other terms in $\mathcal{F}$.

Let B_R be the ball of radius R centered at the origin in $\mathcal{W}^{1+\alpha,2}$. Then, for any pair $u_1,u_2\in B_R$,

$$\|\mathcal{F}(s,u_1)-\mathcal{F}(s,u_2)\|_{\mathcal{L}^2}\le C\|u_1-u_2\|_{\mathcal{W}^{1+\alpha,2}},\quad s\in[0,T], \tag{5.1}$$

where the Lipschitz constant C depends on R, but not on s. The remainder of the proof is standard; see [18, Theorem 3.3.3].

(ii) *Global existence.* The maximum modulus principle (4.1) is a consequence of the maximum principle applied to Eq. (2.5). (Every constant M with $M\ge 1$ is a supersolution of Eq. (2.5).)

The functional $E_\omega[\psi,\boldsymbol{A}]$ defined in Eq. (2.22) is coercive on $\mathcal{W}^{1,2}$; see [22, Chapter I, Eq. (5.45)]. Given a weak solution $(\psi,\boldsymbol{A}')=(\psi,\boldsymbol{A}-\boldsymbol{A}_{\mathbf{H}})$ of the gauged TDGL equations, we let $E_\omega(t)\equiv E_\omega[\psi(t),\boldsymbol{A}(t)]$. The function E_ω is bounded on every interval $[0,T]$, according to Lemma 1. Its coercivity property then implies

$$\psi\in L^\infty(0,T;[W^{1,2}(\Omega)]^2)\quad\text{and}\quad \boldsymbol{A}\in L^\infty(0,T;[W^{1,2}(\Omega)]^n).$$

Also, $\boldsymbol{A}_{\mathbf{H}}\in L^\infty(0,T;[W^{1,2}(\Omega)]^n)$, because of the hypothesis **(H4)**. Hence, $u=(\psi,\boldsymbol{A}')\in L^\infty(0,T;\mathcal{W}^{1,2})$.

It follows from the inequality (2.29) that $(\psi,\boldsymbol{A})\in W^{1,2}(0,T;\mathcal{L}^2)$ and $\nabla\cdot\boldsymbol{A}\in L^2(0,T;[W^{1,2}(\Omega)]^n)$. We also have $\boldsymbol{A}_{\mathbf{H}}\in W^{1,2}(0,T;[L^2(\Omega)]^n)$, again because of the hypothesis **(H4)**. Therefore, $u\in W^{1,2}(0,T;\mathcal{L}^2)$.

We improve this regularity result by taking advantage of the smoothing action of the semigroup $\mathrm{e}^{-\mathcal{A}t}$. This smoothing action has already been demonstrated on the term $\partial_t \boldsymbol{A}_{\mathbf{H}}$ in Section 3.4. We first treat $\boldsymbol{A}'$ and then use the result to improve the regularity of ψ. Each term in $\boldsymbol{J}'_s$ needs to be estimated separately. For example,

$$\|\psi^* \nabla \psi\|_{L^2} \leq \|\psi\|_{L^\infty} \|\psi\|_{W^{1,2}} \leq C \|u\|_{\mathcal{W}^{1,2}}.$$

Here, $C = \max\{1, \|\psi_0\|_{L^\infty}\}$, which is independent of ψ. Similar estimates hold for the other terms in $\boldsymbol{J}'_s$, so $\boldsymbol{J}'_s \in L^\infty(0,T;[L^2(\Omega)]^n)$. Therefore,

$$\left(t \mapsto \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)} \boldsymbol{F}(s)\,\mathrm{d}s\right) \in C(0,T;[W^{1+\alpha,2}(\Omega)]^n),$$

so $\boldsymbol{A}' \in C(0,T;[W^{1+\alpha,2}(\Omega)]^n)$.

Next, we improve the regularity of ψ. Again, each term in φ needs to be estimated separately. For example,

$$\|(\nabla\psi)\cdot(\boldsymbol{A}_{\mathbf{H}} + \boldsymbol{A}')\|_{L^2} \leq \|(\nabla\psi)\cdot \boldsymbol{A}_{\mathbf{H}}\|_{L^2} + \|(\nabla\psi)\cdot\boldsymbol{A}'\|_{L^2},$$

where

$$\|(\nabla\psi)\cdot \boldsymbol{A}_{\mathbf{H}}\|_{L^2} \leq \|\nabla\psi\|_{L^2}\|\boldsymbol{A}_{\mathbf{H}}\|_{L^\infty} \leq C\|u\|_{\mathcal{W}^{1,2}}\|\boldsymbol{A}_{\mathbf{H}}\|_{W^{1+\alpha,2}}$$

and

$$\|(\nabla\psi)\cdot \boldsymbol{A}'\|_{L^2} \leq \|\nabla\psi\|_{L^2}\|\boldsymbol{A}'\|_{L^\infty} \leq C\|u\|_{\mathcal{W}^{1,2}}\|\boldsymbol{A}'\|_{W^{1+\alpha,2}}.$$

(To obtain the last estimate, we used the Sobolev imbedding theorem.) Similar estimates hold for the other terms in φ, so $\varphi \in L^\infty(0,T;[L^2(\Omega)]^2)$ and, therefore, $\psi \in C(0,T;[W^{1+\alpha,2}(\Omega)]^2)$. It follows that $u \in C(0,T;\mathcal{W}^{1+\alpha,2})$, as claimed. ∎

5.2 Proof of Theorem 2

Proof. We use Eq. (3.14) to prove the regularity of the solution u of the initial-value problem (3.13).

Let B_R be the ball of radius R centered at the origin in $\mathcal{W}^{1+\alpha,2}$. Let $u_1 \in B_R$ and $u_2 \in B_R$ satisfy Eq. (3.14) with initial data u_{10} and u_{20}, respectively. Define $v = u_1 - u_2$ and $v_0 = u_{10} - u_{20}$. Combining the inequality (5.1) with Eq. (3.14), we obtain

$$\|v(t)\|_{\mathcal{W}^{1+\alpha,2}} \leq \|\mathrm{e}^{-\mathcal{A}t}\|_{\mathcal{W}^{1+\alpha,2}}\|v_0\|_{\mathcal{W}^{1+\alpha,2}}$$

$$+C\int_0^t \|\mathcal{A}^{(1+\alpha)/2}\mathrm{e}^{-\mathcal{A}(t-s)}\|_{\mathcal{W}^{1+\alpha,2}}\|v(s)\|_{\mathcal{W}^{1+\alpha,2}}\,\mathrm{d}s. \tag{5.2}$$

Applying Gronwall's inequality, we find

$$\|v(t)\|_{\mathcal{W}^{1+\alpha,2}} \leq C\|v_0\|_{\mathcal{W}^{1+\alpha,2}}, \quad 0 \leq t \leq T, \tag{5.3}$$

so the mapping S_0 defined in Eq. (4.2) is Lipschitz continuous on B_R.

Set $f(s) = \mathcal{F}(s, u_1(s)) - \mathcal{F}(s, u_2(s))$. Then, for $0 \leq t < t' \leq T$,

$$v(t') - v(t) = \left(\mathrm{e}^{-\mathcal{A}(t'-t)} - I\right)\mathrm{e}^{-\mathcal{A}t}v_0 + \int_0^{t'-t}\mathrm{e}^{-\mathcal{A}s}f(t'-s)\,\mathrm{d}s$$

$$+\left(\mathrm{e}^{-\mathcal{A}(t'-t)} - I\right)\int_0^t \mathrm{e}^{-\mathcal{A}(t-s)}f(s)\,\mathrm{d}s.$$

Taking α, α', β, and β' subject to the conditions of the theorem, we obtain

$$\mathcal{A}^{(1+\alpha)/2}(v(t') - v(t))$$

$$= \left(\mathrm{e}^{-\mathcal{A}(t'-t)} - I\right)\mathcal{A}^{-\beta}\mathcal{A}^{\beta'}\mathrm{e}^{-\mathcal{A}t}\mathcal{A}^{(1+\alpha')/2}v_0 + \int_0^{t'-t}\mathcal{A}^{(1+\alpha)/2}\mathrm{e}^{-\mathcal{A}s}f(t'-s)\,\mathrm{d}s$$

$$+\left(\mathrm{e}^{-\mathcal{A}(t'-t)} - I\right)\mathcal{A}^{-\beta}\int_0^t \mathcal{A}^{\beta+(1+\alpha)/2}\mathrm{e}^{-\mathcal{A}(t-s)}f(s)\,\mathrm{d}s.$$

Using the inequalities (3.18) and (3.19), we deduce the estimates

$$\|\mathcal{A}^{(1+\alpha)/2}(v(t') - v(t))\|_{\mathcal{W}^2} \leq C_1(t'-t)^{\beta}t^{-\beta'}\|\mathcal{A}^{(1+\alpha')/2}v_0\|_{\mathcal{L}^2}$$

$$+C_2\left((t'-t)^{(1-\alpha)/2} + (t'-t)^{\beta}t^{(1-\alpha)/2-\beta}\right)\operatorname{ess\,sup}\{\|f(s)\|_{\mathcal{L}^2} : 0 < s < T\}$$

$$\leq C(t'-t)^{\beta}t^{-\beta'}\left(\|v_0\|_{\mathcal{W}^{1+\alpha',2}} + C\sup\{\|v(s)\|_{\mathcal{W}^{1,2}} : 0 < s < T\}\right).$$

But, as we have seen, the solution map S_0 defined in (4.2) is Lipschitz continuous, so $\sup\{\|v(s)\|_{\mathcal{W}^{1,2}} : 0 < s < T\} \leq C\|v_0\|_{\mathcal{W}^{1,2}}$. Therefore, the mapping (4.3) is Lipschitz continous, as claimed. ∎

5.3 Proof of Theorem 3

Proof. (i) The continuity of the functional E_ω follows from the continuous imbedding of $\mathcal{W}^{1+\alpha,2}$ into $\mathcal{W}^{1,2} \cap \mathcal{L}^\infty$. The identity (2.23) shows that the function $t \mapsto E_\omega[S(t)u_0]$ is nonincreasing, for every $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$.

Let $u_0 = (\psi, \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}}) \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ be such that $E_\omega[S(t)u_0] = E_\omega[u_0]$ for some $t > 0$. From the inequality (4.5), we obtain immediately the identities

$\partial_t \boldsymbol{A} = \mathbf{0}$ and $\omega\nabla(\nabla \cdot \boldsymbol{A}) = \mathbf{0}$ in $\Omega \times (0,t)$. The first identity implies that $\partial_t(\nabla \cdot \boldsymbol{A}) = 0$ in $\Omega \times (0,t)$. From this and the second identity we deduce that $\omega(\nabla \cdot \boldsymbol{A}) = c$ in $\Omega \times (0,t)$, where c is a real constant. We conclude from Eq. (2.19) that $\nabla \cdot \boldsymbol{J}_s = 0$. Also, the inequality (4.5) implies $\partial_t \psi = i\kappa c \psi$ in $[L^2(\Omega \times (0,t))]^2$, so Eq. (2.20) reduces to $c|\psi|^2 = 0$ in $\Omega \times (0,t)$. We claim that $c = 0$.

Suppose $c \neq 0$. Then it must be the case that $\psi = 0$ in $\Omega \times (0,t)$. Equations (2.16)–(2.17) reduce to the boundary-value problem (3.2)–(3.3) for $\boldsymbol{A}_{\mathbf{H}}$. Therefore, $\boldsymbol{A} = \boldsymbol{A}_{\mathbf{H}}$ and $\boldsymbol{A}' = \mathbf{0}$ in $\Omega \times (0,t)$, so $c = \omega(\nabla \cdot \boldsymbol{A}_{\mathbf{H}}) = 0$, and we have a contradiction.

The identity $\partial_t \psi = 0$ in $\Omega \times (0,t)$, together with the identity $\partial_t \boldsymbol{A} = \mathbf{0}$ established above, implies that $S(t')u_0 = u_0$ for all $t' \in (0,t)$.

(ii) An immediate consequence of Corollary 2.

(iii) It follows from (ii) that the omega-limit set of each $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ is nonempty and compact. We prove by contradiction that $\omega(u_0)$ is connected. Suppose $\omega(u_0)$ is not connected. Then $\omega(u_0) = K_1 \cup K_2$, where K_1 and K_2 are compact and disjoint. Hence, there exist two disjoint open neighborhoods N_1 and N_2 of K_1 and K_2, respectively, in $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ and $t_0 \geq 0$, such that $S(t)u_0 \in N_1 \cup N_2$ for all $t \geq t_0$. But $\{S(t)u_0 : t \geq t_0\}$, being the image of the interval $[t_0, \infty)$, is connected, so we have a contradiction.

The proof that all points of the omega-limit set of u_0 are equilibrium points is standard; cf. [24, Chapter VII, Proof of Theorem 4.1].

If $w = (\psi, \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}}) \in \omega(u_0)$, then $E_\omega[S(t)w] = E_\omega[w]$ for all $t > 0$, and the same argument as in (i) above leads to the conclusion that $\omega(\nabla \cdot \boldsymbol{A}) = 0$ in Ω. Since $\omega > 0$, $\boldsymbol{A}$ must be divergence free. ∎

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