AN EQUIVALENCE RELATION FOR THE GINZBURG-LANDAU EQUATIONS OF SUPERCONDUCTIVITY

Hans G. Kaper[1] and Peter Takáč[2]

[1] Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL 60439, USA (`kaper@mcs.anl.gov`)
[2] Fachbereich Mathematik, Universität Rostock, Universitätsplatz 1, D-18055 Rostock, Germany (`peter.takac@mathematik.uni-rostock.de`)

Abstract. Gauge invariance is used to establish an equivalence relation between solutions of the time-independent and time-dependent Ginzburg-Landau equations that describe the same physical state of a superconductor. The equivalence relation shows how equilibrium configurations are obtained as large-time asymptotic limits of solutions of the time-dependent Ginzburg-Landau equations.

Keywords. Superconductivity, Ginzburg-Landau equations, gauge invariance, large-time asymptotic behavior.

1991 Mathematics Subject Classification. Primary 35K55. Secondary 35B40, 35J65, 35K60, 82D55.

1 Introduction

The physical state of a superconductor at equilibrium corresponds to a critical point of the free-energy functional. In the framework of the Ginzburg-Landau (GL) theory, the state is represented by a solution of the time-independent GL equations of superconductivity. By applying a time-dependent gauge transformation, one can "lift" this representation to a solution of the time-dependent GL (TDGL) equations (which constitute a nontrivial generalization of the time-independent GL equations). The state described by this solution, though explicitly dependent on time, is physically indistinguishable from the equilibrium state. In this article, we address the converse problem. Our objective is to compare the solutions of the time-independent GL equations with those solutions of the TDGL equations that represent the same stationary physical state. The comparison establishes an equivalence relation between solutions of the time-independent and time-dependent GL equations. In particular, a solution of the TDGL equations in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge, which is gauge equivalent with a stationary physical state, tends to a solution of the time-independent GL equation in the London "$\nabla \cdot \boldsymbol{A} = 0$" gauge as time goes to infinity, for all $\omega > 0$. The

"$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge generalizes the classical "$\phi = -\nabla \cdot \boldsymbol{A}$" gauge and reduces to the zero-electric potential gauge ($\phi = 0$) in the limit $\omega = 0$.

In Section 2, we briefly summarize the Ginzburg-Landau model of superconductivity and the gauge invariance properties of the time-independent and time-dependent GL equations. In Section 3, we define the functional framework for the GL equations. In Section 4, we establish the equivalence relation. In Section 5, we analyze the large-time asymptotic limit of a solution of the TDGL equations in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge.

2 Ginzburg-Landau Model of Superconductivity

In the Ginzburg-Landau theory of phase transitions [1], the state of a superconductor is described by a complex-valued *order parameter* ψ and a real vector-valued *vector potential* $\boldsymbol{A}$. The order parameter can be thought of as the wave function for the center-of-mass motion of the "superelectrons" (Cooper pairs), whose density is $n_s = |\psi|^2$. The vector potential determines the magnetization, which is the difference between the *induced* magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and the *applied* magnetic field $\boldsymbol{H}$.

An equilibrium state corresponds to a critical point of the Helmholtz free-energy functional. In the Ginzburg-Landau theory, this functional is given by the expression

$$E_0[\psi, \boldsymbol{A}] = \int_\Omega \left[\left| \left(\frac{i}{\kappa} \nabla + \boldsymbol{A} \right) \psi \right|^2 + \frac{1}{2} \left(1 - |\psi|^2 \right)^2 + |\nabla \times \boldsymbol{A} - \boldsymbol{H}|^2 \right] \mathrm{d}x + \int_{\partial\Omega} \gamma \left| \frac{i}{\kappa} \psi \right|^2 \mathrm{d}\sigma(x). \tag{2.1}$$

Here, Ω is the region occupied by the superconductor; we assume that Ω is a bounded domain in $\mathbf{R}^n$ ($n = 2$ or $n = 3$), with boundary $\partial\Omega$. The vector potential $\boldsymbol{A}$ and the applied magnetic field $\boldsymbol{H}$ take their values in $\mathbf{R}^n$. The (dimensionless) Ginzburg-Landau parameter κ is the ratio of the characteristic length scales for the vector potential and the order parameter. The function γ is defined on $\partial\Omega$, and $\gamma(x) \geq 0$ for $x \in \partial\Omega$. As usual, $\nabla \equiv$ grad, $\nabla \times \equiv$ curl, $\nabla \cdot \equiv$ div, and $\nabla^2 = \nabla \cdot \nabla \equiv \Delta$; i is the imaginary unit, and a superscript * denotes complex conjugation.

A critical point of $E_0 \equiv E_0[\psi, \boldsymbol{A}]$ is a solution of the boundary-value problem

$$-\left(\frac{i}{\kappa} \nabla + \boldsymbol{A} \right)^2 \psi + \left(1 - |\psi|^2 \right) \psi = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$-\nabla \times \nabla \times \boldsymbol{A} + \boldsymbol{J}_s + \nabla \times \boldsymbol{H} = \boldsymbol{0} \quad \text{in } \Omega, \tag{2.3}$$

$$\boldsymbol{n} \cdot \left(\frac{i}{\kappa} \nabla + \boldsymbol{A} \right) \psi + \gamma \frac{i}{\kappa} \psi = 0 \quad \text{and} \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A} - \boldsymbol{H}) = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{2.4}$$

Here, $\boldsymbol{n}$ is the local outer unit normal to $\partial\Omega$. The vector $\boldsymbol{J}_s$ is the *supercurrent density*, which is a nonlinear function of ψ and $\boldsymbol{A}$,

$$\boldsymbol{J}_s \equiv \boldsymbol{J}_s(\psi, \boldsymbol{A}) = \frac{1}{2i\kappa}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) - |\psi|^2 \boldsymbol{A} = -\mathrm{Re}\left[\psi^* \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)\psi\right]. \tag{2.5}$$

We refer to the system of Eqs. (2.2)–(2.4) as the *time-independent* Ginzburg-Landau (GL) equations. The trivial solution, $\psi = 0$ and $\boldsymbol{B} = \boldsymbol{H}$, represents the *normal state*, where all superconducting properties have been lost. The GL equations embody, in a most simple way, the macroscopic quantum-mechanical nature of the superconducting state; see [2, 3, 4].

The time-independent GL equations are invariant under a *gauge transformation*,

$$G_\chi : (\psi, \boldsymbol{A}) \mapsto \left(\psi \mathrm{e}^{i\kappa\chi}, \boldsymbol{A} + \nabla\chi\right). \tag{2.6}$$

The *gauge* χ can be any (sufficiently smooth) real scalar-valued function of position. Solutions that are related through a gauge transformation are physically indistinguishable, because they result in identical observable quantities (the current $\boldsymbol{J} = \boldsymbol{J}_s + \nabla \times \boldsymbol{H}$ and the induced magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$). Mathematically, gauge invariance reflects a nonuniqueness of the solution of the GL equations. Uniqueness is achieved by choosing a gauge and thus constraining the solution to a specified manifold in the solution space. A common choice is the London gauge [4, Chapter 4], where χ is a solution of Poisson's equation, $\Delta\chi = -\nabla \cdot \boldsymbol{A}$ in Ω, satisfying the Neumann condition $\boldsymbol{n} \cdot \nabla\chi = -\boldsymbol{n} \cdot \boldsymbol{A}$ on $\partial\Omega$. In this gauge, the vector potential is divergence free in Ω and tangential at the boundary $\partial\Omega$. In general, one chooses a gauge to suit the problem under investigation; see [5] for an example in a two-dimensional domain with periodic boundary conditions.

A generalization of the time-independent GL equations to the time-dependent case was first proposed by Schmid [6]. The generalization was subsequently analyzed by Gor'kov and Eliashberg [7] in the context of the microscopic Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity and validated in a narrow range of temperatures near the critical temperature. Because of gauge invariance, the generalization is nontrivial.

In addition to the order parameter and the vector potential, a third variable is needed to complete the description of the physical state of the system in a manner consistent with the gauge invariance. This is the *electric potential*, ϕ, a real scalar-valued function of position and time. It is a diagnostic variable, as opposed to the prognostic variables ψ and $\boldsymbol{A}$. The evolution of ψ and $\boldsymbol{A}$ is described by the equations

$$\eta\left(\frac{\partial}{\partial t} + i\kappa\phi\right)\psi = -\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \psi + \left(1 - |\psi|^2\right)\psi \quad \text{in } \Omega \times (0, \infty), \tag{2.7}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\phi = -\nabla \times \nabla \times \boldsymbol{A} + \boldsymbol{J}_s + \nabla \times \boldsymbol{H} \quad \text{in } \Omega \times (0, \infty), \tag{2.8}$$

$$\boldsymbol{n} \cdot \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)\psi + \frac{i}{\kappa}\gamma\psi = 0 \quad \text{and} \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A} - \boldsymbol{H}) = \boldsymbol{0} \quad \text{on } \partial\Omega \times (0, \infty). \tag{2.9}$$

We refer to the system of Eqs. (2.7)–(2.9) as the *time-dependent* Ginzburg-Landau (TDGL) equations.

The TDGL equations also have an invariance property, but the situation is more complicated because of the presence of the electric potential. The TDGL equations are invariant under a gauge transformation

$$G_\chi : (\psi, \boldsymbol{A}, \phi) \mapsto \left(\psi \mathrm{e}^{i\kappa\chi}, \boldsymbol{A} + \nabla\chi, \phi - \partial_t \chi\right), \tag{2.10}$$

where the gauge χ can be any (sufficiently smooth) real scalar-valued function of position and time. One readily verifies that two states that, at any time t, are related by a gauge transformation of the type (2.10) give identical values for the physically observable quantities (the current $\boldsymbol{J} = \boldsymbol{J}_s + \nabla \times \boldsymbol{H}$, the induced magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and the electric field $\boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla\phi$). A gauge choice is necessary to eliminate the lack of uniqueness.

Various gauges have been proposed for the TDGL equations. In the zero-electric potential gauge, one chooses χ so the electric potential vanishes identically in Ω at all times. On the other hand, in the London gauge one chooses χ so the vector potential is divergence free in Ω and tangential at $\partial\Omega$ at all times. It is not possible to satisfy both the zero-electric potential gauge and the London gauge in Ω simultaneously at all times, but one can choose χ so $\phi = -\nabla \cdot \boldsymbol{A}$ in Ω at all times. Again, the particular choice is made to suit the problem under investigation. For example, Du [8] chose the zero-electric potential, while Chen, Hoffmann, and Liang [9] used the "$\phi = -\nabla \cdot \boldsymbol{A}$" gauge to prove existence and uniqueness for the TDGL equations. Liang and Tang [10] and Tang and Wang [11], who studied the dynamics of the TDGL equations, both chose the London gauge, but neither addressed the large-time asymptotic behavior. Consequently, there is no indication how solutions of the TDGL equations relate to solutions of the time-independent GL equations. In fact, we doubt that this problem can be studied in the London gauge. As Takáč [12] first showed, a gauge like the "$\phi = -\nabla \cdot \boldsymbol{A}$" gauge is more appropriate for the study of the dynamics and the large-time asymptotics of the TDGL equations.

In [13], we introduced the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge, which is defined by the solution $\chi \equiv \chi_\omega(x,t)$ of the boundary-value problem

$$(\partial_t - \omega\Delta)\chi = \phi + \omega(\nabla \cdot \boldsymbol{A}) \quad \text{in } \Omega \times (0,\infty), \tag{2.11}$$

$$\omega(\boldsymbol{n} \cdot \nabla\chi) = -\omega(\boldsymbol{n} \cdot \boldsymbol{A}) \quad \text{on } \partial\Omega \times (0,\infty). \tag{2.12}$$

Here, ω is a real nonnegative parameter. This gauge generalizes the "$\phi = -\nabla \cdot \boldsymbol{A}$" gauge and reduces to the zero-electric potential gauge in the limit $\omega = 0$. It results in the constraints

$$\phi = -\omega(\nabla \cdot \boldsymbol{A}) \quad \text{in } \Omega \times (0,\infty), \quad \omega(\boldsymbol{n} \cdot \boldsymbol{A}) = 0 \quad \text{on } \partial\Omega \times (0,\infty). \tag{2.13}$$

The zero-electric potential gauge presents a somewhat exceptional case. In this gauge, one obtains the TDGL equations from the time-independent GL equations by simply adding the time derivatives of the order parameter and vector potential to the respective differential equations. In this sense, the TDGL equations represent a trivial lifting of the time-independent GL equations into the time-dependent domain. However, the analysis of the TDGL equations in the zero-electric potential gauge is complicated by the fact that the expression $-\nabla \times \nabla \times$ does not define a uniformly elliptic operator. Consequently, the solution lacks regularity, and the standard techniques to prove existence and uniqueness do not apply. For these reasons we exclude this case from further consideration and consider only the case $\omega > 0$.

The TDGL equations in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge define a *dynamical process* in the general case where the applied magnetic field $\boldsymbol{H}$ varies with time; if $\boldsymbol{H}$ is time independent, the dynamical process becomes a *dynamical system*. In the following analysis, we restrict ourselves to the latter case, so the TDGL equations define a dynamical system and every solution is attracted to a set of stationary solutions, which are divergence free [14].

3 Functional Formulation

In this section, we establish a functional framework for the time-independent and time-dependent Ginzburg-Landau equations.

We briefly explain our notation. All Banach spaces are taken to be real. Complex-valued functions are interpreted as vector-valued functions with two real components. The Banach spaces in this investigation are the standard ones (Lebesgue, Sobolev, etc.); definitions are given in [14]. Functions of space and time are considered as mappings from the time domain $[0, \infty)$ into a Banach space of functions on the spatial domain Ω.

The framework for the time-independent GL equations is a function space $\mathcal{X}$ of ordered pairs $(\psi, \boldsymbol{A})$, where the first element is a complex-valued (that is, a two-dimensional real vector-valued) function and the second an n-dimensional real vector-valued function (with $n = 2$ or $n = 3$), both defined on Ω. The regularity requirements for ψ and $\boldsymbol{A}$ are the same, so $\mathcal{X}$ is of the form $\mathcal{X} = X^2 \times X^n$, where X is a space of real scalar-valued functions. A suitable framework for the functional analysis of the time-independent GL equations is

$$\mathcal{W}^{1+\alpha,2} \equiv [W^{1+\alpha,2}(\Omega)]^2 \times [W^{1+\alpha,2}(\Omega)]^n,$$

with $\frac{1}{2} < \alpha < 1$. This space is continuously imbedded in $\mathcal{W}^{1,2} \cap \boldsymbol{L}^{\infty}$. A *weak solution* of the time-independent GL equations is a function $(\psi, \boldsymbol{A}) \in \mathcal{W}^{1+\alpha,2}$ that satisfies Eqs. (2.2)–(2.4) in the sense of distributions.

The TDGL equations give rise to an abstract initial-value problem for the function

$$(\psi, \boldsymbol{A}, \phi) : \ [0, \infty) \to \mathcal{W}^{1,2} \times L^2$$

in $\mathcal{W}^{1,2}\times L^2$. A *weak solution* of the TDGL equations on the interval $(0,T)$, for some $T>0$, is a function $(\psi, \boldsymbol{A}, \phi) \in C(0,T;\mathcal{W}^{1+\alpha,2}\times L^2)$, with values $(\psi,\boldsymbol{A},\phi)(t) \equiv (\psi(t),\boldsymbol{A}(t),\phi(t)) \in \mathcal{W}^{1+\alpha,2}\times L^2$, which satisfies Eqs. (2.7)–(2.9) in the sense of distributions for each $t\in(0,T)$.

The functional analog of the gauge transformation (2.6) is a continuous affine transformation G_χ of $\mathcal{W}^{1,2}$ into itself, which is defined in terms of a gauge $\chi \in W^{2,2}(\Omega)$,

$$(G_\chi(\psi,\boldsymbol{A}))(x) = \left(\psi(x)\mathrm{e}^{i\kappa\chi(x)}, \boldsymbol{A}(x)+\nabla\chi(x)\right), \quad x\in\Omega,\ (\psi,\boldsymbol{A})\in\mathcal{W}^{1,2}. \tag{3.1}$$

It has the property $G_\chi G_{-\chi} = G_{-\chi}G_\chi = I$, where I is the identity mapping in $\mathcal{W}^{1,2}$. Two states $(\psi,\boldsymbol{A})\in\mathcal{W}^{1,2}$ and $(\psi',\boldsymbol{A}')\in\mathcal{W}^{1,2}$ are *gauge equivalent* (and physically indistinguishable) if there exists a gauge χ such that $(\psi',\boldsymbol{A}') = G_\chi(\psi,\boldsymbol{A})$ in $\mathcal{W}^{1,2}$.

The generalization to a time-dependent gauge is obtained by considering χ as a mapping from the time domain into an appropriate space of functions on Ω. Specifically, we take $\chi \in L^2_{\text{loc}}([0,\infty);W^{2,2}(\Omega)) \cap W^{1,2}_{\text{loc}}([0,\infty);L^2(\Omega))$. Thus, the gauge transformation (2.10) becomes a mapping G_χ from $[0,\infty)$ into the space of continuous affine mappings of $\mathcal{W}^{1,2}\times L^2$ into itself,

$$(G_\chi(\psi,\boldsymbol{A},\phi))(t) \equiv G_{\chi(t)}(\psi,\boldsymbol{A},\phi)(t), \quad t\geq 0,\ (\psi,\boldsymbol{A},\phi)\in\mathcal{W}^{1,2}\times L^2, \tag{3.2}$$

where $\chi(t)\equiv\chi(\cdot\,,t)$ and

$$\left(G_{\chi(t)}(\psi,\boldsymbol{A},\phi)(t)\right)(x) = \left(\psi(x,t)\mathrm{e}^{i\kappa\chi(x,t)}, \boldsymbol{A}(x,t)+\nabla\chi(x,t), \phi(x,t)-\partial_t\chi(x,t)\right), \quad x\in\Omega.$$

Obviously, $G_{\chi(t)}G_{-\chi(t)} = G_{-\chi(t)}G_{\chi(t)} = I$, for each t. Note that gauge equivalence is a property shared by two states at a *fixed* instance. If the gauge equivalence holds at all times $t\geq 0$ (with respect to the same gauge χ), we may write $(\psi',\boldsymbol{A}',\phi') = G_\chi(\psi,\boldsymbol{A},\phi)$.

Unless indicated otherwise, we assume that the data satisfy the following conditions: $\Omega\subset\mathbf{R}^n$ ($n=2$ or $n=3$) is bounded, with $\partial\Omega$ of class $C^{1,1}$ (that is, $\partial\Omega$ is a compact $(n-1)$-manifold described by Lipschitz-continuously differentiable charts); $\gamma:\partial\Omega\to\mathbf{R}$ is Lipschitz continuous, with $\gamma(x)\geq 0$ for all $x\in\partial\Omega$; ω is a constant, $\omega>0$; and $\boldsymbol{H}\in[L^2(\Omega)]^n$.

4 Equivalence Relation

To motivate the discussion, assume that $(\psi_0,\boldsymbol{A}_0)\in\mathcal{W}^{1,2}$ is a weak solution of the time-independent GL equations (2.2)–(2.4). We extend $(\psi_0,\boldsymbol{A}_0)$ trivially to $(\psi_0,\boldsymbol{A}_0,0)\in\mathcal{W}^{1,2}\times L^2$ and define the function $(\psi,\boldsymbol{A},\phi):[0,\infty)\to\mathcal{W}^{1,2}\times L^2$ by the expression

$$(\psi,\boldsymbol{A},\phi)(t) = G_{\chi(t)}(\psi_0,\boldsymbol{A}_0,0) \equiv \left(\psi_0\mathrm{e}^{i\kappa\chi(t)}, \boldsymbol{A}_0+\nabla\chi(t), -\partial_t\chi(t)\right), \quad t\geq 0. \tag{4.1}$$

Here, $\chi\in L^2_{\text{loc}}([0,\infty);W^{2,2}(\Omega))\cap W^{1,2}_{\text{loc}}([0,\infty);L^2(\Omega))$ is any time-dependent gauge. Then it is trivial to verify that $(\psi,\boldsymbol{A},\phi)$ is a weak solution of the TDGL equations (2.7)–(2.9).

This solution is gauge equivalent with and therefore physically indistinguishable from the stationary solution $(\psi_0, \boldsymbol{A}_0, 0) \in \mathcal{W}^{1,2} \times L^2$.

The following theorem shows that we are, in fact, dealing with an "if and only if" situation.

Theorem 1 *A stationary state $(\psi_0, \boldsymbol{A}_0, \phi_0) \in \mathcal{W}^{1,2} \times L^2$ with $\psi_0 \not\equiv 0$ is a weak solution of the TDGL equations (2.7)–(2.9) if and only if $\phi_0 = 0$ a.e. in Ω and $(\psi_0, \boldsymbol{A}_0)$ is a weak solution of the time-independent GL equations (2.2)–(2.4).*

Proof. It suffices to prove the "only if" part. The fact that $(\psi_0, \boldsymbol{A}_0, \phi_0)$ is a solution of the TDGL equations implies that the functions ψ_0, $\boldsymbol{A}_0$, and ϕ_0 must be such that the equations

$$i\eta\kappa\phi_0\psi_0 = -\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}_0\right)^2 \psi_0 + \left(1 - |\psi_0|^2\right)\psi_0 \quad \text{in } \Omega, \tag{4.2}$$

$$\nabla\phi_0 = -\nabla \times \nabla \times \boldsymbol{A}_0 + \boldsymbol{J}_{s,0} + \nabla \times \boldsymbol{H} \quad \text{in } \Omega, \tag{4.3}$$

$$\boldsymbol{n} \cdot \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}_0\right)\psi_0 + \frac{i}{\kappa}\gamma\psi_0 = 0 \quad \text{and} \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A}_0 - \boldsymbol{H}) = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{4.4}$$

are satisfied. Here, $\boldsymbol{J}_{s,0}$ is a nonlinear function of ψ_0 and $\boldsymbol{A}_0$,

$$\boldsymbol{J}_{s,0} \equiv \boldsymbol{J}_{s,0}(\psi_0, \boldsymbol{A}_0) = \frac{1}{2i\kappa}\left(\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*\right) - |\psi_0|^2\boldsymbol{A}_0 = -\mathrm{Re}\left[\psi_0^*\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}_0\right)\psi_0\right]; \tag{4.5}$$

cf. Eq. (2.5). From these equations we obtain a differential equation for ϕ_0 in the following way. First we take the divergence of Eq. (4.3) and obtain the equation $\Delta\phi_0 = \nabla \cdot \boldsymbol{J}_{s,0}$ in Ω. An expression for $\nabla \cdot \boldsymbol{J}_{s,0}$ follows readily from Eq. (4.5),

$$\nabla \cdot \boldsymbol{J}_{s,0} = -\kappa \ \mathrm{Im}\left[\psi_0^*\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}_0\right)^2\psi_0\right]. \tag{4.6}$$

We use Eq. (4.2) to evaluate this expression and find $\nabla \cdot \boldsymbol{J}_{s,0} = \eta\kappa^2|\psi_0|^2\phi_0$. Thus,

$$\Delta\phi_0 = \eta\kappa^2|\psi_0|^2\phi_0 \quad \text{in } \Omega. \tag{4.7}$$

We obtain a boundary condition for ϕ_0 at any point $x \in \partial\Omega$, where the local unit normal vector is $\boldsymbol{n}(x)$, by considering the component of Eq. (4.3) in the direction of $\boldsymbol{n}(x)$ at an interior point $y \in \Omega$ and letting y approach x. Because $\partial\Omega$ is locally the level surface (or curve) of a $C^{1,1}$-function $\Phi : \mathbf{R}^n \to \mathbf{R}$, the unit normal vector is $\boldsymbol{n} = |\nabla\Phi|^{-1}\nabla\Phi$, where $\nabla\Phi$ is nonvanishing and Lipschitz continuous near every point of $\partial\Omega$. Hence, $\boldsymbol{n} \cdot (\nabla \times \boldsymbol{n}) = 0$ on $\partial\Omega$. But $\nabla \times \boldsymbol{A}_0 - \boldsymbol{H}$ and $\boldsymbol{n}$ are colinear on $\partial\Omega$, by the second boundary condition in Eq. (4.4), so it must be the case that $\boldsymbol{n} \cdot \nabla \times (\nabla \times \boldsymbol{A}_0 - \boldsymbol{H}) = 0$ on $\partial\Omega$. Also, $\boldsymbol{n} \cdot \boldsymbol{J}_{s,0} = 0$ on

$\partial\Omega$, as follows immediately from the definition (4.5) of $\boldsymbol{J}_{s,0}$ and the first of the boundary conditions (4.4). Thus,

$$\boldsymbol{n} \cdot \nabla\phi_0 = 0 \quad \text{on } \partial\Omega. \tag{4.8}$$

Obviously, the Neumann problem (4.7)–(4.8) forces the energy equation

$$\int_\Omega |\nabla\phi_0|^2 \, \mathrm{d}x + \int_\Omega \eta\kappa^2 |\psi_0|^2 \phi_0^2 \, \mathrm{d}x = 0, \tag{4.9}$$

so $\nabla\phi_0 = \mathbf{0}$ and $\psi_0\phi_0 = 0$ a.e. in Ω. Thus, Eqs. (4.2)–(4.4) reduce to the time-independent GL equations (2.2)–(2.4). Furthermore, because the state $(\psi_0, \boldsymbol{A}_0, \phi_0)$ is nontrivial, it must be the case that $\phi_0 = 0$ a.e. in Ω. ∎

The theorem gives a complete characterization of those stationary states in $\mathcal{W}^{1,2} \times L^2$ that are gauge equivalent with a solution $(\psi, \boldsymbol{A}, \phi)$ of the TDGL equations. They must be of the form $(\psi_0, \boldsymbol{A}_0, 0)$, where $(\psi_0, \boldsymbol{A}_0)$ is a solution of the time-independent GL equations. The gauge equivalence is expressed by Eq. (4.1).

5 Large-Time Asymptotics

We now refine the result of the preceding section by imposing an additional constraint, namely that the TDGL equations be considered in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge. We recall the characteristic constraints of this gauge, Eq. (2.13); if $\omega > 0$, they simplify to

$$\phi = -\omega(\nabla \cdot \boldsymbol{A}) \quad \text{in } \Omega \times (0, \infty), \quad \boldsymbol{n} \cdot \boldsymbol{A} = 0 \quad \text{on } \partial\Omega \times (0, \infty). \tag{5.1}$$

Theorem 2 *Let $(\psi, \boldsymbol{A}, \phi) : (0, \infty) \to \mathcal{W}^{1,2} \times L^2$ be a weak solution of the TDGL equations* (2.7)–(2.9) *in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge. If $(\psi, \boldsymbol{A}, \phi)$ is gauge equivalent with a nontrivial stationary state $(\psi_0, \boldsymbol{A}_0, \phi_0) \in \mathcal{W}^{1,2} \times L^2$, then $\phi_0 = 0$ a.e. in Ω, $\lim_{t\to\infty} (\psi, \boldsymbol{A}, \phi)(t)$ exists in $\mathcal{W}^{1,2} \times L^2$, and*

$$\lim_{t\to\infty} (\psi, \boldsymbol{A}, \phi)(t) = (\psi_\infty, \boldsymbol{A}_\infty, 0) \quad \text{in } \mathcal{W}^{1,2} \times L^2. \tag{5.2}$$

There exists a time-independent gauge $\chi_0 \in W^{1,2}(\Omega)$, such that

$$(\psi_\infty, \boldsymbol{A}_\infty) = G_{\chi_0}(\psi_0, \boldsymbol{A}_0) \equiv \left(\psi_0 \mathrm{e}^{i\kappa\chi_0}, \boldsymbol{A}_0 + \nabla\chi_0\right). \tag{5.3}$$

Furthermore,

$$\nabla \cdot \boldsymbol{A}_\infty = 0 \quad \text{in } \Omega, \quad \boldsymbol{n} \cdot \boldsymbol{A}_\infty = 0 \quad \text{on } \partial\Omega. \tag{5.4}$$

Proof. It follows from Theorem 1 that $\phi_0 = 0$ a.e. in Ω. Furthermore, there exists a gauge $\chi \in L^2_{\text{loc}}([0,\infty); W^{2,2}(\Omega)) \cap W^{1,2}_{\text{loc}}([0,\infty); L^2(\Omega))$ such that

$$(\psi, \boldsymbol{A}, \phi)(t) = G_{\chi(t)}(\psi_0, \boldsymbol{A}_0, 0) \equiv \left(\psi_0 \mathrm{e}^{i\kappa\chi(t)}, \boldsymbol{A}_0 + \nabla\chi(t), -\partial_t\chi(t)\right), \quad t > 0. \tag{5.5}$$

The fact that $(\psi, \boldsymbol{A}, \phi)$ satisfies the characteristic constraints (5.1) of the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge implies that the gauge χ in Eq. (5.5) satisfies the boundary-value problem

$$\partial_t\chi - \omega\Delta\chi = \omega(\nabla \cdot \boldsymbol{A}_0) \quad \text{in } \Omega \times (0,\infty), \tag{5.6}$$

$$\boldsymbol{n} \cdot \nabla\chi = -\boldsymbol{n} \cdot \boldsymbol{A}_0 \quad \text{on } \partial\Omega \times (0,\infty). \tag{5.7}$$

We look for a solution of the form

$$\chi(x,t) = \chi_0(x) + z(x,t), \tag{5.8}$$

where $\chi_0 \in W^{1,2}(\Omega)$ satisfies the equations

$$-\Delta\chi_0 = \nabla \cdot \boldsymbol{A}_0 \quad \text{in } \Omega, \quad \boldsymbol{n} \cdot \nabla\chi_0 = -\boldsymbol{n} \cdot \boldsymbol{A}_0 \quad \text{on } \partial\Omega. \tag{5.9}$$

We render χ_0 unique by imposing the normalization condition

$$\int_\Omega \chi_0(x)\,\mathrm{d}x = \int_\Omega \chi(x,0)\,\mathrm{d}x. \tag{5.10}$$

The function χ_0 thus defined satisfies the regularity condition $\chi_0 \in W^{2,2}(\Omega)$.

With these definitions, z must satisfy the homogeneous boundary-value problem

$$\partial_t z - \omega\Delta z = 0 \quad \text{in } \Omega \times (0,\infty), \tag{5.11}$$

$$\boldsymbol{n} \cdot \nabla z = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{5.12}$$

with initial data $z(x,0) = \chi(x,0) - \chi_0(x)$ for $x \in \Omega$.

The boundary-value problem (5.11)–(5.12), with the initial data $z(\cdot\,,0) = \chi(\cdot\,,0) - \chi_0$, gives rise to an abstract initial-value problem in $L^2(\Omega)$. If Δ_N denotes the self-adjoint Neumann Laplacian in $L^2(\Omega)$, then

$$z(t) = \mathrm{e}^{\omega\Delta_N t}(\chi(0) - \chi_0), \quad t \ge 0. \tag{5.13}$$

From the spectral decomposition of Δ_N and the Kreĭn-Rutman theorem we infer that $0 \in \mathbf{C}$ is the principal eigenvalue of Δ_N, which is isolated and simple with the constant 1 eigenfunction. Then it follows from the spectral mapping theorem for the exponential function and the normalization condition (5.10) that $\|z(t)\|_{L^2(\Omega)} \to 0$ as $t \to \infty$; hence, $\|z(t)\|_{W^{2,2}(\Omega)} \to 0$ as $t \to \infty$, by regularity, and therefore

$$\lim_{t\to\infty} \|z(t)\|_{L^2(\Omega)} = 0, \quad \lim_{t\to\infty} \|(\nabla z)(t)\|_{W^{1,2}(\Omega)} = 0, \quad \lim_{t\to\infty} \|(\partial_t z)(t)\|_{L^2(\Omega)} = 0. \tag{5.14}$$

It follows that

$$\lim_{t\to\infty} (\psi, \boldsymbol{A}, \phi)(t) = \left(\psi_0 e^{i\kappa\chi_0}, \boldsymbol{A}_0 + \nabla\chi_0, 0\right) \quad \text{in } \mathcal{W}^{1,2} \times L^2. \tag{5.15}$$

The identities (5.4) are an immediate consequence of Eqs. (5.9). ∎

Theorem 2 shows that the large-time asymptotic limit $(\psi_\infty, \boldsymbol{A}_\infty, \phi_\infty)$ of $(\psi, \boldsymbol{A}, \phi)(t)$ in $\mathcal{W}^{1,2} \times L^2$ satisfies the zero-electric potential gauge ($\phi_\infty = 0$ in Ω) for all values of ω ($\omega > 0$). Moreover, the element $(\psi_\infty, \boldsymbol{A}_\infty) \in \mathcal{W}^{1,2}$ is a weak solution of the time-independent GL equations in the London gauge ($\nabla \cdot \boldsymbol{A}_\infty = 0$ in Ω, $\boldsymbol{n} \cdot \boldsymbol{A}_\infty = 0$ on $\partial\Omega$).

6 Summary

In this article we have given a characterization of those stationary states in $\mathcal{W}^{1,2} \times L^2$ that are gauge equivalent with a solution $(\psi, \boldsymbol{A}, \phi)$ of the TDGL equations. They must be of the form $(\psi_0, \boldsymbol{A}_0, 0)$, where $(\psi_0, \boldsymbol{A}_0)$ is a solution of the time-independent GL equations. We have also shown that a weak solution of the TDGL equations in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge ($\omega > 0$) defines a weak solution of the time-independent GL equations in the limit of large times. The latter satisfies the London gauge.

Acknowledgments

The work of H. G. Kaper is supported by the Mathematical, Information, and Computational Sciences Division subprogram of the Office of Computational and Technology Research, U.S. Department of Energy, under Contract W-31-109-Eng-38. The work of P. Takáč is supported in part by the U.S. National Science Foundation under Grant DMS-9401418 to Washington State University, Pullman, WA 99164–3113, U.S.A.

References

[1] V. L. Ginzburg and L. D. Landau, *On the theory of superconductivity*, Zh. Eksp. Teor. Fiz. (USSR) **20** (1950), 1064–1082; Engl. transl. in D. ter Haar, *L. D. Landau; Men of Physics*, Vol. I, Pergamon Press, Oxford, 1965, 138–167.

[2] A. A. Abrikosov, *Fundamentals of the Theory of Metals*, North-Holland Publ. Co., Amsterdam, 1988.

[3] P. DeGennes, *Superconductivity in Metals and Alloys*, Benjamin, New York, 1966.

[4] M. Tinkham, *Introduction to Superconductivity*, McGraw-Hill, Inc., New York, 1975.

[5] H. G. Kaper and M. K. Kwong, *Vortex configurations in type-II superconducting films*, J. Comput. Phys. **119** (1995), 120–131.

[6] A. Schmid, *A time dependent Ginzburg-Landau equation and its application to the problem of resistivity in the mixed state*, Phys. Kondens. Mater. **5** (1966), 302–317.

[7] L. P. Gor'kov and G. M. Eliashberg, *Generalizations of the Ginzburg-Landau equations for non-stationary problems in the case of alloys with paramagnetic impurities*, Zh. Eksp. Teor. Fiz. **54** (1968), 612–626; Soviet Phys. JETP **27** (1968), 328–334.

[8] Q. Du, *Global existence and uniqueness of solutions of the time-dependent Ginzburg-Landau model for superconductivity*, Appl. Anal. **53** (1994), 1–18.

[9] Z. M. Chen, K.-H. Hoffmann, and J. Liang, *On a nonstationary Ginzburg-Landau superconductivity model*, Math. Methods Appl. Sci. **16** (1993), 855–875.

[10] J. Liang and Q. Tang, *Asymptotic behavior of the solutions of an evolutionary Ginzburg-Landau superconductivity model*, J. Math. Anal. Appl. **195** (1995), 92–107.

[11] Q. Tang and S. Wang, *Time dependent Ginzburg-Landau equations of superconductivity.* Physica D **88** (1995), 139–166.

[12] P. Takáč, *On the dynamical process generated by a superconductivity model*, Proc. ICIAM 95. Z. Angew. Math. Mech. (1996), to appear.

[13] J. Fleckinger–Pellé and H. G. Kaper, *Gauges for the Ginzburg-Landau equations of superconductivity*, Proc. ICIAM 95. Z. Angew. Math. Mech. (1996), to appear. Preprint ANL/MCS-P527-0795, Mathematics and Computer Science Division, Argonne National Laboratory, 1995.

[14] J. Fleckinger–Pellé, H. G. Kaper and P. Takáč, *Dynamics of the Ginzburg-Landau equations of superconductivity.* Preprint ANL/MCS-P565-0296, Mathematics and Computer Science Division, Argonne National Laboratory, 1996.