Further Properties of a Continuum of Model Equations with Globally Defined Flux

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Abstract

To develop an understanding of singularity formation in vortex sheets, we consider model equations that exhibit shared characteristics with the vortex sheet equation but are slightly easier to analyze. A model equation is obtained by replacing the flux term in Burgers' equation by alternatives that contain contributions depending globally on the solution. We consider the continuum of partial differential equations $u_t = \theta(H(u)u)_x + (1-\theta)H(u)u_x + \nu u_{xx}$, $0 \leq \theta \leq 1$, $\nu \geq 0$, where $H(u)$ is the Hilbert transform of u. We show that when $\theta = 1/2$, for $\nu > 0$, the solution of the equation exists for all time and is unique. We also show with a combination of analytical and numerical means that the solution when $\theta = 1/2$ and $\nu > 0$ is analytic. Using a pseudo-spectral method in space and the Adams-Moulton fourth-order predictor-corrector in time, we compute the numerical solution of the equation with $\theta = 1/2$ for various viscosities. The results confirm that for $\nu > 0$, the solution is well behaved and analytic. The numerical results also confirm that for $\nu = 0$ and $\theta = 1/2$, the solution becomes singular in finite time and finite viscosity prevents singularity formation. We also present, for a certain class of initial conditions, solutions of the equation, with $0 < \theta < 1/3$ and $\theta = 1$, that become infinite for $\nu \geq 0$ in finite time.

1 Introduction

We consider the continuum of partial differential equations

$$\text{(1a)} \qquad u_t = \theta(H(u)u)_x + (1-\theta)H(u)u_x + \nu u_{xx},$$
$$\text{(1b)} \qquad u(x,0) = f(x),$$

with $0 \leq \theta \leq 1$, $\nu \geq 0$, $H(u)$ the Hilbert transform of u, and 2π-periodic initial and boundary conditions. This equation is of interest because it models the motion of vortex sheets. Here θ is a parameter varying between 0 and 1 that helps us recast the two partial differential equations

$$\begin{aligned} u_t &= (H(u)u)_x + \nu u_{xx}, \\ u_t &= H(u)u_x + \nu u_{xx}, \end{aligned}$$

as a single one. By varying θ between 0 and 1, we gain further insight about the results previously obtained in [1] and [8].

Note that (1a) with $\theta = 0$ and $\theta = 1$ has already been considered in [1] and [8]. In [1], it was shown that the solution with $\theta = 1$ develops singularities in finite time for a certain class of initial

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conditions; it was also shown that the solution with $\theta = 0$ exists for all time provided $\nu > 0$. In [8], it was shown that the solution of (1a) with $\theta = 0$ has a weak limit in the limit of zero viscosity, the weak limit belongs to $L^\infty([0,2\pi]) \cap BV([0,2\pi])$, and the temporal derivative of the weak limit is locally measurable. Here we extend the results presented in [1] and [8]; we show that the solution of (1a) with $\theta = 1/2$ and $\nu > 0$ exists for all time and is unique. This existence proof is obtained first by using Ball's theorem [2] to obtain short-term existence and then by proceeding as for Burgers' equation [7] to obtain long-term existence and uniqueness. Proceeding along the lines of the work presented in [5], we show that when $\nu > 0$ and $\theta = 1/2$, the solution of (1a) is analytic. We numerically compute the solution of (1a), $\theta = 1/2$, for various viscosities, with a pseudo-spectral method in space and the Adams-Moulton fourth-order predictor corrector-method in time. The numerical results confirm that the solution is analytic when $\nu > 0$ and that the viscosity prevents the solution from becoming singular in finite time. The numerical results also indicate that for $\nu = 0$ in finite time, the derivatives of the solution become infinite.

In Section 3, we show, that for $0 < \theta < 1/3$ and for symmetric initial conditions, the solution of (1a) blows up in finite time; the result is obtained by using the Fourier space method of Palais [3], [4], and [9]: If the system of Fourier modes is cooperative, then the evolution of the $\hat{u}_k$ for any finite subsystem of the original one serves as a lower bound for the $\hat{u}_k$ of the full system. We also show, using a different approach from that in [1], that in finite time the solution of (1a) with $\theta = 1$ becomes infinite. In both cases, instead of transforming the original equation into Burgers' equation for the function $z = H(u) - iu$, we derive an infinite system of differential equations for the Fourier coefficients and show that, for a certain class of initial conditions, the L^2 norm of the solution becomes infinite in finite time.

2 Equation with $\theta = 1/2$

Here we investigate the behavior of the solution of (1a) with 2π-periodic initial and boundary conditions. We first prove short-term existence with Ball's theorem [2], then long-term existence proceeding as in [1] and [7]. We also show uniqueness of the solution of (1a). Proceeding as in [5] for 3D Navier-Stokes equations, we show that the solution of (1a) is analytic when $\nu > 0$. The difference between our approach and that in [5] is that we use numerical results to confirm that the differential equation whose solution is obtained by analytical means is representative of the general behavior of the original equation. We present numerical solutions of (1a) for $\nu > 0$. The equation is approximated with a pseudo-spectral method in space and the Adams-Moulton predictor-corrector in time. The numerical results confirm that for $\nu > 0$, the solution is well behaved and analytic and that when $\nu = 0$ the derivatives of the solution become infinite in finite time.

2.1 Existence and Uniqueness

To prove short-term existence of the solution of (1a) with $\theta = 1/2$, we use Ball's theorem [2], given below. in [1].

Theorem 2.1 *Consider the equation*

$$\frac{d}{dt}u = Au + f(u), \tag{2}$$

where A is the generator of a holomorphic semigroup $S(t)$ of bounded operators on a Banach space X. Suppose that $||S(t)|| \leq M$ for some constant $M > 0$ and all $t \in \mathbb{R}^+$. Under these hypotheses, the fractional powers $(-A)^{-\alpha}$ can be defined for $0 \leq \alpha < 1$, and $(-A)^\alpha$ is a closed linear operator with domain $X_\alpha = Domain((-A)^\alpha)$ dense in X. Let $f(u)$ be locally Lipschitz; that is, for each bounded subset U of X_α there exists a constant C_U so that

$$||f(u) - f(v)|| \leq C_U ||u - v||_\alpha \quad \forall \;\; u, \; v \in U.$$

Then, given $u_0 \in X$, there exists a finite time interval $[0,t)$ and a unique solution to (2) with $u(\cdot, 0) = u_0$ on that time interval, and the solution can be continued uniquely on a maximal interval *of existence* $[0, T^\star)$*. Moreover, if* $T^\star < \infty$*, then necessarily*

$$\lim_{t \to T^\star} ||u(t)||_\alpha = \infty.$$

We directly apply Theorem 2.1 to (1a) with $\theta = 1/2$, $A = \nu \partial^2/\partial x^2$, $X = L^2([0,1])$, and

$$f(u) = \frac{1}{2}(H(u)u)_x + \frac{1}{2}H(u)u_x.$$

Then $X_\alpha = H^2$ and

$$\begin{aligned} ||f(u) - f(v)|| &\leq ||H(u)||_\infty ||u_x - v_x|| + ||v_x||_\infty ||H(u) - H(v)|| \\ &\quad + \frac{1}{2}||u||_\infty ||H(u_x) - H(v_x)|| + \frac{1}{2}||H(v_x)||_\infty ||u - v||, \\ &\leq C\left(||u||_{H^2} + ||v||_{H^2}\right) ||u - v||_{H^1}. \end{aligned}$$

In the above relations and the rest of the paper, $||u||$ denotes the L^2 norm of u and C denotes a constant. To obtain the above inequalities, we use the following properties of the Hilbert transform [10]:

- Let g be a C^∞, 2π-periodic function. Then $H(g)$ is a C^∞, 2π-periodic function and

$$(H(g))_x = H(g_x).$$

- The L^2 norm of $H(g)$ satisfies the bound

$$||H(g)|| \leq ||g||.$$

- $H(e^{ikx}) = i\text{sign}(k)e^{ikx}$.

The final bound for the L^2 norm of $f(u) - f(v)$ is a direct consequence of Sobolev's inequalities. Hence, f is locally Lipschitz continuous on H^2. Theorem 2.1 implies that a solution exists in any time interval in which the H^2 norm of the solution is controlled.

To prove the existence of a solution for all time, we require bounds on the solution to hold independent of the length of the time interval. We can obtain such bounds using the following lemma which establishes an L^2 norm bound for the solution of (1a), $\theta = 1/2$.

Lemma 2.1 *Let the initial data* f *be* C^∞ *and* 2π*-periodic. Let* u *be a solution of (1a),* $\theta = 1/2$*, that exists for some* $[0, T]$*. Then*

$$||u(\cdot, t)|| \leq ||f||. \tag{3}$$

Proof: Taking the inner product of u with the equation it satisfies leads to

$$\frac{d}{dt}||u||^2 = (u, (H(u)u)_x) + (u, H(u)u_x) - 2\nu||u_x||^2 \tag{4a}$$

$$= -2\nu||u_x||^2. \tag{4b}$$

We are led to (4b) because $(u, (H(u)u)_x) = -(u_x, H(u)u)$. Integration with respect to time of (4b) gives us the desired bound. ∎

The following theorem replaces Theorem 4.2.1 in [7] and Theorem 5 in [1].

Theorem 2.2 *Let* f *be a* 2π*-periodic* C^∞ *initial condition, and let* u *be a* C^∞ *solution of (1a),* $\theta = 1/2$*, defined on* $[0, T]$*. Then there is a constant* K*, dependent on the* H^2 *norm of the initial condition and on the viscosity* ν*, but independent of* T*, such that*

$$||u(\cdot, t)||_{H^2} \leq K, \tag{5}$$

with $t \in [0, T]$*.*

Proof: Let $v = u_x$. The equation for v is

$$v_t = \frac{1}{2}(H(u)u)_{xx} + \frac{1}{2}(H(u)v)_x + \nu v_{xx}. \tag{6a}$$

Taking the inner product of v with the equation it satisfies, we are led to

$$\frac{d}{dt}||v||^2 = (v, (H(u)u)_{xx}) + (v, (H(u)v)_x) - 2\nu||v_x||^2. \tag{6b}$$

Integrating by parts the inner product $(v, (H(u)v)_x)$, expanding $(H(u)u)_{xx}$, and integrating by parts the scalar product $(v, H(v)v)$, to decrease the derivative of v, we obtain

$$\frac{d}{dt}||v||^2 = -(v, H(v_x)u) - 2(v_x, H(v)u) - 2\nu||v_x||^2. \tag{6c}$$

We need to estimate both inner products in (6c)

$$\begin{aligned} |(v, H(v_x)u)| &\leq ||u||_\infty||v||\,||v_x||, \\ |(v_x, H(v)u)| &\leq ||u||_\infty||v||\,||v_x||. \end{aligned}$$

Sobolev's inequality then tells us

$$||u||_\infty||v||\,||v_x|| \leq \sqrt{2}||u||^{1/2}||v||^{3/2}||v_x|| \leq 2^{5/4}||u||^{5/4}||v_x||^{7/4}.$$

Recall Young's inequality, which states that for a, $b > 0$

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, \qquad \text{with} \qquad \frac{1}{p} + \frac{1}{q} = 1. \tag{7}$$

Take $a = 3\,2^{3/8}||u||^{5/4}/\nu^{7/8}$ and $b = 2^{7/8}\nu^{7/8}||v_x||^{7/4}$. Then, with $p = 8$ and $q = 8/7$, Equation (7) becomes

$$3||u||_\infty||v||\,||v_x|| \leq \frac{3^8\,2^3}{\nu^7}||u||^{10} + \frac{7}{4}\nu||v_x||^2. \tag{8}$$

Since $||v||^2 \leq ||v_x||^2$, we are led to the differential inequality

$$\frac{d}{dt}||v||^2 + \frac{1}{4}\nu||v||^2 \leq \frac{3^8\,2^3}{\nu^7}||u||^{10}.$$

Since from Lemma 2.1, $||u|| \leq ||f||$, we may integrate this differential inequality in a standard way to obtain

$$||v||^2 \leq ||f_x||^2 + \frac{3^8\,2^5}{\nu^8}||f||^{10}. \tag{9}$$

Now, we need an estimate for the L^2 norm of $w = u_{xx}$, that satisfies the equation

$$w_t = \frac{1}{2}H(w_x)u + 2H(w)v + \frac{5}{2}H(v)w + H(u)w_x + \nu w_{xx}. \tag{10}$$

To proceed, we need estimates for several inner products. First we crudely estimate $(w, H(w_x)u)$:

$$\begin{aligned} |(w, H(w_x)u)| &\leq ||u||_\infty||w||\,||w_x||, \\ &\leq \sqrt{2}||u||^{1/2}||v||^{1/2}||w||\,||w_x||, \\ &\leq 4||u||^{7/6}||w_x||^{11/6}. \end{aligned}$$

The above inequalities are a direct consequence of Sobolev's inequalities and of Lemma 8 of [1].

Then we integrate by parts $(w, H(w)v)$ and $(w, H(v)w)$, decreasing the derivative order of v and $H(v)$, respectively, and crudely estimate the resulting inner products to obtain

$$\begin{aligned} |(w, H(w)v)| &\leq 2||u||_\infty||w||\,||w_x|| \leq 2\sqrt{2}||u||^{1/2}||v||^{1/2}||w||\,||w_x||, \\ |(w, H(v)w)| &\leq 2||H(u)||_\infty||w||\,||w_x|| \leq 2\sqrt{2}||u||^{1/2}||v||^{1/2}||w||\,||w_x||. \end{aligned}$$

As for the first inner product, we crudely estimate $(w, H(u)w_x)$ to obtain

$$|(w, H(u)w_x)| \leq ||H(u)||_\infty ||w|| \, ||w_x|| \leq \sqrt{2}||u||^{1/2}||v||^{1/2}||w|| \, ||w_x||.$$

Therefore, taking the inner product of w with the equation it satisfies, (10), we are led to the differential inequality

$$\frac{d}{dt}||w||^2 \leq 24||u||^{7/6}||w_x||^{11/6} - 2\nu||w_x||^2.$$

Applying Young's inequality (7) to $24||u||^{7/6}||w_x||^{11/6}$ with $a = 24||u||^{7/6}/\nu^{11/12}$, $b = \nu^{11/12}||w_x||^{11/6}$, $p = 12$, and $q = 12/11$, we may rewrite the differential inequality as

$$\frac{d}{dt}||w||^2 \leq \frac{2^{34}\,3^{11}}{\nu^{11}}||u||^{14} - \nu||w_x||^2.$$

Using $||w||^2 \leq ||w_x||^2$ and integrating the differential inequality

$$\frac{d}{dt}||w||^2 + \nu||w||^2 \leq \frac{2^{34}\,3^{11}}{\nu^{11}}||f||^{14},$$

we are led to

$$||w||^2 \leq \frac{2^{34}\,3^{11}}{\nu^{12}}||f||^{14} + ||f_{xx}||^2. \tag{11}$$

Combining (3), (9), and (11), we obtain

$$||u(\cdot, t)||_{H^2} \leq K,$$

where K depends on ν and $||f||_{H^2}$ but not on T.

Note that if $||v|| \leq ||u||$, the inequality (9) would read

$$||v||^2 \leq ||f_x||^2 + \frac{9}{2\nu^2}||f||^4,$$

and (11) would read

$$||w||^2 \leq ||f_{xx}||^2 + \frac{3^5 2^{14}}{\nu^6}||f||^8.$$

To obtain the above inequality, we bounded $||u||_\infty ||w|| \, ||w_x||$ by $\sqrt{2}||u|| \, ||w|| \, ||w_x||$. We then applied Lemma 8 of [1] and Young's inequality with $a = 12\sqrt{2}||u||^{4/3}/\nu^{5/6}$, $b = \nu^{5/6}||w_x||^{5/3}$, $p = 6$, and $q = 6/5$. The inequality $||u(\cdot, t)||_{H^2} \leq K$ again holds. ∎

We are now in a position to prove the major result of this section.

Theorem 2.3 *Let the initial condition be C^∞ and 2π-periodic, and let $\nu > 0$. Equation (1a) with $\theta = 1/2$ has a unique, 2π-periodic solution u on $[0, \infty)$, which is infinitely many times differentiable.*

Proof: Proof of existence follows directly from the arguments in Theorem 4.2.2 in [7]. We have only to show uniqueness. Let u and v be solutions of (1a), $\theta = 1/2$, that satisfy the same initial condition. Their difference $w = u - v$ satisfies

$$\begin{aligned} w_t &= H(u)w_x + \frac{1}{2}H(w_x)v + \frac{1}{2}H(u_x)w + H(w)v_x + \nu w_{xx}, \\ w(x,0) &= 0. \end{aligned}$$

Integration by parts of the inner product $(w, H(u)w_x)$, decreasing the derivative order of w_x, leads us to $(w, H(u)w_x) = -(w, H(u_x)w)/2$. The inner products $(w, H(w_x)v)$ and $(w, H(w)v_x)$ are crudely estimated by

$$\begin{aligned} |(w, H(w_x)v)| &\leq ||v||_\infty ||w|| \, ||w_x|| \leq \frac{\nu}{2}||w_x||^2 + \frac{1}{2\nu}||v||_\infty^2 ||w||^2, \\ |(w, H(w)v_x)| &\leq ||v_x||_\infty ||w||^2. \end{aligned}$$

The above estimates may be used to obtain the differential inequality

$$\frac{d}{dt}||w||^2 \leq \left(\frac{1}{4\nu}||v||_\infty^2 + ||v_x||_\infty\right)||w||^2.$$

Gronwall-Bellman's inequality then implies that $w = 0$. ∎

Finally we can show that all spatial and temporal derivatives of the solution of (1a), $\theta = 1/2$, remain bounded for all time provided $\nu > 0$. The proof follows the ideas already expressed. This result indicates that there is a limit to how distorted the solution can become.

2.2 Analyticity

Now that we have shown that the solution of (1a), $\theta = 1/2$, exists for all time when $\nu > 0$, we want to prove that it is analytic. To do so, we proceed as in [5] for the three-dimensional Navier-Stokes equations and we use a combination of analytical and numerical techniques. The method used in [5] is based on Foias and Teman's work [6]; we can prove that the Fourier coefficients of the solution of (1a), $\theta = 1/2$, decay exponentially. We first derive an evolution inequality for

$$||e^{\alpha|\frac{\partial}{\partial x}|t}u_x||^2 = \sum_{k\neq 0} e^{2\alpha|k|t}k^2|\hat{u}_k|^2$$

for some $\alpha > 0$ and for some time interval $[0, t)$.

We derive the exact differential equation for the evolution of the square of the L^2 norm of $e^{\alpha|\frac{\partial}{\partial x}|t}v$, where v is the solution of (6a). Using (6a), we are led to

$$\begin{aligned}
\text{(12a)}\qquad \frac{d}{dt}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^2 &= 2\alpha\left(e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}\left|\frac{\partial}{\partial x}\right|v\right) + 2\left(e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}v_t\right),\\
&= 2\alpha\left(e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}\left|\frac{\partial}{\partial x}\right|v\right) + \left(e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)v\right)\\
\text{(12b)}\qquad &\quad -\left(e^{\alpha|\frac{\partial}{\partial x}|t}v_x, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)u\right) - 2\nu||e^{\alpha|\frac{\partial}{\partial x}|t}v_x||^2.
\end{aligned}$$

We obtain (12b) from (12a), the equation v satisfies, the properties of the Hilbert transform, and integration by parts, since

$$\begin{aligned}
2(e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}H(u)v_x) &= -(e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)v),\\
(e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}H(v_x)u) &= -(e^{\alpha|\frac{\partial}{\partial x}|t}v_x, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)u) - (e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)v).
\end{aligned}$$

The action of the operator $\left|\frac{\partial}{\partial x}\right|$ is defined in terms of the Fourier transform as

$$\left|\frac{\partial}{\partial x}\right|u(x,t) = \sum_{k\neq 0}|k|\hat{u}_k(t)e^{ikx}.$$

Since u_x and $\left|\frac{\partial}{\partial x}\right|u$ have the same norm, the first scalar product in (12b) can be estimated as

$$\begin{aligned}
\alpha\left(e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}\left|\frac{\partial}{\partial x}\right|v\right) &\leq \alpha||e^{\alpha|\frac{\partial}{\partial x}|t}v||\,||e^{\alpha|\frac{\partial}{\partial x}|t}v_x||,\\
&\leq \frac{\nu}{2}||e^{\alpha|\frac{\partial}{\partial x}|t}v_x||^2 + \frac{\alpha^2}{2\nu}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^2.
\end{aligned}$$

Hence (12b) can be rewritten as

$$\begin{aligned}
\frac{d}{dt}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^2 &\leq \frac{\alpha^2}{\nu}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^2 + (e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)v)\\
\text{(12c)}\qquad &\quad -(e^{\alpha|\frac{\partial}{\partial x}|t}v_x, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)u) - \nu||e^{\alpha|\frac{\partial}{\partial x}|t}v_x||^2.
\end{aligned}$$

Now we bound the two scalar products in (12c) that arise from the nonlinear terms in (6a). For the first product, using the Fourier expansion of the solution, we obtain

$$\begin{aligned}\left|(e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)v)\right| &= \left|\sum_{k\neq 0} e^{2\alpha|k|t}k \sum_{k'+k''=k} k'k''\mathrm{sign}(k')\overline{(\hat{u}_k)}\hat{u}_{k'}\hat{u}_{k''}\right|, \\ &\leq \sum_{k\neq 0} e^{\alpha|k|t}|k||\hat{u}_k| \sum_{k'+k''=k} \left(e^{\alpha|k'|t}|k'||\hat{u}_{k'}|\right)\left(e^{\alpha|k''|t}|k''||\hat{u}_{k''}|\right),\end{aligned}$$

where $\overline{(\hat{u}_k)}$ is the complex conjugate of $\hat{u}_k$ and the triangular inequality $|k| \leq |k'| + |k''|$ has been used. Let us define the periodic function r by its Fourier transform $\hat{r}_k$ as $\hat{r}_k = e^{\alpha|k|t}|\hat{u}_k|$. Then the first scalar product in (12c) is bounded by

$$\begin{aligned}\left|\left(e^{\alpha|\frac{\partial}{\partial x}|t}v, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)v\right)\right| &\leq \sum_{k\neq 0}|k|\hat{r}_k \sum_{k'+k''=k} |k'|\,|k''|\hat{r}_{k'}\hat{r}_{k''}, \\ &\leq \left(\left|\frac{\partial}{\partial x}\right|r, \left[\left|\frac{\partial}{\partial x}\right|r\right]^2\right), \\ &\leq ||r_x||_\infty ||r_x||^2, \\ &\leq \sqrt{2}||r_x||^{5/2}||r_{xx}||^{1/2}.\end{aligned}$$

The second inequality above is directly derived from the definition of r and its derivatives in terms of its Fourier coefficients; the third inequality is derived from the fact that r_x and $\left|\frac{\partial}{\partial x}\right| r$ have the same L^2 norm and brute force estimation of the scalar product; the last inequality is derived from Sobolev's inequality.

The second scalar product can be bounded similarly. We obtain

$$\begin{aligned}\left|(e^{\alpha|\frac{\partial}{\partial x}|t}v_x, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)u)\right| &= \left|\sum_{k\neq 0} e^{2\alpha|k|t}k^2 \sum_{k'+k''=k} k'\mathrm{sign}(k')\overline{(\hat{u}_k)}\hat{u}_{k'}\hat{u}_{k''}\right|, \\ &\leq \sum_{k\neq 0} e^{\alpha|k|t}k^2|\hat{u}_k| \sum_{k'+k''=k} \left(e^{\alpha|k'|t}|k'||\hat{u}_{k'}|\right)\left(e^{\alpha|k''|t}|\hat{u}_{k''}|\right).\end{aligned}$$

Then, proceeding as for the first scalar product, we get

$$\begin{aligned}\left|(e^{\alpha|\frac{\partial}{\partial x}|t}v_x, e^{\alpha|\frac{\partial}{\partial x}|t}H(v)u)\right| &\leq \sum_{k\neq 0}k^2\hat{r}_k \sum_{k'+k''=k} |k'|\hat{r}_{k'}\hat{r}_{k''}, \\ &\leq \left(r_{xx}, \tilde{r}\left|\frac{\partial}{\partial x}\right|r\right), \\ &\leq ||\tilde{r}||_\infty ||r_x||||r_{xx}||, \\ &\leq \sqrt{2}||r_x||^2||r_{xx}||.\end{aligned}$$

The second inequality above is derived directly from the definition of r and its derivatives in terms of its Fourier coefficients. The function $\tilde{r}$ is obtained from r by setting the zero Fourier mode to 0 (note that the function whose Fourier coefficients are $\sum_{k'+k''=k}|k'|\hat{r}_{k'}\hat{r}_{k''}$ has zero average). The third inequality is derived from the fact that r_x and $\left|\frac{\partial}{\partial x}\right| r$ have the same L^2 norm and brute force estimation of the scalar product. The last inequality is derived from Sobolev's inequality and the fact that $||\tilde{r}|| \leq ||r_x||$. Therefore (12c) becomes

$$\begin{aligned}\frac{d}{dt}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^2 &\leq \frac{\alpha^2}{\nu}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^2 + \sqrt{2}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^{5/2}||e^{\alpha|\frac{\partial}{\partial x}|t}v_x||^{1/2} \\ &\quad +\sqrt{2}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^2||e^{\alpha|\frac{\partial}{\partial x}|t}v_x|| - \nu||e^{\alpha|\frac{\partial}{\partial x}|t}v_x||^2, \qquad \text{(12d)} \\ &\leq \frac{\alpha^2}{\nu}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^2 + \frac{3}{2^{5/3}\nu^{1/3}}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^{10/3} + \frac{1}{\nu}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^4. \qquad \text{(12e)}\end{aligned}$$

Inequality (12e) is obtained from (12d), Young's inequality, with $a = 2^{1/4}||e^{\alpha|\frac{\partial}{\partial x}|t}v||^{5/2}/\nu^{1/4}$, $b = 2^{1/4}\nu^{1/4}||e^{\alpha|\frac{\partial}{\partial x}|t}v_x||^{1/2}$, $p = 4/3$, and $q = 4$, and the Cauchy-Schwartz inequality. We want to solve (12e) with initial condition $||f_x||^2$. The function $z = ||e^{\alpha|\frac{\partial}{\partial x}|t}v||^2$ satisfies the differential inequality

$$\frac{d}{dt}z \leq \frac{\alpha^2}{\nu}z + \frac{3}{2^{5/3}\nu^{1/3}}z^{5/3} + \frac{1}{\nu}z^2. \tag{13}$$

If $z \geq 1$ and $\nu \leq 1$, then $z^{5/3}/\nu^{1/3} \leq z^2/\nu$, and the solution y of

$$\frac{d}{dt}y \leq \frac{\alpha^2}{\nu}y + \left(1 + \frac{3}{2^{5/3}}\right)\frac{1}{\nu}y^2, \tag{14}$$

is an upper bound for z. The closed-form solution of (14) may be obtained by rewriting (14) as

$$\frac{d}{dt}\tilde{y} \leq \left(1 + \frac{3}{2^{5/3}}\right)\frac{1}{\nu}e^{\alpha^2 t/\nu}[\tilde{y}]^2, \tag{15}$$

with $\tilde{y} = e^{-\alpha^2 t/\nu}y$, and integrating (15). We are led to

$$\tilde{y}(t) \leq \frac{||f_x||^2\alpha^2}{\alpha^2 - (1 + 3/2^{5/3})||f_x||^2(e^{\alpha^2 t/\nu} - 1)}.$$

Therefore, $z = ||e^{\alpha t|\frac{\partial}{\partial x}|}v||^2$ is bounded by

$$||e^{\alpha t|\frac{\partial}{\partial x}|}v||^2 \leq \frac{\alpha^2 e^{\alpha^2 t/\nu}||f_x||^2}{\alpha^2 - (1 + 3/2^{5/3})||f_x||^2(e^{\alpha^2 t/\nu} - 1)}, \tag{16}$$

which is finite on the interval $[0, t^*)$, with

$$t^* = \frac{\nu}{\alpha^2}\ln\left(1 + \frac{\alpha^2}{(1 + 3/2^{5/3})||f_x||^2}\right). \tag{17}$$

We first want to compare the breakdown times of (13) and (14) and check whether (17) is a good lower bound for the breakdown time of (13). To do so, we solve equations (13) and (14) with MATLAB's function ode45, fourth-/fifth-order Runge-Kutta-Fehlberg method, an initial condition of 1, and a tolerance of 10^{-12}. The parameters α and ν vary from 10^{-2} to 1, with a step of 10^{-2}. We measure the time t at which the solution of the equations is equal to 1000; we use 1000 to give a lower bound for the time of breakdown.

In the left and right graphs of Figure 1, we respectively plot the time at which the amplitude of the solutions of (13) and (14) equal 1000 versus α and ν, α and ν varying between 10^{-2} and 1 with a step of 10^{-2}. We see that as α increases, ν being fixed, the time at which the amplitude of the solution is 1000 decreases and that as ν increases, α being fixed, the time at which the amplitude of the solution is 1000 also increases. From Figure 1, we see that the term y^2/ν has the most influence on the time at which the amplitude of the solution of the differential equation reaches the value 1000, since both figures in Figure 1 are similar and that (14) seems to be a legitimate approximation of (13) in the case considered.

To better quantify the results, we take cross sections of the surfaces presented in Figure 1, fixing one of the two parameters, and we compare these with the breakdown time of (14) given in (17). In Figure 2, we plot the estimated times at which the amplitude of the solutions of (13) and (14) is 1000, as well as the breakdown time of (14) for different values of α, $\nu = .01$ (the left figure) and for different values of α, $\nu = 1$ (the right figure). The solid curve is the breakdown time of (14), the x curve is the estimated time at which the amplitude of the solution of (14) is 1000, and the o curve is the estimated time at which the amplitude of the solution of (13) is 1000. Note that the solid and x curves are in quite good agreement and that the o curve is nearly self-similar to the solid and x curves. The increase in viscosity translates into larger values of the breakdown time and estimated times. We show in Figure 3 the log-log plot of the estimated times at which the amplitude of the

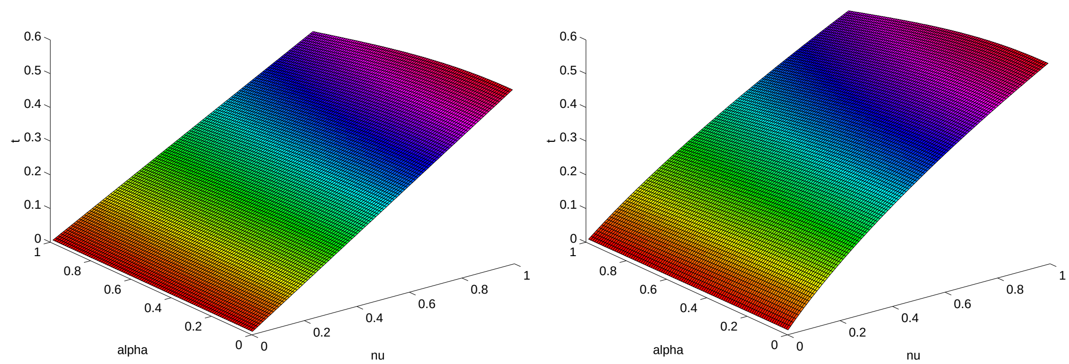

Figure 1: Time at which the amplitude of the solutions of (13) and (14) equals 1000 versus α and ν.

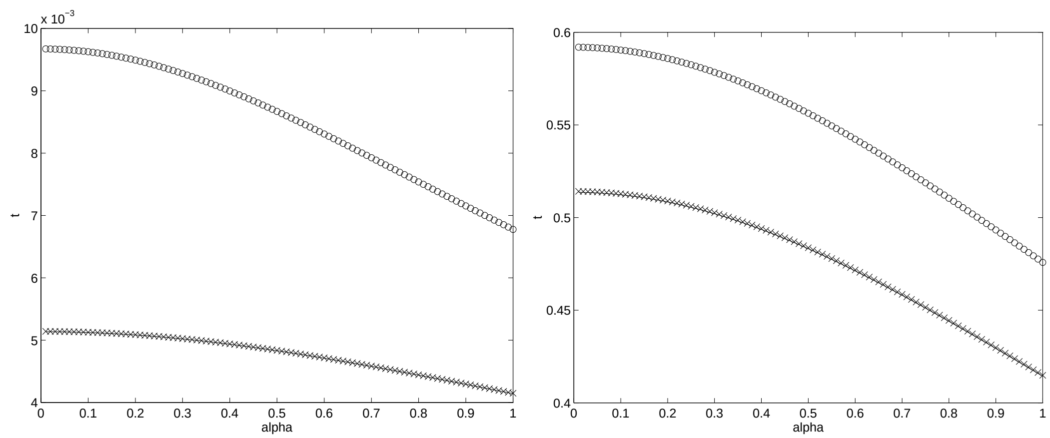

Figure 2: Comparison of the breakdown time of (14) with the estimated times at which the amplitude of the solutions of (13) and (14) is 1000, given by o and x, for different values of α, $\nu = .01$ (left) and for different values of α, $\nu = 1$ (right).

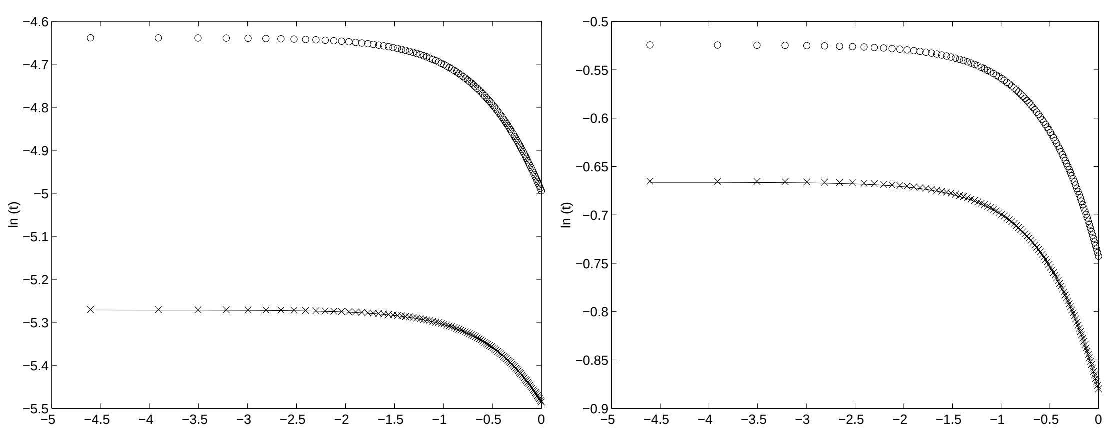

Figure 3: Log-log plot of the breakdown time of (14) and of the estimated times at which the amplitude of the solutions of (13) and (14) is 1000 for different values of α, $\nu = .01$ (left) and for different values of α, $\nu = 1$ (right).

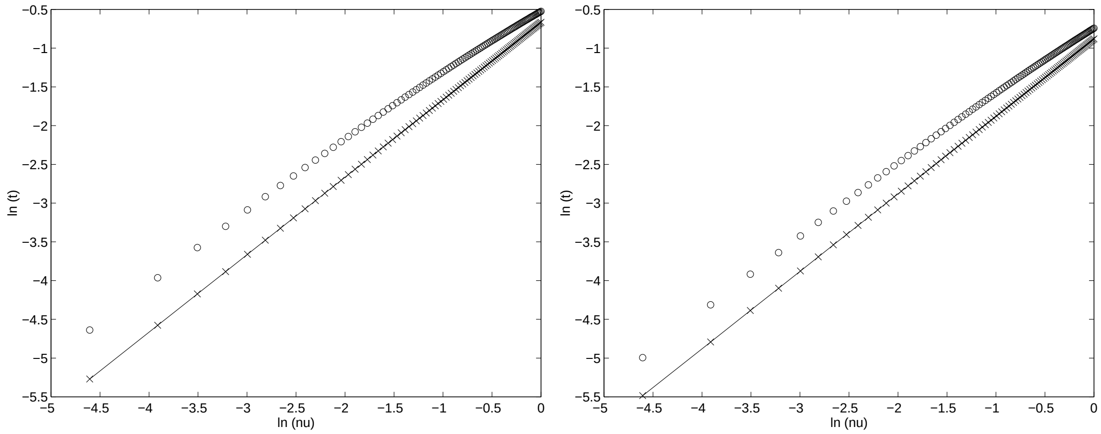

Figure 4: Log-log plot of the breakdown time of (14) and of the estimated times at which the amplitude of the solutions of (13) and (14) is 1000 for different values of ν, $\alpha = .01$ (left) and for different values of ν, $\alpha = 1$ (right).

solutions of (13) and (14) is 1000 as well as the breakdown time of (14) for different values of α, $\nu = .01$ (left) and for different values of α, $\nu = 1$ (right). As in Figure 2, we see that the solid and x curves are superimposed and that for small α, the three curves are self-similar. We also find, for our purposes, that (14) is a better approximation of (13) in the regime small α, independently of the value of ν.

Now we look at sections where α is fixed and ν varies. In Figure 4, we present the log-log plot of the estimated times at which the amplitude of the solutions of (13) and (14) is 1000, as well as the breakdown time of (14) for different values of ν, $\alpha = .01$ (left) and for different values of ν, $\alpha = 1$ (right). We see that the solid and x curves are superimposed and that for small ν, the three curves are parallel to each other. Note that the increase in the parameter α translates into smaller values of breakdown time and estimated times.

From this numerical comparison of the solutions of (13) and (14), we find that (14) gives a rather good estimate of the breakdown time of (13) despite the fact that it underestimates it. Estimating the breakdown time by finding the time at which the amplitude of the solution is 1000 thus appears to be acceptable because the solid and x curves superimpose.

From the expression (16), we see that each Fourier mode amplitude can be individually controlled, since

$$e^{2\alpha|k|t}k^2|\hat{u}(k,t)|^2 \leq \sum_{k=-\infty}^{\infty} e^{2\alpha|k|t}k^2|\hat{u}(k,t)|^2 = ||e^{\alpha|\frac{\partial}{\partial x}|t}v(\cdot,t)||^2.$$

Hence, the amplitude of the kth Fourier is bounded explicitly by

$$|\hat{u}(k,t)|^2 \leq \frac{1}{k^2}\frac{\alpha^2 e^{\alpha^2 t/\nu - 2\alpha|k|t}||f_x||^2}{\alpha^2 - (1+3/2^{5/3})||f_x||^2(e^{\alpha^2 t/\nu}-1)}, \tag{18}$$

on the interval $[0,t^*)$ with t^* defined in (17). The upper bound in (18) exhibits a local mininum for $t \in [0,t^*)$, the location and value of which depend on the value of k considered as well as the choice of α. We first choose t, then adjust α to get explicit bounds.

The exponential decay length of the spectrum at time $t < t^*$ is αt. Let us choose $t = t^*/2$ and define the associated length $\lambda^* = \alpha t^*/2 = \frac{\nu}{2\alpha}\ln\left(1+\frac{\alpha^2}{(1+3/2^{5/3})||f_x||^2}\right)$. Note that since λ^* is a concave function of $\alpha > 0$, we may maximize it over the choices of α. The extremum satisfies the equation

$$\frac{\partial\lambda^*}{\partial\alpha} = \frac{\nu}{2(1+3/2^{5/3})||f_x||^2}\left[-\frac{(1+3/2^{5/3})||f_x||^2}{\alpha^2}\ln\left(1+\frac{\alpha^2}{(1+3/2^{5/3})||f_x||^2}\right)\right.$$

$$+\frac{2}{1+\alpha^2/((1+3/2^{5/3})||f_x||^2))}\Bigg] = 0$$

This fixes α as

$$\alpha = \sqrt{1+3/2^{5/3}}||f_x||\gamma,$$

with γ the positive root of

$$-\frac{1}{\gamma^2}\log(1+\gamma^2) + \frac{2}{1+\gamma^2} = 0.$$

Hence, an estimate of the "best" length scale associated with the exponential decay at $t = t^*/2$ is

$$\lambda^* = \frac{\nu}{2\gamma\sqrt{1+3/2^{5/3}}||f_x||}\ln\left(1+\gamma^2\right).$$

Therefore,

$$|\hat{u}(k,t^*/2)|^2 \leq \frac{1}{k^2}\frac{\gamma^2 e^{-2\lambda^*|k|}||f_x||^2\sqrt{1+\gamma^2}}{[\sqrt{1+\gamma^2}-1]}. \tag{19}$$

Since we know that when $\nu > 0$, the L^2 norm of v is uniformly bounded in time, we know that a bound similar to (19) holds at a later time. If we define the uniform length scale $\overline{\lambda}$ in the spirit of λ^*,

$$\overline{\lambda} = c\frac{\nu}{\sqrt{||f_x||^2 + 3^8\, 2^5||f||^{10}/\nu^8}}, \tag{20}$$

where c is a constant independent of time and ν, we obtain

$$|\hat{u}(k,t)| \leq C_0\frac{e^{-\overline{\lambda}|k|}}{|k|}\sqrt{||f_x||^2 + \frac{3^8\, 2^5}{\nu^8}||f||^{10}}.$$

We now present the following theorem.

Theorem 2.4 *Let f be a 2π-periodic C^∞ solution and let u be a 2π-periodic C^∞ solution of (1a), $\theta = 1/2$, defined on $[0,\infty)$. Then the solution of (1a), $\theta = 1/2$, is analytic and the Fourier coefficients satisfy*

$$|\hat{u}(k,t)| \leq C_0\frac{e^{-\overline{\lambda}|k|}}{|k|}\sqrt{||f_x||^2 + \frac{3^8\, 2^5}{\nu^8}||f||^{10}},$$

with $\overline{\lambda}$ given by (20) and C_0 a constant independent of t, ν, and k.

2.3 Numerical Results

Since we are interested in periodic solutions of (1a) with $\theta = 1/2$, we use spectral methods to construct the solutions numerically. The coefficients of a Fourier series satisfy the system

$$\frac{d}{dt}A_k = -\nu k^2 A_k + \hat{G}_k, \tag{21}$$

where $\hat{G}_k$ are the coefficients for $[(H(u)u)_x + H(u)u_x]/2$. For large k and $\nu \neq 0$, this system is stiff, so we use the alternative form,

$$\frac{d}{dt}(e^{\nu k^2 t}A_k) = e^{\nu k^2 t}\hat{G}_k. \tag{22}$$

We apply the Adams-Moulton fourth-order predictor-corrector to either form, evaluating $\hat{G}_k$ by pseudo-spectral techniques. Given the Fourier coefficients at some time level, we use the fast inverse Fourier transform to obtain u at evenly spaced points. To obtain u_x and $H(u)$ at evenly spaced points, we must first multiply the Fourier coefficients by ik and $i\text{sign}(k)$, respectively, before using the inverse transform. We then form the products $H(u)u_x$ and $H(u)u$ at evenly spaced points. By using the fast Fourier transform, we obtain the Fourier coefficients for these products. Finally,

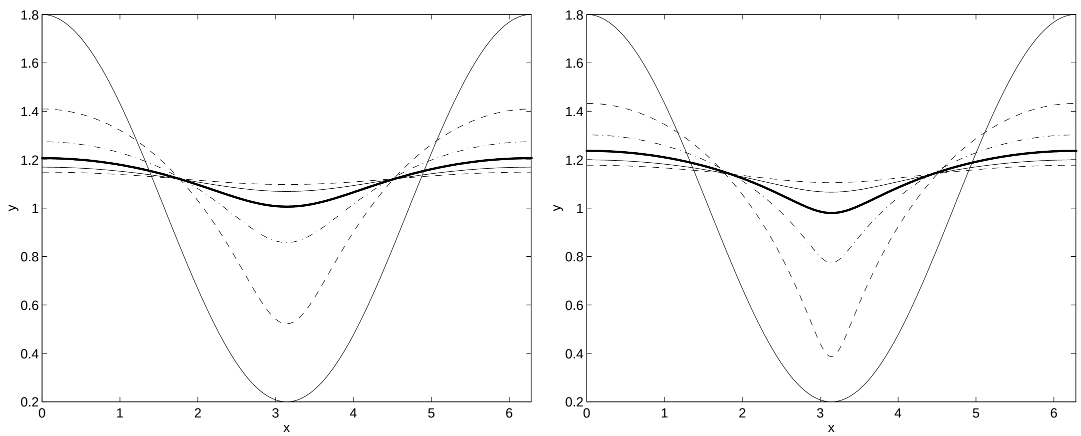

Figure 5: Solution of (1a) at times $t = 0, 1, 2, 3, 4$, and 5 with $\theta = .5$, $dt = .001$, $N = 512$, initial condition $1 + .8 \cos x$, and $\nu = .1$ (left) and $\nu = .009$ (right).

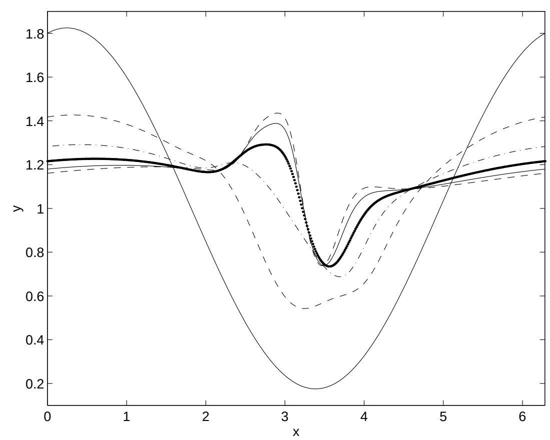

Figure 6: Solution of (1a) at times $t = 0, 1, 2, 3, 4$, and 5 with $\theta = 1/2$, $\nu = .1$, $dt = .001$, $N = 512$, and initial condition $1 + .8 \cos x + .2 \sin x$.

to obtain the Fourier coefficient of $(H(u)u)_x$, we multiply the Fourier coefficient of $H(u)u$ by ik. We must use a truncated Fourier series to perform these operations, and we update only those coefficients in the truncated series through either (21) or (22). For all calculations reported in this paper, we took $N = 2^p$ Fourier coefficients.

In Figure 5, we plot the solution of (1a), $\theta = 1/2$ and $\nu = .1$ and $\nu = .009$ at different times, with initial condition $1 + .8 \cos x$, a perturbation of a flat sheet. We use 512 Fourier coefficients to advance the solution with time step 10^{-3}. The solid curve closest to the $y = .2$ line is the initial condition $1 + .8 \cos x$; the dash curve closest to the $y = .2$ line is the solution of (1a), $\theta = 1/2$, at time $t = 1$; the dash-dot curve is the solution at time $t = 2$; the point curve is the solution at time $t = 3$; the solid curve closest to the $y = 1$ line is the solution at $t = 4$; and the dash curve closest to the $y = 1$ line is the solution at $t = 5$. As time evolves, the solution of (1a) tends to a straight line, which is not 1. This is not surprising because the average is not a conserved quantity. Comparing the solutions of (1a), $\theta = 1/2$, for $\nu = .1$ and $\nu = .009$, we see that the viscosity has smoothing effects on the solution of the equation, since the solution with $\nu = .009$ has steeper slope in the vicinity of π than the solution with $\nu = .1$.

In Figure 6, we plot the solution of (1a), $\theta = 1/2$, and $\nu = .1$, with initial condition $1 + .8 \cos x + .2 \sin x$. We use 512 Fourier coefficients to advance the solution with time step 10^{-3}. The solid curve closest to the $y = .1$ line is the initial condition $1 + .8 \cos x + .2 \sin x$; the dash curve closest to the

$y = .1$ line is the solution of (1a), $\theta = 1/2$, at time $t = 1$; the dash-dot curve is the solution at time $t = 2$, the point curve is the solution at $t = 3$; the solid curve closest to the $y = 1.1$ line is the solution at $t = 4$; and the dash curve closest to the $y = 1.1$ line is the solution at $t = 5$. It appears that viscosity stops the solution from having an infinite slop at $x = \pi$. The average of the solution is an increasing function of time.

In Figure 7, we plot the solutions of (1a) at time $t = 1$, 2, 3, 4, and 5, with $\theta = 1/2$, initial condition $1 + .8\cos x$, and different values of ν, $\nu = .1$, .07, .04, and .01. We use 512 Fourier coefficients to advance the solutions with time step 10^{-3}. For the top left figure, the solution of the equation at $t = 1$, the solid curve is the solution of (1a), with $\theta = 1/2$, and $\nu = .1$; the dash-dot curve is the solution with $\nu = .07$; the dash curve is the solution with $\nu = .04$; and the point curve is the solution with $\nu = .01$. As ν is decreased in the vicinity of π, the slope of the solution of (1a), $\theta = 1/2$, gets steeper and steeper. For the top right figure, the solution of the equation at $t = 2$, the solid curve is the solution of (1a), with $\theta = 1/2$, and $\nu = .1$; the dash-dot curve is the solution with $\nu = .07$; the dash curve is the solution with $\nu = .04$; and the point curve is the solution with $\nu = .01$. As at time $t = 1$, as ν is decreased in the vicinity of π, the slope of the solution of (1a), $\theta = 1/2$ gets steeper and steeper; the solutions at $t = 2$ for a given ν are not as steep as at $t = 1$. Also, at this time the four curves intersect at points closer to the tip of the curve than at the earlier time. For the left middle figure, the solution of the equation at $t = 3$, the solid curve is the solution of (1a), with $\theta = 1/2$, and $\nu = .1$; the dash-dot curve is the solution with $\nu = .07$; the dash curve is the solution with $\nu = .04$, and the point curve is the solution with $\nu = .01$. As ν is decreased, in the vicinity of π the slope of the solution of (1a), $\theta = 1/2$ gets steeper and steeper; nevertheless, the solutions do not present gradients as sharp as at earlier times. As pointed out, as time evolves, the location at which the four curves intersect moves toward the tip of the curves. For the middle right figure, the solution of the equation at $t = 4$, the solid curve is the solution of (1a), with $\theta = 1/2$, and $\nu = .1$; the dash-dot curve is the solution with $\nu = .07$; the dash curve is the solution with $\nu = .04$; and the point curve is the solution with $\nu = .01$. As ν is decreased in the vicinity of π, the slope of the solution of (1a), $\theta = 1/2$ gets steeper and steeper. Compared to earlier times, the slopes of the solutions are not as steep; the location at which the four curves intersect is very close to the tip of the curves. For the lowest figure, the solution of the equation at $t = 5$, the solid curve is the solution of (1a), with $\theta = 1/2$, and $\nu = .1$; the dash-dot curve is the solution with $\nu = .07$; the dash curve is the solution with $\nu = .04$; and the point curve is the solution with $\nu = .01$. As ν is decreased in the vicinity of π, the slope of the solution of (1a), $\theta = 1/2$ gets steeper and steeper, but for fixed ν, they are not as steep as at earlier times. Now, the curves no longer intersect.

From the study of the equation (1a), $\theta = 1/2$, with 2π-periodic initial and boundary conditions, we see that despite the fact that the solution exists for all time and is analytic when $\nu > 0$, the analyticity bandwidth is proportional to ν^5. Hence, one may encounter difficulties when one computes numerical solution of (1a) for small ν. Despite the analyticity result obtained here, the solution of (1a), $\theta = 1/2$, does not seem to be as well as behaved as the solution of (1a), $\theta = 0$: the solution of (1a), $\theta = 1/2$ and $\nu > 0$, satisfies only an L^2 norm bound independent of ν; the solution of (1a), $\theta = 0$ and $\nu > 0$, satisfies a maximum norm bound independent of ν.

3 Singularities Formation with $0 < \theta < 1/3$ and $\theta = 1$

In the preceding section and in [1], we have shown that the solution of (1a), $\theta = 0$ and $\theta = 1/2$, exists for all time if $\nu > 0$. We also have shown in [1] that, for certain initial conditions, the solution of (1a), $\theta = 1$ blows up in finite time. In this section, we show, for a certain class of initial conditions, and for $0 < \theta < 1/3$ and $\theta = 1$, that the solution of (1a) forms singularities in finite time. Instead of using the properties of the Hilbert transform and showing that the function $H(u) - iu$ satisfies Burgers' equation as in [1], we derive the system of ordinary differential equations for the Fourier coefficients of the solution of (1a). For $0 < \theta < 1/3$, we show that for symmetric initial conditions, the solution of (1a) blows up in finite time using the Fourier space method of Palais [3], [4], and [9]. We show, that if the Fourier modes of the initial condition are all strictly positive and such that $\hat{u}_k(0) = \hat{u}_{-k}(0)$, the system for the Fourier modes is cooperative. To prove that some solutions blow

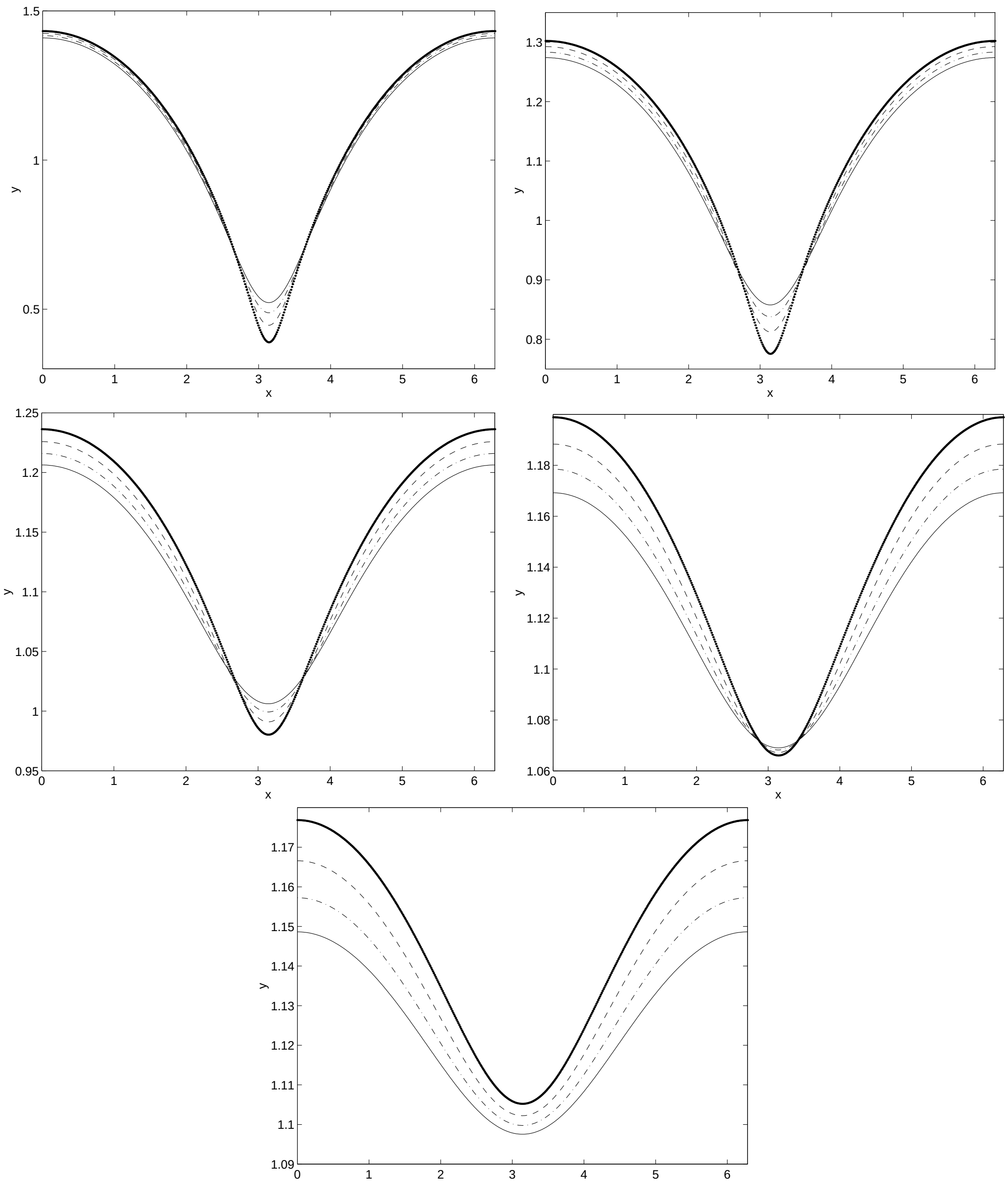

Figure 7: Solution of (1a) with $\theta = .5$ at time $t = 1$ (top left), $t = 2$ (top right), $t = 3$ (middle left), $t = 4$ (middle right), and $t = 5$, (lower), and different values of ν, $\nu = .1$, $.07$, $.04$, and $.01$ and initial condition $1 + .8\cos x$.

up in finite time, we then use Palais' result that the evolution of the $\hat{u}_k$ for any finite subsystem of the original one serves as a lower bound for the $\hat{u}_k$ of the full system. Hence, using Palais' method, we can prove that, for a certain class of initial conditions, the solution of the equation blows up in finite time. For $\theta = 1$, we explicitly solve the system of differential equations the Fourier coefficients of the solution of (1a) satisfy. The method, in the case $\theta = 1$, has limited application, since for most of the nonlinear partial differential equations of interest, we cannot obtain an explicit expression for the Fourier coefficients.

The Fourier coefficients of the solution of (1a) satisfy the infinite system of differential equations, if $k \neq 0$,

$$\frac{d}{dt}\hat{u}_k(t) = -\theta k \sum_{l\neq 0} \operatorname{sign}(l)\hat{u}_l\hat{u}_{k-l} - (1-\theta)\sum_{l\neq 0,k-l\neq 0} \operatorname{sign}(l)(k-l)\hat{u}_l\hat{u}_{k-l} - \nu k^2\hat{u}_k, \tag{23a}$$

and

$$\frac{d}{dt}\hat{u}_0(t) = (1-\theta)\sum_{l\neq 0} \operatorname{sign}(l) l\hat{u}_l\hat{u}_{-l} = 2(1-\theta)\sum_{l>0} l|\hat{u}_l|^2. \tag{23b}$$

To obtain the equations for the Fourier coefficients of (1a), we used the fact that

$$\begin{aligned} H(u) &= i\sum_{k\neq 0}\operatorname{sign}(k)\hat{u}_k e^{ikx}, \\ u_x &= i\sum_{k\neq 0} k\hat{u}_k e^{ikx}, \\ H(u)u &= i\sum_{k=-\infty}^{\infty}\sum_{l\neq 0}\operatorname{sign}(l)\hat{u}_l\hat{u}_{k-l}e^{ikx}, \\ H(u)u_x &= -\sum_{k=-\infty}^{\infty}\sum_{l\neq 0,k-l\neq 0}\operatorname{sign}(l)(k-l)\hat{u}_l\hat{u}_{k-l}e^{ikx}, \\ (H(u)u)_x &= -\sum_{k\neq 0} k\sum_{l\neq 0}\operatorname{sign}(l)\hat{u}_l\hat{u}_{k-l}e^{ikx}. \end{aligned}$$

Rewriting the expression for the kth Fourier coefficient of $(H(u)u)_x$, we have

$$((H(\widehat{u)u})_x)_k = -k\sum_{l>0}\hat{u}_l\hat{u}_{k-l} + k\sum_{l>0}\hat{u}_{-l}\hat{u}_{k+l}, \tag{24a}$$

$$= -k\sum_{l>0}\hat{u}_l\hat{u}_{k-l} + k\sum_{l>k}\hat{u}_l\hat{u}_{k-l}, \tag{24b}$$

$$= \begin{cases} 0 & \text{if } k=0 \\ -k\sum_{0<l<k}\hat{u}_l\hat{u}_{k-l} & \text{if } k>0 \\ k\sum_{k<l<0}\hat{u}_l\hat{u}_{k-l} & \text{if } k<0 \end{cases}. \tag{24c}$$

We obtain (24b) by making the change of variable $l' = k + l$ in the second sum of (24a) and by dropping the $'$ on l. We arrive at (24c) by considering the three cases $k = 0$, $k > 0$, and $k < 0$. Also note that if $\hat{u}_{-l} = \hat{u}_l$ for $l \geq 0$, then $((H(\widehat{u)u})_x)_k = ((H(\widehat{u)u})_x)_{-k}$.

We also obtain for the kth Fourier coefficient of $H(u)u_x$,

$$(H(\widehat{u)u}_x)_k = -\sum_{l>0,k-l\neq 0}(k-l)\hat{u}_l\hat{u}_{k-l} + \sum_{l>0,k-l\neq 0}(k+l)\hat{u}_{-l}\hat{u}_{k+l}, \tag{25a}$$

$$= -\sum_{l>0,k-l\neq 0}(k-l)\hat{u}_l\hat{u}_{k-l} + \sum_{l>k} l\hat{u}_l\hat{u}_{k-l}, \tag{25b}$$

$$= \begin{cases} 2\sum_{l>0} l|\hat{u}_l|^2 & \text{if } k=0, \\ \sum_{0<l<k}(k-l)\hat{u}_l\hat{u}_{k-l} + \sum_{l>k}(2l-k)\hat{u}_l\hat{u}_{k-l} & \text{if } k>0, \\ \sum_{k<l<0} l\hat{u}_l\hat{u}_{k-l} + \sum_{l>0}(2l-k)\hat{u}_l\hat{u}_{k-l} & \text{if } k<0. \end{cases} \tag{25c}$$

We obtain (25b) by making the change of variable $l' = k + l$ in the second sum of (25a) and by dropping the $'$ on l. We arrive at (25c) by considering the three cases $k = 0$, $k > 0$, and $k < 0$. Also note that if $\hat{u}_{-l} = \hat{u}_l$ for $l \geq 0$, then $(\widehat{H(u)u_x})_k = (\widehat{H(u)u_x})_{-k}$.

If we take into account the above expressions for $((\widehat{H(u)u})_x)_k$ and $(\widehat{H(u)u_x})_k$, the equation (23a) becomes

$$\frac{d}{dt}\hat{u}_k + \nu k^2 \hat{u}_k = \tag{26}$$
$$\begin{cases} \sum_{0<l<k}[(1-2\theta)k - (1-\theta)l]\hat{u}_l\hat{u}_{k-l} + (1-\theta)\sum_{l>k}(2l-k)\hat{u}_l\hat{u}_{k-l}, & \text{if } k > 0, \\ \sum_{k<l<0}[\theta(k-l)+l]\hat{u}_l\hat{u}_{k-l} + (1-\theta)\sum_{l>0}(2l-k)\hat{u}_l\hat{u}_{k-l}. & \text{if } k < 0. \end{cases}$$

From now on, we restrict ourselves to symmetric initial conditions, that is, initial conditions for which $\hat{u}_k(0) = \hat{u}_{-k}(0)$. Then, from the symmetry properties, (26) reduces to

$$\frac{d}{dt}\hat{u}_0 = 2(1-\theta)\sum_{l>0} l|\hat{u}_l|^2, \tag{27a}$$

$$\begin{aligned} \frac{d}{dt}\hat{u}_{2k+1} + \nu(2k+1)^2\hat{u}_{2k+1} &= (2k+1)(1-3\theta)\sum_{1\leq l\leq k}\hat{u}_l\hat{u}_{2k+1-l} \\ &\quad +(1-\theta)\sum_{l>2k+1}(2l-(2k+1))\hat{u}_l\hat{u}_{l-2k-1}, \quad k \geq 0 \end{aligned} \tag{27b}$$

$$\begin{aligned} \frac{d}{dt}\hat{u}_{2k} + 4\nu k^2\hat{u}_{2k} &= 2k(1-3\theta)\sum_{1\leq l\leq k-1}\hat{u}_l\hat{u}_{2k-l} + k(1-3\theta)[\hat{u}_k]^2 \\ &\quad +2(1-\theta)\sum_{l>2k}(l-k)\hat{u}_l\hat{u}_{l-2k}, \quad k \geq 1. \end{aligned} \tag{27c}$$

Now we can prove the equivalent of Lemma 3.2 of [4] for the solution of (27a), (27b), and (27c).

Lemma 3.1 *If the solution is such that $\hat{u}_k(0) = \hat{u}_{-k}(0) > 0$, for all k, and if $0 < \theta \leq 1/3$, then for all subsequent times (such that a solution exists),*

$$\hat{u}_k(t) > 0. \tag{28}$$

Proof: Suppose that $\hat{u}_m$ is the first coefficient to violate (28) at time t^* such that $\hat{u}_m(t^*) = 0$. Then it follows from (27a), (27b), and (27c) that

$$\hat{u}_m(t^*) = \hat{u}_m(0)e^{-\nu m^2 t^*} + e^{-\nu m^2 t^*}\int_0^{t^*} f(\tau)d\tau, \tag{29}$$

with the function f given by

$$2(1-\theta)\sum_{l>0} l|\hat{u}_l|^2, \qquad \text{if } m = 0,$$

$$e^{\nu(2k+1)^2 t}\left[(2k+1)(1-3\theta)\sum_{1\leq l\leq k}\hat{u}_l\hat{u}_{2k+1-l} \right.$$
$$\left. +(1-\theta)\sum_{l>2k+1}(2l-2k-1)\hat{u}_l\hat{u}_{l-2k-1}\right], \qquad \text{if } m = 2k+1,\ k \geq 0,$$

$$e^{4\nu k^2 t}\left[2k(1-3\theta)\sum_{1\leq l\leq k-1}\hat{u}_l\hat{u}_{2k-l} + k(1-3\theta)[\hat{u}_k]^2 \right.$$
$$\left. +2(1-\theta)\sum_{l>2k}(l-k)\hat{u}_l\hat{u}_{l-2k}\right], \qquad \text{if } m = 2k,\ k \geq 1,$$

Note that if $0 < \theta \leq 1/3$, all the terms in the right-hand side of (29) are positive for $0 \leq t < t^*$. Thus, (29) is strictly positive, which is a contradiction. ∎

Now that we have shown that the system for the Fourier modes $\hat{u}_k$ is cooperative [3], we use Palais' method [9] to prove that the solution of (27a), (27b), and (27c) becomes infinite in finite time. Palais shows that the evolution of the $\hat{u}_k$ for any finite subsystem of (27a), (27b), and (27c) serves as a lower bound for the $\hat{u}_k$ of the full system provided that the full system is cooperative and that the initial condition satisfies $\hat{u}_k(0) \geq 0$ for all k.

Lemma 3.2 *Let u be a solution to*

$$u_t = \theta(H(u)u)_x + (1-\theta)H(u)u_x + \nu u_{xx}, \tag{30a}$$

$$u(x,0) = f(x), \tag{30b}$$

f a 2π-periodic function, and $0 < \theta < 1/3$. There exists initial data f that produces a solution that blows up in finite time.

Proof: From Palais' work and Lemma 3.1, we need to show there exists a subsystem of (27a), (27b), and (27c) whose solution becomes infinite in finite time. If $0 < \theta < 1/3$, consider the system

$$\frac{d}{dt}p = (1-\theta)pq - \nu p, \tag{31a}$$

$$\frac{d}{dt}q = (1-3\theta)p^2 - 4\nu q, \tag{31b}$$

which is obtained from (27b) and (27c) by taking $k = 0$ and 1. If $\hat{u}_1(0) = p(0)$ and $\hat{u}_2(0) = q(0)$, then $\hat{u}_1(t) > p(t)$ and $\hat{u}_2(t) > q(t)$ $\forall t > 0$. The system (31a) and (31b) has an attractive node at (0,0) (the eigenvalues of the Jacobian are $-\nu$ and -4ν) and an instable fixed point at $(2\nu/\sqrt{(1-3\theta)(1-\theta)}, \nu/(1-\theta))$ (the eigenvalues of the Jacobian are $2\nu(1+i)$ and $2\nu(1-i)$). If $p = q$, then

$$\frac{d}{dt}p = (1-\theta)q^2 - \nu q, \qquad \frac{d}{dt}q = (1-3\theta)q^2 - 4\nu q.$$

So, if $q > 4\nu/(1-3\theta)$, then $0 < dq/dp < 1$ and if at $t = 0$, $p > q$, then $p(t) > q(t)$ for all $t > 0$ provided $q(t) > 4\nu/(1-3\theta)$. Suppose that $4\nu/(1-3\theta) < q < p$. Then

$$\frac{d}{dt}q > (1-3\theta)q^2 - 4\nu q. \tag{32}$$

Therefore $dq/dt > 0$ and $q(t) > 4\nu/(1-3\theta)$ for all $t > 0$. Integration of (32) gives us

$$q(t) > \frac{4\nu q(0)}{(1-3\theta)q(0) + (4\nu - (1-3\theta)q(0))e^{4\nu t}}. \tag{33}$$

So $q(t)$ blows up at a finite time T, with

$$T < \frac{1}{4\nu}\ln\left(\frac{(1-3\theta)q(0)}{(1-3\theta)q(0) - 4\nu}\right).$$

Therefore p and q solution of (31a) and (31b) blow up in finite time; The solution of (27a), (27b), and (27c) blows up at a time $t \leq T$. ∎

We can also show that the solution of (1a) with $\theta = 1$ blows up in finite time using a different approach from Palais' comparison method and the explicit construction of a "traveling wave" solution, presented in [1], that blows up in finite time. Instead, we derive the system the Fourier coefficients satisfy, and we solve the system.

Lemma 3.3 *Let u be a solution to*

$$u_t = (H(u)u)_x + \nu u_{xx},$$

$$u(x,0) = f(x),$$

f a 2π-periodic function. There exists initial data f that produces a solution that blows up in finite time.

Proof: If $\theta = 1$, (23b) and (26) reduce to

$$\frac{d}{dt}\hat{u}_0(t) = 0, \tag{34a}$$

$$\frac{d}{dt}\hat{u}_k(t) + \nu k^2 \hat{u}_k(t) = \begin{cases} -k\sum_{0<l<k} \hat{u}_l \hat{u}_{k-l}, & \text{if } k > 0, \\ k\sum_{k<l<0} \hat{u}_l \hat{u}_{k-l}, & \text{if } k < 0. \end{cases} \tag{34b}$$

Since we have previously shown that, if $\hat{u}_k(0) = \hat{u}_{-k}(0)$, $\hat{u}_k$ real, then $\hat{u}_k(t) = \hat{u}_{-k}(t)$, we construct by induction an explicit solution of (34a) and (34b). More precisely, we look for the solution of the system with initial condition $\hat{u}_0(0) \neq 0$, $\hat{u}_1(0) = \hat{u}_{-1}(0) \neq 0$, and $\hat{u}_k(0) = \hat{u}_{-k}(0) = 0$ for $k \geq 2$. Integration of (34a) and (34b) for $k = 1$ and 2 gives

$$\begin{aligned} \hat{u}_0(t) &= \hat{u}_0(0), \\ \hat{u}_1(t) &= \hat{u}_1(0)\exp(-\nu t), \\ \hat{u}_2(t) &= -[\hat{u}_1(0)]^2 \frac{e^{-2\nu t} - e^{-4\nu t}}{\nu}. \end{aligned}$$

Now, we can use an induction process to derive the expression for $\hat{u}_k$, $k \geq 3$. Assume that

$$\hat{u}_k(t) = -\frac{[\hat{u}_1(0)]^k}{\nu^{k-1}} \sum_{0<l<k} f_l(t) f_{k-l}(t),$$

with $f_l(t)$, $1 \leq l < k$, positive function for $t \geq 0$, $f_l(t) = \sum_m C_m^l e^{\nu \alpha_m^l t}$, C_m^l and α_m^l constants depending only on m and l. From the differential equation (34b), we find that

$$\frac{d}{dt}\left(e^{\nu(k+1)^2 t}\hat{u}_{k+1}(t)\right) = -e^{\nu(k+1)^2 t}\frac{[\hat{u}_1(0)]^{k+1}}{\nu^{k-1}}(k+1)\sum_{0<l<k+1} f_l(t) f_{k+1-l}(t),$$

with the function $e^{\nu(k+1)^2 t}\sum_{0<l<k+1} f_l(t) f_{k+1-l}(t)$ again a positive function for $t \geq 0$. From the expression of $f_l(t)$, we conclude that

$$\hat{u}_{k+1}(t) = -\frac{[\hat{u}_1(0)]^{k+1}}{\nu^k}\sum_{0<l<k+1} g_l(t),$$

with

$$g_l(t) = e^{-\nu(k+1)^2 t}(k+1)\int_0^t e^{\nu(k+1)^2\tau} f_l(\tau) f_{k+1-l}(\tau) d\tau,$$

which is of the form $\sum_m D_m^l e^{\nu \alpha_m^l t}$, with D_m^l a constant depending only on m and l.

So the L^2 norm of the solution of (1a) with $\theta = 1$ and the initial condition $\hat{u}_0(0) \neq 0$, $\hat{u}_1(0) = \hat{u}_{-1}(0)$ is

$$||u(\cdot,t)||^2 = |\hat{u}_0(0)|^2 + 2\sum_{k=1}^{\infty}\frac{[\hat{u}_1(0)]^{2k}}{\nu^{2(k-1)}}\left[\sum_{0<l<k} f_l(t) f_{k-l}(t)\right]^2. \tag{35}$$

The function $f_1(t)$ is strictly positive for $t \geq 0$ and tends to 0 as $t \to \infty$; the function $f_l(t)$, $l \geq 2$, is 0 at $t = 0$, is strictly positive for $t > 0$, and tends to 0 as $t \to \infty$. Let $a(t)$ be the minimum of $f_k(t)$ over k; the minimum is strictly positive when $t > 0$, since each term is strictly positive. Then (35) becomes

$$||u(\cdot,t)||^2 \geq |\hat{u}_0(0)|^2 + 2[\hat{u}_1(0)]^2[a(t)]^2\sum_{k=0}^{\infty} k^2\left(\frac{[\hat{u}_1(0)]^2}{\nu^2}\right)^k. \tag{36}$$

The infinite series in the right-hand side of (36) converges provided $|\hat{u}_1(0)|/\nu < 1$. Hence, if $|\hat{u}_1(0)| > \nu$, there exists a time $t^* < \infty$ for which the L^2 norm of u is infinite.

Note that to simplify the algebra, we look for a solution with an initial condition of the form $a_0 + a_1\cos x$; the result extends to any initial condition of the form $\sum_{k=0}^{\infty} a_k\cos(kx)$. ∎

4 Conclusion and Open Questions

From the study of the continuum of partial differential equation (1a) and the work obtained in [1] and [8], we conclude that second-order viscous regularization prevents singularity formation in finite time only in two cases: when the nonlinear contribution only contains the flux term in nonconservative form and when the contribution of the flux term in conservative form balances the contribution of the flux term in nonconservative form for the L^2 scalar product with the solution of the equation. From the results obtained in Section 3, it seems that the flux term in conservative form cannot be balanced by the flux term in nonconservative form and second-order viscous regularization, even when $(1-\theta) > \theta$, $0 < \theta < 1/3$.

A few open questions remain, whose answers would shed some light on the behavior of the solutions of the continuum of partial differential equations:

- Does a weak limit of the sequence of solutions $(u^{\nu,1/2})_{\nu>0}$ exist? If so, to which space does it belong?
- What is the decay rate of the Fourier coefficients of the solution of (1a), $\theta = 1/2$, when $\nu = 0$ before singularity forms? What type of singularity forms when $\nu = 0$?
- What kind of viscous regularization should be introduced to control the nonlinear flux term in conservative form?
- Do singularities form in finite time when $1/3 \leq \theta < 1/2$ and $1/2 < \theta < 1$?

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