DETERMINING NODES FOR THE GINZBURG-LANDAU EQUATIONS OF SUPERCONDUCTIVITY

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Abstract. It is shown that a solution of the time-independent Ginzburg-Landau equations of superconductivity is determined completely and exactly by its values at a finite but sufficiently dense set of determining nodes in the domain. If the applied magnetic field is time dependent and asymptotically stationary, the large-time asymptotic behavior of a solution of the time-dependent Ginzburg-Landau equations of superconductivity is determined similarly by its values at a finite set of determining nodes, whose positions may vary with time.

Key words. Ginzburg-Landau equations, superconductivity, large-time asymptotic behavior, determining nodes.

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1 Introduction

In this article, we prove the existence of a set of determining nodes for the Ginzburg-Landau equations of superconductivity.

The concept of "determining nodes" was first introduced in the context of the Navier-Stokes equations for viscous incompressible fluids by FOIAS and TEMAM [8]. In an effort to identify a finite number of parameters controlling a turbulent flow, these authors showed that there exists a set of points (*determining nodes*) having the property that the values of the velocity vector at these points (*nodal values*) completely and exactly determine the large-time asymptotic behavior of the flow. The cardinality of the set of determining nodes (N) is unknown but can be estimated in terms of the physical parameters, at least in principle; Jones and Titi [15] give an estimate of N in terms of the Grashof number, G. It has been conjectured on the basis of the Takens imbedding theorem [20] that, for dissipative partial differential equations like the two-dimensional Navier-Stokes equations, N is in fact independent of the physical parameters and determined entirely by the dimensionality of the spatial domain. This conjecture has been verified for the one-dimensional Ginzburg-Landau equation by Kukavica [18] ($N = 2$) and for the one-dimensional Kuramoto-Sivashinsky equation by Foias and Kukavica [7] ($N = 4$).

In this article, we explore the notion of determining nodes for the Ginzburg-Landau equations of superconductivity. These equations are more complicated than the Ginzburg-Landau or amplitude equation commonly considered in the mathematical community and studied, for example, in [18]. The Ginzburg-Landau equations of superconductiviy are two coupled partial differential equations for the unknown (complex-valued) order parameter and (real vector-valued) vector potential; in time-dependent problems there is a third unknown, the real scalar-valued electric potential, which is a diagnostic variable somewhat similar to a Lagrange multiplier.

We show that in the time-independent case a solution of the Ginzburg-Landau equations of superconductivity is determined completely and exactly by a finite number of nodal values, while in the time-dependent case the large-time asymptotic behavior of the solution is determined completely and exactly by the nodal values. In the latter case, we allow for a time-varying but asymptotically stationary applied magnetic field; the determining nodes themselves may also vary with time.

The Ginzburg-Landau equations of superconductivity admit vortex solutions, which are characterized by singularities of the complex order parameter; see [3]. Vortex solutions are of particular interest in solid-state physics, as well as for techno-

logical applications. It is tempting to speculate (but by no means certain) that the determining nodes are related to the vortices. If so, the dynamics of the Ginzburg-Landau equations of superconductivity are determined entirely by the dynamics of the vortices—a significant reduction of dimensionality. Important results in this direction were obtained recently by Jerrard [14].

The remainder of this article consists of four sections. In Section 2 we give the classical formulation of the Ginzburg-Landau equations of superconductivity, in Section 3 the functional formulation in a Hilbert space. Section 4 contains some auxiliary estimates of the size of a function in terms of its values on a finite point set. In Section 5 we present our results on determining nodes.

2 Ginzburg-Landau Equations

In this section we present the classical formulation of the Ginzburg-Landau equations of superconductivity; see [10, 1, 4, 22]. We use the symbol Ω to denote the region occupied by the superconducting material, assuming that Ω is an open bounded subset of $\mathbf{R}^n$ $(n = 2, 3)$ with boundary $\partial\Omega$ of class $C^{1,1}$.

2.1 Time-Independent Equations

The state of a superconductor is described by a complex-valued *order parameter* ψ and a real vector-valued *vector potential* $\boldsymbol{A}$. Identifying the complex plane with $\mathbf{R}^2$ in the usual way, we have $\psi : \Omega \to \mathbf{R}^2$ and $\boldsymbol{A} : \Omega \to \mathbf{R}^n$. An equilibrium state corresponds to a critical point of the Helmholtz free-energy functional,

$$E[\psi, \boldsymbol{A}] = \int_\Omega \left[\left| \left(\frac{i}{\kappa}\nabla + \boldsymbol{A} \right) \psi \right|^2 + \frac{1}{2}\left(1 - |\psi|^2\right)^2 + |\nabla \times \boldsymbol{A} - \boldsymbol{H}|^2 \right] \mathrm{d}x$$

$$+ \int_{\partial\Omega} \gamma \left| \frac{i}{\kappa}\psi \right|^2 \mathrm{d}\sigma(x). \tag{2.1}$$

Here, κ is the (dimensionless) Ginzburg-Landau parameter, which is the ratio of the characteristic length scales for the vector potential and the order parameter. The function γ is defined on $\partial\Omega$, and $\gamma(x) \geq 0$ for $x \in \partial\Omega$. The other symbols have their usual meaning: $\nabla \equiv$ grad, $\nabla\times \equiv$ curl, $\nabla\cdot \equiv$ div, and $\nabla^2 = \nabla \cdot \nabla \equiv \Delta$; i is the imaginary unit, and the superscript * denotes complex conjugation. The vector $\boldsymbol{H}$

represents the *applied magnetic field*; it is a given function of position, which takes its values in $\mathbf{R}^n$.

A critical point of $E \equiv E[\psi, \boldsymbol{A}]$ is a solution of the boundary-value problem

$$-\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \psi + \left(1 - |\psi|^2\right)\psi = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$-\nabla \times \nabla \times \boldsymbol{A} + \boldsymbol{J}_s + \nabla \times \boldsymbol{H} = \boldsymbol{0} \quad \text{in } \Omega, \tag{2.3}$$

$$\boldsymbol{n} \cdot \left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)\psi + \gamma\frac{i}{\kappa}\psi = 0 \quad \text{and} \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A} - \boldsymbol{H}) = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{2.4}$$

Here, $\boldsymbol{n}$ is the local outer unit normal to $\partial\Omega$. The vector $\boldsymbol{J}_s$ is the *supercurrent density*, which is a nonlinear function of ψ and $\boldsymbol{A}$,

$$\boldsymbol{J}_s \equiv \boldsymbol{J}_s(\psi, \boldsymbol{A}) = \frac{1}{2i\kappa}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - |\psi|^2\boldsymbol{A} = -\mathrm{Re}\left[\psi^*\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)\psi\right]. \tag{2.5}$$

Equations (2.2)–(2.4) are the Ginzburg-Landau (GL) equations of superconductivity. They embody, in a most simple way, the macroscopic quantum-mechanical nature of the superconducting state. The trivial solution, $\psi = 0$ and $\nabla \times \boldsymbol{A} = \boldsymbol{H}$, represents the *normal state*, where all superconducting properties have been lost. Note the identities

$$-\left(\frac{i}{\kappa}\nabla + \boldsymbol{A}\right)^2 \psi = \frac{1}{\kappa^2}\Delta\psi - \frac{2i}{\kappa}(\nabla\psi)\cdot\boldsymbol{A} - \frac{i}{\kappa}\psi(\nabla\cdot\boldsymbol{A}) - \psi|\boldsymbol{A}|^2 \tag{2.6}$$

and

$$-\nabla \times \nabla \times \boldsymbol{A} = \Delta\boldsymbol{A} - \nabla(\nabla\cdot\boldsymbol{A}). \tag{2.7}$$

The GL equations are invariant under a gauge transformation,

$$G_\chi : (\psi, \boldsymbol{A}) \mapsto \left(\psi \mathrm{e}^{i\kappa\chi}, \boldsymbol{A} + \nabla\chi\right). \tag{2.8}$$

For the present investigation we adopt the London gauge [22, Chapter 4], where any solution $(\psi, \boldsymbol{A})$ of the GL equations satisfies the constraints

$$\nabla\cdot\boldsymbol{A} = 0 \quad \text{in } \Omega, \qquad \boldsymbol{n}\cdot\boldsymbol{A} = 0 \quad \text{on } \partial\Omega. \tag{2.9}$$

Under these constraints, Eqs. (2.2)–(2.4) reduce to

$$-\frac{1}{\kappa^2}\Delta\psi = -\frac{2i}{\kappa}(\nabla\psi)\cdot\boldsymbol{A} - \psi|\boldsymbol{A}|^2 + \left(1 - |\psi|^2\right)\psi \quad \text{in } \Omega, \tag{2.10}$$

$$\nabla \times \nabla \times \boldsymbol{A} = \boldsymbol{J}_s + \nabla \times \boldsymbol{H} \quad \text{and} \quad \nabla\cdot\boldsymbol{A} = 0 \quad \text{in } \Omega, \tag{2.11}$$

$$\boldsymbol{n}\cdot\nabla\psi + \gamma\psi = 0, \quad \boldsymbol{n}\cdot\boldsymbol{A} = 0, \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A} - \boldsymbol{H}) = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{2.12}$$

Henceforth, we refer to the system of Eqs. (2.10)–(2.12) as the "gauged GL equations." This system will be the starting point for the functional formulation in Section 3.2.

2.2 Time-Dependent Equations

A generalization of the GL equations to the time-dependent case was first proposed by SCHMID [19] and subsequenly validated by GOR'KOV and ELIASHBERG [12] in the context of the microscopic Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity. Because of gauge invariance, the generalization is nontrivial. In addition to the order parameter and the vector potential, a third variable is needed to complete the description of the physical state of the system in a manner consistent with the gauge invariance. This is the *electric* or *scalar potential* ϕ, a real scalar-valued function of position and time. It is a diagnostic variable, as opposed to the prognostic variables ψ and $\boldsymbol{A}$. The evolution of ψ and $\boldsymbol{A}$ is described by the equations

$$\eta\left(\frac{\partial}{\partial t}+i\kappa\phi\right)\psi=-\left(\frac{i}{\kappa}\nabla+\boldsymbol{A}\right)^2\psi+\left(1-|\psi|^2\right)\psi \quad \text{in } \Omega\times(0,\infty), \tag{2.13}$$

$$\frac{\partial\boldsymbol{A}}{\partial t}+\nabla\phi=-\nabla\times\nabla\times\boldsymbol{A}+\boldsymbol{J}_s+\nabla\times\boldsymbol{H} \quad \text{in } \Omega\times(0,\infty), \tag{2.14}$$

$$\boldsymbol{n}\cdot\left(\frac{i}{\kappa}\nabla+\boldsymbol{A}\right)\psi+\frac{i}{\kappa}\gamma\psi=0 \quad \text{and} \quad \boldsymbol{n}\times(\nabla\times\boldsymbol{A}-\boldsymbol{H})=\boldsymbol{0} \quad \text{on } \partial\Omega\times(0,\infty). \tag{2.15}$$

The parameter η measures the ratio of the relaxation times for the vector potential and the order parameter. Equations (2.13)–(2.15), together with the expression (2.5) for $\boldsymbol{J}_s$, are the time-dependent Ginzburg-Landau (TDGL) equations of superconductivity.

The TDGL equations are invariant under a gauge transformation

$$G_\chi:(\psi,\boldsymbol{A},\phi)\mapsto\left(\psi\mathrm{e}^{i\kappa\chi},\boldsymbol{A}+\nabla\chi,\phi-\partial_t\chi\right). \tag{2.16}$$

In the present investigation we adopt the generalized Lorentz or "$\phi=-\omega(\nabla\cdot\boldsymbol{A})$" ($\omega>0$) gauge, where any solution $(\psi,\boldsymbol{A},\phi)$ of the TDGL equations satisfies the constraints

$$\phi+\omega(\nabla\cdot\boldsymbol{A})=0 \quad \text{in } \Omega\times(0,\infty), \qquad \boldsymbol{n}\cdot\boldsymbol{A}=0 \quad \text{on } \partial\Omega\times(0,\infty). \tag{2.17}$$

Clearly, $\int_\Omega\phi\,\mathrm{d}x=0$ at all times. The "$\phi=-\omega(\nabla\cdot\boldsymbol{A})$" gauge generalizes the classical Lorentz gauge ($\omega=1$) and reduces to the zero-electric potential ($\phi=0$) as $\omega\to0$ [5].

Under the constraints (2.17), the TDGL equations (2.13)–(2.15) reduce to a system of equations for ψ and $\boldsymbol{A}$,

$$\eta\frac{\partial\psi}{\partial t}=-\left(\frac{i}{\kappa}\nabla+\boldsymbol{A}\right)^2\psi+i\eta\kappa\omega\psi(\nabla\cdot\boldsymbol{A})+\left(1-|\psi|^2\right)\psi \quad \text{in } \Omega\times(0,\infty), \tag{2.18}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \times \nabla \times \boldsymbol{A} + \omega \nabla(\nabla \cdot \boldsymbol{A}) + \boldsymbol{J}_s + \nabla \times \boldsymbol{H} \quad \text{in } \Omega \times (0, \infty), \tag{2.19}$$

$$\boldsymbol{n} \cdot \nabla \psi + \gamma \psi = 0, \quad \boldsymbol{n} \cdot \boldsymbol{A} = 0, \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A} - \boldsymbol{H}) = \boldsymbol{0} \quad \text{on } \partial\Omega \times (0, \infty). \tag{2.20}$$

The system of Eqs. (2.18)–(2.20) is supplemented by a set of initial data, which may assume the form

$$\psi = \psi_0 \quad \text{and } \boldsymbol{A} = \boldsymbol{A}_0 \quad \text{on } \Omega \times \{0\}. \tag{2.21}$$

Henceforth, we use the term "gauged TDGL equations" to refer to the system of Eqs. (2.18)–(2.20). This system will be the starting point for the functional formulation in Section 3.3.

The gauged TDGL equations generate a dynamical process [6]. If the applied magnetic field is time independent, the dynamical process is a dynamical system [6, 21]. If the applied magnetic field is asymptotically stationary, the dynamical process is asymptotically autonomous and approaches a dynamical system; in this case, the omega-limit set of the dynamical process coincides with the attractor of the dynamical system [17]. Finally, any solution of the TDGL equation in the "$\phi = -\omega(\nabla \cdot \boldsymbol{A})$" gauge that is gauge equivalent with a stationary physical state tends to a solution of the GL equations in the "$\nabla \cdot \boldsymbol{A} = 0$" London gauge as time goes to infinity [21, 16]. These notions will be made more precise in Section 3.

3 Functional Formulation

In this section we rewrite the Ginzburg-Landau equations in functional form.

3.1 Definitions and Notation

We begin with a summary of our notational conventions. Throughout the following analysis, $\Omega \subset \mathbf{R}^n$ ($n = 2$ or $n = 3$), Ω is bounded, and $\partial\Omega$ is of class $C^{1,1}$. The function $\gamma : \partial\Omega \to \mathbf{R}$ is Lipschitz continuous, with $\gamma(x) \geq 0$ for all $x \in \partial\Omega$. The symbol C denotes a generic positive constant, not necessarily the same at different occurrences; in general, these constants depend on the parameters of the problem (γ, η, κ, Ω, and $\boldsymbol{H}$), but we do not indicate this dependence explicitly.

All Banach spaces are real; the (real) dual of a Banach space X is denoted by X'. The Banach spaces in this investigation are the standard ones [2, 13]: the Lebesgue spaces $L^p(\Omega)$ for $1 \leq p < \infty$, with norm $\|\cdot\|_{L^p}$; the Sobolev spaces $H^m(\Omega) = W^{m,2}(\Omega)$

for nonnegative integer m, with norm $\|\cdot\|_{H^m}$; the fractional Sobolev spaces $H^s(\Omega) = W^{s,2}(\Omega)$ for noninteger s, with norm $\|\cdot\|_{H^s}$; and the spaces $C^\nu(\overline{\Omega})$, for $\nu \geq 0$, $\nu = m+\lambda$ with $0 \leq \lambda < 1$, of m times continuously differentiable functions on the closure of Ω, whose mth-order derivatives satisfy a Hölder condition with exponent λ if ν is not an integer, with norm $\|\cdot\|_{C^\nu}$. The definitions extend to spaces of vector-valued functions in the usual way. Complex-valued functions are interpreted as vector-valued functions with two real components.

Because the regularity requirements for the order parameter ψ and the vector potential $\boldsymbol{A}$ are the same, we use the notation $\mathcal{X} = X^2 \times X^n$ for any Banach space X of real-valued functions defined on Ω. Thus, X^2 and X^n are the underlying spaces for ψ and $\boldsymbol{A}$, respectively. The basic framework for the functional analysis of the Ginzburg-Landau equations is the Cartesian product

$$\mathcal{H}^{1+\alpha} = [H^{1+\alpha}(\Omega)]^2 \times [H^{1+\alpha}(\Omega)]^n$$

for some $\alpha \in (\frac{1}{2}, 1)$. This space is continuously imbedded in $\mathcal{H}^1 \cap \mathcal{L}^\infty$.

Functions of space and time defined on $\Omega \times [0,T]$, for some $T > 0$, are considered as mappings from the time domain $[0,T]$ into a Banach space $X = (X, \|\cdot\|_X)$ of functions on the spatial domain Ω and may be considered as elements of $L^p(0,T;X)$ for $1 \leq p \leq \infty$, or $H^s(0,T;X)$ for nonnegative s, or $C^\nu([0,T];X)$ for $\nu \geq 0$.

A solution of the gauged GL equations is a function $(\psi, \boldsymbol{A}) \in \mathcal{H}^{1+\alpha}$, for some $\alpha \in (\frac{1}{2}, 1)$, which satisfies Eqs. (2.2)–(2.4) and the constraint (2.9) in the sense of distributions.

A solution of the gauged TDGL equations on the interval $[0,T]$, for some $T > 0$, is a function $(\psi, \boldsymbol{A}) \in C([0,T]; \mathcal{H}^{1+\alpha})$, with values $(\psi, \boldsymbol{A})(t) \equiv (\psi(t), \boldsymbol{A}(t)) \in \mathcal{H}^{1+\alpha}$, for some $\alpha \in (\frac{1}{2}, 1)$, which satisfies Eqs. (2.18)–(2.20) in the sense of distributions for each $t \in (0,T)$.

3.2 Time-Independent Case

The functional formulation of the gauged GL equations requires the homogenization of the boundary conditions (2.12).

Assume $\boldsymbol{H} \in [L^2(\Omega)]^n$. Let $\boldsymbol{A}_{\mathbf{H}}$ be the (unique) minimizer of the convex quadratic form $J \equiv J[\boldsymbol{A}]$,

$$J[\boldsymbol{A}] = \int_\Omega |\nabla \times \boldsymbol{A} - \boldsymbol{H}|^2 \, \mathrm{d}x \tag{3.1}$$

on the domain

$$\mathcal{D}(J) = \{\boldsymbol{A} \in [H^1(\Omega)]^n : \nabla \cdot \boldsymbol{A} = 0 \text{ in } \Omega,\ \boldsymbol{n} \cdot \boldsymbol{A} = 0 \text{ on } \partial\Omega\}. \tag{3.2}$$

The vector $\boldsymbol{A}_{\mathbf{H}}$ is a weak solution of the boundary-value problem

$$-\nabla \times \nabla \times \boldsymbol{A}_{\mathbf{H}} + \nabla \times \boldsymbol{H} = \mathbf{0} \quad \text{and} \quad \nabla \cdot \boldsymbol{A}_{\mathbf{H}} = 0 \quad \text{in } \Omega, \tag{3.3}$$

$$\boldsymbol{n} \cdot \boldsymbol{A}_{\mathbf{H}} = 0 \quad \text{and} \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A}_{\mathbf{H}} - \boldsymbol{H}) = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3.4}$$

The mapping $\boldsymbol{H} \mapsto \boldsymbol{A}_{\mathbf{H}}$ is linear and continuous from $[H^\theta(\Omega)]^n$ to $[H^{1+\theta}(\Omega)]^n$, for $0 \le \theta \le 1$; see [6, Lemma 4].

Given $\boldsymbol{A}_{\mathbf{H}}$, we introduce the *reduced vector potential* $\boldsymbol{A}'$,

$$\boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}}, \tag{3.5}$$

and reformulate the gauged GL equations in terms of ψ and $\boldsymbol{A}'$,

$$-\frac{1}{\kappa^2}\Delta\psi = -\frac{2i}{\kappa}(\nabla\psi) \cdot (\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}) - \psi|\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}|^2 + \left(1 - |\psi|^2\right)\psi \quad \text{in } \Omega, \tag{3.6}$$

$$\nabla \times \nabla \times \boldsymbol{A}' = \frac{1}{2i\kappa}\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right) - |\psi|^2(\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}) \quad \text{and} \quad \nabla \cdot \boldsymbol{A}' = 0 \quad \text{in } \Omega, \tag{3.7}$$

$$\boldsymbol{n} \cdot \nabla\psi + \gamma\psi = 0, \quad \boldsymbol{n} \cdot \boldsymbol{A}' = 0, \quad \boldsymbol{n} \times \nabla \times \boldsymbol{A}' = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3.8}$$

The boundary conditions (3.8) are homogeneous.

The abstract analog of the system of Eqs. (3.6)–(3.8) is a functional equation in $\mathcal{L}^2$ for the vector $u = (\psi, \boldsymbol{A}')$,

$$\mathcal{A}u = \mathcal{F}(u), \quad u \in \mathcal{D}(\mathcal{A}), \tag{3.9}$$

where $\mathcal{A}$ is the linear self-adjoint operator associated with the quadratic form $Q \equiv Q[u]$,

$$Q[u] = \int_\Omega \left[\frac{1}{\kappa^2}|\nabla\psi|^2 + |\nabla \times \boldsymbol{A}'|^2\right] \mathrm{d}x + \int_{\partial\Omega} \frac{\gamma(x)}{\kappa^2}|\psi|^2 \,\mathrm{d}\sigma(x), \tag{3.10}$$

on the domain

$$\mathcal{D}(Q) = \mathcal{D}(\mathcal{A}^{1/2}) = \{u = (\psi, \boldsymbol{A}') \in \mathcal{H}^1 : \nabla \cdot \boldsymbol{A}' = 0 \text{ in } \Omega,\ \boldsymbol{n} \cdot \boldsymbol{A}' = 0 \text{ on } \partial\Omega\}, \tag{3.11}$$

and $\mathcal{F}$ is a nonlinear function, which depends parametrically on $\boldsymbol{A}_{\mathbf{H}}$,

$$\mathcal{F}(u) \equiv \mathcal{F}(\psi, \boldsymbol{A}') = \Big(-\frac{2i}{\kappa}(\nabla\psi) \cdot (\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}) - \psi|\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}|^2 + (1 - |\psi|^2)\psi,$$

$$\frac{1}{2i\kappa}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - |\psi|^2(\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}})\Big). \tag{3.12}$$

The equation $\mathcal{A}u = f$, with $f = (\varphi, \boldsymbol{F})$ a given element of $\mathcal{L}^2$, is equivalent with two separated boundary-value problems for ψ and $\boldsymbol{A}'$,

$$-\frac{1}{\kappa^2}\Delta\psi = \varphi \text{ in } \Omega, \quad \boldsymbol{n}\cdot\nabla\psi + \gamma\psi = 0 \text{ on } \partial\Omega, \tag{3.13}$$

$$\nabla\times\nabla\times\boldsymbol{A}' = \boldsymbol{F} \quad \text{and} \quad \nabla\cdot\boldsymbol{A}' = 0 \quad \text{in } \Omega, \tag{3.14}$$

$$\boldsymbol{n}\cdot\boldsymbol{A}' = 0 \quad \text{and} \quad \boldsymbol{n}\times(\nabla\times\boldsymbol{A}') = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{3.15}$$

Boundary-value problems of this type have been studied by Georgescu [9]. From his results we infer that $\mathcal{D}(\mathcal{A})$ is a closed linear subspace of $\mathcal{H}^2$. Furthermore, $\mathcal{A}$ is positive definite in $\mathcal{L}^2$, because Q is nonnegative and $Q[\psi, \boldsymbol{A}'] + c\|\psi\|_{L^2}^2$ is coercive on $\mathcal{H}^1$, for any positive constant c [11, Chapter I, Eq. (5.45)]. The fractional powers $\mathcal{A}^\theta$ are therefore well defined for all real θ: $\mathcal{A}^\theta$ is unbounded for $\theta > 0$; see [13, Section 1.4]. Interpolation theory shows that $\mathcal{D}(\mathcal{A}^\theta)$ is a closed linear subspace of $\mathcal{H}^{2\theta}$ for all $\theta \in (0,1)$; see [2, Chapter IV].

A solution $u = (\psi, \boldsymbol{A}')$ of the functional equation (3.9) defines a solution $(\psi, \boldsymbol{A}) = (\psi, \boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}})$ of the gauged GL equations. Notice, however, that even though u is an element of the space $\mathcal{H}^2$, we may not conclude that $(\psi, \boldsymbol{A})$ is in $\mathcal{H}^2$. If $\boldsymbol{H} \in [H^\alpha(\Omega)]^n$ for some $\alpha \in (\frac{1}{2}, 1)$, we only have $\boldsymbol{A}_{\mathbf{H}} \in [H^{1+\alpha}(\Omega)]^n$, so $(\psi, \boldsymbol{A}) \in \mathcal{H}^{1+\alpha}$.

3.3 Time-Dependent Case

The procedure for the gauged TDGL equations is similar. First, we homogenize the boundary conditions by introducing, at each instant t, the applied vector potential $\boldsymbol{A}_{\mathbf{H}(t)}$ as in Eq. (3.1). In general, $\boldsymbol{A}_{\mathbf{H}}$ varies with time, $\boldsymbol{A}_{\mathbf{H}}(t) = \boldsymbol{A}_{\mathbf{H}(t)}$ for $t \geq 0$. The mapping $\boldsymbol{H}(t) \mapsto \boldsymbol{A}_{\mathbf{H}}(t)$ is linear, time independent, and continuous from $[H^\theta(\Omega)]^n$ to $[H^{1+\theta}(\Omega)]^n$, for $0 \leq \theta \leq 1$. Furthermore, $\partial_t \boldsymbol{A}_{\mathbf{H}} = \boldsymbol{A}_{\partial_t \mathbf{H}}$.

Given $\boldsymbol{A}_{\mathbf{H}}(t)$ for each $t \geq 0$, we introduce the reduced vector potential as in Eq. (3.5) and reformulate the gauged TDGL equations in terms of ψ and $\boldsymbol{A}'$,

$$\begin{aligned}\eta\frac{\partial\psi}{\partial t} - \frac{1}{\kappa^2}\Delta\psi &= -\frac{2i}{\kappa}(\nabla\psi)\cdot(\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}) - \frac{i}{\kappa}(1 - \eta\kappa^2\omega)\psi(\nabla\cdot\boldsymbol{A}') \\ &\quad -\psi|\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}|^2 + \left(1 - |\psi|^2\right)\psi \quad \text{in } \Omega\times(0,\infty), \end{aligned} \tag{3.16}$$

$$\frac{\partial\boldsymbol{A}'}{\partial t} + \nabla\times\nabla\times\boldsymbol{A}' - \omega\nabla(\nabla\cdot\boldsymbol{A}') = \frac{1}{2i\kappa}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$$

$$-|\psi|^2(\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}) - \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t} \quad \text{in } \Omega \times (0, \infty), \tag{3.17}$$

$$\boldsymbol{n} \cdot \nabla\psi + \gamma\psi = 0, \quad \boldsymbol{n} \cdot \boldsymbol{A}' = 0, \quad \boldsymbol{n} \times \nabla \times \boldsymbol{A}' = \boldsymbol{0} \quad \text{on } \partial\Omega \times (0, \infty). \tag{3.18}$$

In addition, we have the initial conditions

$$\psi = \psi_0 \quad \text{and} \quad \boldsymbol{A}' = \boldsymbol{A}_0 - \boldsymbol{A}_{\mathbf{H}}(0) \quad \text{on } \Omega \times \{0\}. \tag{3.19}$$

The abstract analog of the system of Eqs. (3.16)–(3.19) is an initial-value problem in $\mathcal{L}^2$ for the vector $u = (\psi, \boldsymbol{A}')$,

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u = \mathcal{F}(t, u(t)) \quad \text{for } t > 0, \quad u(0) = u_0, \tag{3.20}$$

where $\mathcal{A}$ is the linear self-adjoint operator associated with the quadratic form $Q \equiv Q[u]$,

$$Q[u] = \int_\Omega \left[\frac{1}{\eta\kappa^2}|\nabla\psi|^2 + \omega(\nabla \cdot \boldsymbol{A}')^2 + |\nabla \times \boldsymbol{A}'|^2\right] \mathrm{d}x + \int_{\partial\Omega} \frac{\gamma(x)}{\eta\kappa^2}|\psi|^2 \,\mathrm{d}\sigma(x), \tag{3.21}$$

on the domain

$$\mathcal{D}(Q) = \mathcal{D}(\mathcal{A}^{1/2}) = \{u = (\psi, \boldsymbol{A}') \in \mathcal{H}^1 : \boldsymbol{n} \cdot \boldsymbol{A}' = 0 \text{ on } \partial\Omega\}. \tag{3.22}$$

and $\mathcal{F}$ is a nonlinear function of u, which depends explicitly on t through the applied vector potential $\boldsymbol{A}_{\mathbf{H}}$,

$$\begin{aligned}\mathcal{F}(t, u) \equiv \mathcal{F}(t, \psi, \boldsymbol{A}') = \Bigg(\frac{1}{\eta}\Bigg[&-\frac{2i}{\kappa}(\nabla\psi) \cdot (\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}) \\ &-\frac{i}{\kappa}(1 - \eta\kappa^2\omega)\psi(\nabla \cdot \boldsymbol{A}') - \psi|\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}|^2 + (1 - |\psi|^2)\psi\Bigg], \\ &\frac{1}{2i\kappa}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2(\boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}}) - \frac{\partial \boldsymbol{A}_{\mathbf{H}}}{\partial t}\Bigg).\end{aligned} \tag{3.23}$$

The initial data in Eq. (3.20) are $u_0 = (\psi_0, \boldsymbol{A}'_0) = (\psi_0, \boldsymbol{A}_0 - \boldsymbol{A}_{\mathbf{H}}(0))$.

The equation $\mathcal{A}u = f$, where $f = (\varphi, \boldsymbol{F})$ is a given element in $\mathcal{L}^2$, is equivalent with two separated boundary-value problems,

$$-\frac{1}{\eta\kappa^2}\Delta\psi = \varphi \text{ in } \Omega, \quad \boldsymbol{n} \cdot \nabla\psi + \gamma\psi = 0 \text{ on } \partial\Omega, \tag{3.24}$$

$$\nabla \times \nabla \times \boldsymbol{A}' - \omega\nabla(\nabla \cdot \boldsymbol{A}') = \boldsymbol{F} \quad \text{in } \Omega, \tag{3.25}$$

$$\boldsymbol{n} \cdot \boldsymbol{A}' = 0 \quad \text{and} \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A}') = \boldsymbol{0} \quad \text{on } \partial\Omega. \tag{3.26}$$

The operator $\mathcal{A}$ has the same properties as in the time-independent case: $\mathcal{D}(\mathcal{A})$ is a closed linear subspace of $\mathcal{H}^2$, $\mathcal{A}$ is positive definite in $\mathcal{L}^2$, the fractional powers $\mathcal{A}^\theta$ are well defined for all real θ, and $\mathcal{D}(\mathcal{A}^\theta)$ is a closed linear subspace of $\mathcal{H}^{2\theta}$ for all $\theta \in (0,1)$. Furthermore, $-\mathcal{A}$ is the generator of a holomorphic semigroup $\{\mathrm{e}^{-\mathcal{A}t} : t \geq 0\}$ in $\mathcal{L}^2$.

A mild solution of the intial-value problem (3.20) on an interval $[0,T]$ is a continuous function $u : [0,T] \to \mathcal{H}^{1+\alpha}$, for some $\alpha \in (\frac{1}{2}, 1)$, that satisfies the equation

$$u(t) = \mathrm{e}^{-\mathcal{A}t} u_0 + \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)} \mathcal{F}(s, u(s))\, \mathrm{d}s \quad \text{for } 0 \leq t \leq T. \tag{3.27}$$

A mild solution $u = (\psi, \boldsymbol{A}')$ of Eq. (3.20) defines a solution $(\psi, \boldsymbol{A}) = (\psi, \boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}})$ of the gauged TDGL equations.

If $\boldsymbol{H} \in L^\infty([0,T]; [H^\alpha(\Omega)]^n) \cap H^1([0,T]; [L^2(\Omega)]^n)$ for some $T > 0$ and $\alpha \in (\frac{1}{2}, 1)$, then the initial-value problem (3.20) has a unique mild solution $u = (\psi, \boldsymbol{A}')$ on $[0,T]$ for any initial data $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ [6, Theorem 1]. The mild solutions of Eq. (3.20) generate a *dynamical process* $U = \{U(t,s) : 0 \leq s \leq t \leq T\}$ on $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ by the definition

$$u(t) = U(t,s) u(s), \quad 0 \leq s \leq t \leq T; \tag{3.28}$$

see [6, Corollary 2]. The process is such that $u(t) \in \mathcal{D}(\mathcal{A})$ for any $t > 0$. A solution $u = (\psi, \boldsymbol{A}')$ of Eq. (3.20) defines a solution $(\psi, \boldsymbol{A}) = (\psi, \boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}})$ of the gauged TDGL equations.

Two cases deserve special mention. First, if the applied magnetic field is time independent, the dynamical process U is, in fact, a *dynamical system* [6, Theorem 3]. This dynamical system, $S = \{S(t) : t \geq 0\}$, is related to the dynamical process U by the identity

$$S(t-s) = U(t,s), \quad t \geq s \geq 0. \tag{3.29}$$

Second, if the applied magnetic field varies with time but is asymptotically stationary, the dynamical process U is *asymptotically autonomous* [17, Corollary 1]. This case arises if $\boldsymbol{H} \in L^\infty([0,\infty); [H^{\alpha'}(\Omega)]^n)$ for some $\alpha' \in (\alpha, 1)$ and $\partial_t \boldsymbol{H} \in [L^1([0,\infty); [L^2(\Omega)]^n) \cap [L^2([0,\infty); [L^2(\Omega)]^n)$. Then $\lim_{t\to\infty} \boldsymbol{H}(t)$ exists in $[H^\alpha(\Omega)]^n$, defining the element $\boldsymbol{H}_\infty \in [H^\alpha(\Omega)]^n$,

$$\boldsymbol{H}_\infty = \lim_{t\to\infty} \boldsymbol{H}(t). \tag{3.30}$$

The solution of the initial-value problem (3.20) can be compared, in the limit of large time, with the solution of the autonomous problem

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u = \mathcal{F}_\infty(u(t)), \quad t > 0; \quad u(0) = u_0, \tag{3.31}$$

where $\mathcal{F}_\infty(u)$ is defined by the same expression (3.23), with $\boldsymbol{A}_{\mathbf{H}}$ replaced by $\boldsymbol{A}_{\mathbf{H}_\infty}$ (and $\partial_t \boldsymbol{A}_{\mathbf{H}}$ by 0). This autonomous problem defines a dynamical system S_∞ on $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$. The orbit of each $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ under the dynamical process $U(s+t,s)$ (s fixed, $s \geq 0$) and under the limiting dynamical system $S_\infty(t)$, $t \geq 0$, has compact closure in $\mathcal{H}^{1+\alpha}$, and the omega-limit set of each $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ under $U(t+s,s)$ (s fixed, $s \geq 0$) is a nonempty compact connected set of divergence-free equilibria for S_∞ [17, Theorem 2].

4 Auxiliary Estimates

In this section we present some auxiliary estimates of the size of a function in terms of its values on a finite point set in the domain.

Let N be a fixed integer, and let $\mathcal{E}$ be a *finite* point set in Ω,

$$\mathcal{E} = \{x_j \in \Omega : j = 1, \ldots, N\}. \tag{4.1}$$

The distance from a point $x \in \Omega$ to $\mathcal{E}$ is

$$\operatorname{dist}(x;\mathcal{E}) = \min\{|x - x_j| : j = 1, \ldots, N\}. \tag{4.2}$$

The mapping $x \mapsto \operatorname{dist}(x;\mathcal{E})$ defines a continuous function in Ω, whose supremum,

$$d \equiv d_\Omega(\mathcal{E}) = \sup\{\operatorname{dist}(x;\mathcal{E}) : x \in \Omega\}, \tag{4.3}$$

measures how well the set $\mathcal{E}$ "covers" the domain Ω: the smaller the d, the better the coverage of Ω. The number d is positive, and there is a point $x_0 \in \overline{\Omega}$ such that $\operatorname{dist}(x_0;\mathcal{E}) = d$.

Theorem 4.1 *Let θ be fixed, $\theta \in (\frac{1}{2}, \frac{3}{2})$. There exists a positive constant C such that, for any $u \in \mathcal{H}^{1+\theta}$,*

$$|u(x)| \leq |u|_{\mathcal{E}} + C d^{\theta - 1/2} \|u\|_{\mathcal{H}^{1+\theta}}, \quad x \in \Omega, \tag{4.4}$$

and

$$\|u\|_{\mathcal{H}^{(1+\theta)/2}} \leq C \left(|u|_{\mathcal{E}}^{1/2} \|u\|_{\mathcal{H}^{1+\theta}}^{1/2} + d^{(\theta-1/2)/2} \|u\|_{\mathcal{H}^{1+\theta}} \right), \quad u \in \mathcal{H}^{1+\theta}, \tag{4.5}$$

where $|u|_{\mathcal{E}} = \max\{|u(x_j)| : x_j \in \mathcal{E}\}$.

Proof. Since $n = 2$ or $n = 3$, the space $\mathcal{H}^{1+\theta}(\Omega)$ is continuously imbedded in the Hölder space $\mathcal{C}^{\theta-1/2}(\overline{\Omega})$. Hence, there exists a constant C such that

$$|u(x) - u(y)| \leq C\|u\|_{\mathcal{H}^{1+\theta}}|x-y|^{\theta-1/2},$$

for all $x, y \in \Omega$. Given any $x \in \Omega$, we can find a point $x_j \in \mathcal{E}$ such that $|x - x_j| = \mathrm{dist}(x; \mathcal{E})$. Taking $y = x_j$, we conclude that

$$|u(x)| \leq |u(x_j)| + C(\mathrm{dist}(x;\mathcal{E}))^{\theta-1/2}\|u\|_{\mathcal{H}^{1+\theta}}.$$

The estimate (4.4) follows if we replace the upper bound by its supremum.

To prove the estimate (4.5), we start from the interpolation inequality

$$\|u\|_{\mathcal{H}^{(1+\theta)/2}} \leq C\|u\|_{\mathcal{L}^2}^{1/2}\|u\|_{\mathcal{H}^{1+\theta}}^{1/2};$$

see, for example, [13, Theorem 1.4.4 and Exercise 5]. Because Ω is bounded, it follows immediately from the estimate (4.4) that

$$\|u\|_{\mathcal{L}^2} \leq C\left(|u|_{\mathcal{E}} + d^{\theta-1/2}\|u\|_{\mathcal{H}^{1+\theta}}\right), \quad u \in \mathcal{H}^{1+\theta}.$$

Applying the elementary inequality $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$, we obtain the inequality (4.5). ∎

In time-dependent problems, we will allow for the possibility that the points of $\mathcal{E}$ change with time,

$$\mathcal{E}(t) = \{x_j(t) \in \Omega : j = 1, 2, \ldots, N\}, \quad t \geq 0. \tag{4.6}$$

(Of course, the results remain true if $\mathcal{E}$ is time independent.) The estimate (4.5) extends in an obvious way. For any $u \in L^\infty([0,\infty); \mathcal{H}^{1+\theta})$ and any $\tau \geq 0$,

$$\|u(t)\|_{\mathcal{H}^{(1+\theta)/2}} \leq C\left(|u|_{\mathcal{E},\tau}^{1/2}\|u(t)\|_{\mathcal{H}^{1+\theta}}^{1/2} + d_\tau^{(\theta-1/2)/2}\|u(t)\|_{\mathcal{H}^{1+\theta}}\right), \quad t \geq \tau. \tag{4.7}$$

where

$$|u|_{\mathcal{E},\tau} = \sup\{\max\{|u(x_j(t),t)| : x_j(t) \in \mathcal{E}(t)\} : t \geq \tau\}$$

and

$$d_\tau = \sup\{\mathrm{dist}(x, \mathcal{E}(t)) : (x,t) \in \Omega \times [\tau, \infty)\}.$$

5 Results

In this section we present our results. We show how and in what sense a solution of the Ginzburg-Landau equations is, at least in principle, determined completely and exactly by its values on a finite point set. The qualifier "in principle" refers to the fact that Ω must be covered sufficiently well by the point set, but we do not have an estimate of the cardinality of the point set in terms of the parameters of the problem.

5.1 Time-Independent Case

Throughout this section we assume that $\boldsymbol{H}$ satsfies the hypothesis

$$\boldsymbol{H} \in [H^{\alpha}(\Omega)]^n \quad \text{for some } \alpha \in (\tfrac{1}{2}, 1). \tag{5.1}$$

The vector $\boldsymbol{A}_{\mathbf{H}}$ is defined by the quadratic form (3.1) on the domain (3.2), $\boldsymbol{A}_{\mathbf{H}} \in [H^{1+\alpha}(\Omega)]^n$. The linear self-adjoint operator $\mathcal{A}$ is defined by the quadratic form (3.10) on the domain (3.11). Any vector $u = (\psi, \boldsymbol{A}') \in \mathcal{D}(\mathcal{A})$ that satisfies Eq. (3.9) defines a solution $(\psi, \boldsymbol{A}) = (\psi, \boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}})$ of the gauged GL equations. We recall that $\mathcal{D}(\mathcal{A})$ is a closed linear subspace of $\mathcal{H}^2$. Hence, while u is actually an element of $\mathcal{H}^2$, $(\psi, \boldsymbol{A})$ is only in $\mathcal{H}^{1+\alpha}$.

Lemma 5.1 *Let B_R be the ball of radius R ($R > 0$) centered at the origin in $\mathcal{H}^{1+\alpha}$. Let $(\psi_1, \boldsymbol{A}_1), (\psi_2, \boldsymbol{A}_2) \in B_R$ be two solutions of the gauged GL equations such that $u_1 = (\psi_1, \boldsymbol{A}_1 - \boldsymbol{A}_{\mathbf{H}})$ and $u_2 = (\psi_2, \boldsymbol{A}_2 - \boldsymbol{A}_{\mathbf{H}})$ belong to $\mathcal{D}(\mathcal{A})$ and satisfy Eq. (3.9). Then the difference $u = (\psi_1, \boldsymbol{A}_1) - (\psi_2, \boldsymbol{A}_2)$ belongs to $\mathcal{D}(\mathcal{A})$ and satisfies the norm inequality*

$$\|u\|_{\mathcal{H}^2} \leq C \|u\|_{\mathcal{H}^1}, \tag{5.2}$$

for some positive constant C that depends on R.

Proof. Let $\psi = \psi_1 - \psi_2$ and $\boldsymbol{A} = \boldsymbol{A}_1 - \boldsymbol{A}_2$, so $u = (\psi, \boldsymbol{A})$. Then

$$u = (\psi_1, \boldsymbol{A}_1) - (\psi_2, \boldsymbol{A}_2) = (\psi_1, \boldsymbol{A}_1 - \boldsymbol{A}_{\mathbf{H}}) - (\psi_2, \boldsymbol{A}_2 - \boldsymbol{A}_{\mathbf{H}}) = u_1 - u_2,$$

so $u \in \mathcal{D}(\mathcal{A})$. A straightforward calculation shows that $\mathcal{A}u$ is a linear function of u,

$$\mathcal{A}u = \mathcal{F}(u_1) - \mathcal{F}(u_2) = \mathcal{B}(\psi_1, \boldsymbol{A}_1; \psi_2, \boldsymbol{A}_2)\, u, \tag{5.3}$$

where $\mathcal{B}$ depends quadratically on ψ_1, $\boldsymbol{A}_1$, ψ_2, and $\boldsymbol{A}_2$,

$$\mathcal{B}(\psi_1, \boldsymbol{A}_1; \psi_2, \boldsymbol{A}_2)\, u = (\psi, 0)$$

$$+\frac{1}{i\kappa}\left(2\left[\boldsymbol{A}_1\cdot\nabla\psi+(\nabla\psi_2)\cdot\boldsymbol{A}\right],\frac{1}{2}\left[(\nabla\psi_1)\psi^*-(\nabla\psi_1^*)\psi+\psi_2^*\nabla\psi-\psi_2\nabla\psi^*\right]\right)$$
$$-\Big(|\boldsymbol{A}_1|^2\psi+\psi_2(\boldsymbol{A}_1+\boldsymbol{A}_2)\cdot\boldsymbol{A}+(|\psi_1|^2+|\psi_2|^2)\psi+\psi_1\psi_2\psi^*,$$
$$\boldsymbol{A}_1(\psi_1^*\psi+\psi_2\psi^*)+|\psi_2|^2\boldsymbol{A}\Big)\,. \tag{5.4}$$

Because

$$\|u\|_{\mathcal{H}^2}=\|\mathcal{A}u\|_{\mathcal{L}^2}=\|\mathcal{B}(\psi_1,\boldsymbol{A}_1;\psi_2,\boldsymbol{A}_2)\,u\|_{\mathcal{L}^2},$$

the inequality (5.2) follows if we can show that the operator norm of $\mathcal{B}:\mathcal{H}^1\to\mathcal{L}^2$ is bounded.

Take any $u=(\psi,\boldsymbol{A})\in\mathcal{H}^1$. Then, trivially,

$$\|(\psi,0)\|_{\mathcal{L}^2}\le\|u\|_{\mathcal{L}^2}\le C\|u\|_{\mathcal{H}^1},$$

for some positive constant C. Next, we estimate the terms in (5.4) that depend linearly on ψ_1, $\boldsymbol{A}_1$, ψ_2, and $\boldsymbol{A}_2$. Using the triangle inequality and Hölder's inequality for integrals, we have

$$\|\boldsymbol{A}_1\cdot\nabla\psi+(\nabla\psi_2)\cdot\boldsymbol{A}\|_{L^2}\le\|\boldsymbol{A}_1\|_{L^\infty}\|\nabla\psi\|_{L^2}+\|\nabla\psi_2\|_{L^3}\|\boldsymbol{A}\|_{L^6}.$$

For $\alpha\in(\frac{1}{2},1)$, $H^{1+\alpha}(\Omega)$ is continuously imbedded in $L^\infty(\Omega)$, so

$$\|\boldsymbol{A}_1\|_{L^\infty}\le C\|\boldsymbol{A}_1\|_{H^{1+\alpha}}\le C\|(\psi_1,\boldsymbol{A}_1)\|_{\mathcal{H}^{1+\alpha}}.$$

Also, $H^\alpha(\Omega)$ is continuously imbedded in $L^3(\Omega)$, so

$$\|\nabla\psi_2\|_{L^3}\le C\|\nabla\psi_2\|_{H^\alpha}\le C\|\psi_2\|_{H^{1+\alpha}}\le C\|(\psi_2,\boldsymbol{A}_2)\|_{\mathcal{H}^{1+\alpha}}.$$

Therefore,

$$\|\boldsymbol{A}_1\cdot\nabla\psi+(\nabla\psi_2)\cdot\boldsymbol{A}\|_{L^2}\le C\left(\|\nabla\psi\|_{L^2}+\|\boldsymbol{A}\|_{L^6}\right),$$

for some constant C that depends on R. But $\|\nabla\psi\|_{L^2}\le\|\psi\|_{H^1}$ and $H^1(\Omega)$ is continuously imbedded in $L^6(\Omega)$, so

$$\|\boldsymbol{A}_1\cdot\nabla\psi+(\nabla\psi_2)\cdot\boldsymbol{A}\|_{L^2}\le C\left(\|\psi\|_{H^1}+\|\boldsymbol{A}\|_{H^1}\right)\le C\|u\|_{\mathcal{H}^1}.$$

Similarly,

$$\|(\nabla\psi_1)\psi^*-(\nabla\psi_1^*)\psi+\psi_2^*\nabla\psi-\psi_2\nabla\psi^*\|_{L^2}\le 2\left(\|\nabla\psi_1\|_{L^3}\|\psi\|_{L^6}+\|\psi_2\|_{L^\infty}\|\nabla\psi\|_{L^2}\right)$$
$$\le C\left(\|\psi\|_{L^6}+\|\nabla\psi\|_{L^2}\right)\le C\|u\|_{\mathcal{H}^1}.$$

The remaining terms in Eq. (5.4) depend quadratically on ψ_1, $\boldsymbol{A}_1$, ψ_2, and $\boldsymbol{A}_2$. We estimate each of them separately. For example, using the triangle inequality, Hölder's inequality, and the continuous imbedding of $H^{1+\alpha}(\Omega)$ in $L^6(\Omega)$, we have

$$\begin{aligned}\||\boldsymbol{A}_1|^2\psi + \psi_2(\boldsymbol{A}_1+\boldsymbol{A}_2)\cdot\boldsymbol{A}\|_{L^2} &\le \|\boldsymbol{A}_1\|_{L^6}^2\|\psi\|_{L^6} + \|\psi_2\|_{L^6}\left(\|\boldsymbol{A}_1\|_{L^6}+\|\boldsymbol{A}_2\|_{L^6}\right)\|\boldsymbol{A}\|_{L^6}\\ &\le C\left(\|\boldsymbol{A}_1\|_{H^{1+\alpha}}^2\|\psi\|_{L^6} + \|\psi_2\|_{H^{1+\alpha}}\left(\|\boldsymbol{A}_1\|_{H^{1+\alpha}}+\|\boldsymbol{A}_2\|_{H^{1+\alpha}}\right)\|\boldsymbol{A}\|_{L^6}\right)\\ &\le C\left(\|\psi\|_{L^6}+\|\boldsymbol{A}\|_{L^6}\right),\end{aligned}$$

for some positive constant C that depends on R. But $H^1(\Omega)$ is continuously imbedded in $L^6(\Omega)$, so

$$\||\boldsymbol{A}_1|^2\psi + \psi_2(\boldsymbol{A}_1+\boldsymbol{A}_2)\cdot\boldsymbol{A}\|_{L^2} \le C\|u\|_{\mathcal{H}^1}.$$

Similarly,

$$\|(|\psi_1|^2+|\psi_2|^2)\psi + \psi_1\psi_2\psi^*\|_{L^2} \le C\|u\|_{\mathcal{H}^1}$$

and

$$\|\boldsymbol{A}_1(\psi_1^*\psi+\psi_2\psi^*) + |\psi_2|^2\boldsymbol{A}\|_{L^2} \le C\|u\|_{\mathcal{H}^1}.$$

Combining the various estimates, we conclude that the linear operator $\mathcal{B}:\mathcal{H}^1\to\mathcal{L}^2$ is indeed bounded. ∎

Theorem 5.1 *Let B_R be the ball of radius R ($R>0$) centered at the origin in $\mathcal{H}^{1+\alpha}$. Let $(\psi_1,\boldsymbol{A}_1),(\psi_2,\boldsymbol{A}_2)\in B_R$ be two solutions of the gauged GL equations such that $u_1=(\psi_1,\boldsymbol{A}_1-\boldsymbol{A}_{\mathbf{H}})$ and $u_2=(\psi_2,\boldsymbol{A}_2-\boldsymbol{A}_{\mathbf{H}})$ belong to $\mathcal{D}(\mathcal{A})$ and satisfy Eq. (3.9). Let $\mathcal{E}=\{x_j\in\Omega : j=1,\dots,N\}$ be a finite point set in Ω whose density $d=d_\Omega(\mathcal{E})$ is defined in Eq. (4.3).*

There exists a positive number δ_0 such that, if $d\le\delta_0$ and

$$(\psi_1,\boldsymbol{A}_1)(x_j) = (\psi_2,\boldsymbol{A}_2)(x_j), \quad j=1,\dots,N, \tag{5.5}$$

then

$$(\psi_1,\boldsymbol{A}_1)(x) = (\psi_2,\boldsymbol{A}_2)(x), \quad x\in\Omega. \tag{5.6}$$

Proof. Let $\psi=\psi_1-\psi_2$ and $\boldsymbol{A}=\boldsymbol{A}_1-\boldsymbol{A}_2$, and define $u=(\psi,\boldsymbol{A})$. Then $u(x_j)=0$ for all $x_j\in\mathcal{E}$. Furthermore, as shown in Lemma 5.1, $u\in\mathcal{D}(\mathcal{A})$ and $\|u\|_{\mathcal{H}^2}\le C\|u\|_{\mathcal{H}^1}$ for some positive constant C.

If Eq. (5.5) is satisfied, we have $|u|_{\mathcal{E}}=\max\{|u(x_j)| : x_j\in\mathcal{E}\}=0$, and the inequality (4.5), with $\theta=1$, reduces to $\|u\|_{\mathcal{H}^1}\le Cd^{1/4}\|u\|_{\mathcal{H}^2}$. Combining the two norm inequalities, we conclude that there exists a positive constant C such that

$$\|u\|_{\mathcal{H}^2}\le Cd^{1/4}\|u\|_{\mathcal{H}^2}. \tag{5.7}$$

Given this constant C, we fix $\delta_0 < C^{-4}$; such a choice is certainly possible and can be made *a priori*. Then $Cd^{1/4} < 1$ whenever $d \le \delta_0$. But then the inequality (5.7) cannot be satisfied unless $\|u\|_{\mathcal{H}^2} = 0$. Hence, if $d \le \delta_0$, it must be the case that $u = 0$ in Ω and, therefore, $(\psi_1, \boldsymbol{A}_1)(x) = (\psi_2, \boldsymbol{A}_2)(x)$ for all $x \in \Omega$. ∎

Theorem 5.1 implies that a solution of the gauged GL equations is determined completely and exactly in all of Ω by its values on a finite point set $\mathcal{E}$, provided $\mathcal{E}$ covers Ω sufficiently well. This property explains why the points of $\mathcal{E}$ are called *determining nodes*. The values of the solution at the determining nodes are its *nodal values*.

5.2 Time-Dependent Case

Next, we consider the time-dependent case. Throughout this section we assume that $\boldsymbol{H}$ satsfies the hypothesis

$$\boldsymbol{H} \in L^\infty([0,\infty); [H^\alpha(\Omega)]^n) \cap H^1([0,\infty); [L^2(\Omega)]^n) \quad \text{for some } \alpha \in (\tfrac{1}{2}, 1). \tag{5.8}$$

For each $t \ge 0$, the vector $\boldsymbol{A}_{\mathbf{H}}(t) = \boldsymbol{A}_{\mathbf{H}(t)}$ is defined by the quadratic form (3.1) on the domain (3.2), $\boldsymbol{A}_{\mathbf{H}}(t) \in [H^{1+\alpha}(\Omega)]^n$. The linear self-adjoint operator $\mathcal{A}$ is defined by the quadratic form (3.21) on the domain (3.22). Any vector $u = (\psi, \boldsymbol{A}') \in C([0,\infty); \mathcal{H}^{1+\alpha})$ which satisfies Eq. (3.20) for some $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ defines a solution $(\psi, \boldsymbol{A}) = (\psi, \boldsymbol{A}' + \boldsymbol{A}_{\mathbf{H}})$ of the gauged TDGL equations. We recall that $\mathcal{D}(\mathcal{A})$ is a closed linear subspace of $\mathcal{H}^2$. Hence, while $u(t)$ is actually an element of $\mathcal{H}^2$ for all $t > 0$, $(\psi, \boldsymbol{A})(t)$ is only in $\mathcal{H}^{1+\alpha}$.

Lemma 5.2 *Let $(\psi_1, \boldsymbol{A}_1)$ and $(\psi_2, \boldsymbol{A}_2)$ be two solutions of the gauged TDGL equations, such that $u_1 = (\psi_1, \boldsymbol{A}_1 - \boldsymbol{A}_{\mathbf{H}})$ and $u_2 = (\psi_2, \boldsymbol{A}_2 - \boldsymbol{A}_{\mathbf{H}})$ satisfy Eq. (3.20), and let $u = (\psi_1, \boldsymbol{A}_1) - (\psi_2, \boldsymbol{A}_2)$. Then $u(t) \in \mathcal{D}(\mathcal{A})$ for all $t > 0$, and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathcal{H}^1} + \nu\|u\|^2_{\mathcal{H}^2} \le C\|u\|^2_{\mathcal{H}^1}, \quad t > 0, \tag{5.9}$$

for some positive constants ν and C.

Proof. Let $\psi = \psi_1 - \psi_2$ and $\boldsymbol{A} = \boldsymbol{A}_1 - \boldsymbol{A}_2$, so $u = (\psi, \boldsymbol{A})$. Then $u = u_1 - u_2$, so $u(t) \in \mathcal{D}(\mathcal{A})$ for all $t > 0$. A straightforward calculation shows that u satisfies a linear differential equation,

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u = \mathcal{F}(t, u_1(t)) - \mathcal{F}(t, u_2(t)) = \mathcal{B}(\psi_1(t), \boldsymbol{A}_1(t); \psi_2(t), \boldsymbol{A}_2(t))\, u, \quad t > 0, \tag{5.10}$$

where $\mathcal{B}$ depends quadratically on its arguments (we omit the argument t),

$$\mathcal{B}(\psi_1, \boldsymbol{A}_1; \psi_2, \boldsymbol{A}_2)\, u = \left(\frac{1}{\eta}\psi, 0\right)$$

$$+\frac{1}{i\kappa}\left(\frac{2}{\eta}\left[\boldsymbol{A}_1 \cdot (\nabla\psi) + (\nabla\psi_2)\cdot \boldsymbol{A}\right] + \frac{1}{\eta}(1-\eta\kappa^2\omega)\left[(\nabla\cdot\boldsymbol{A}_1)\psi + \psi_2(\nabla\cdot\boldsymbol{A})\right],\right.$$

$$\left.\frac{1}{2}\left[(\nabla\psi_1)\psi^* - (\nabla\psi_1^*)\psi + \psi_2^*\nabla\psi - \psi_2\nabla\psi^*\right]\right)$$

$$-\left(\frac{1}{\eta}\left[|\boldsymbol{A}_1|^2\psi + \psi_2(\boldsymbol{A}_1+\boldsymbol{A}_2)\cdot\boldsymbol{A} + (|\psi_1|^2+|\psi_2|^2)\psi + \psi_1\psi_2\psi^*\right],\right.$$

$$\left.\boldsymbol{A}_1(\psi_1^*\psi + \psi_2\psi^*) + |\psi_2|^2\boldsymbol{A}\right). \tag{5.11}$$

Choosing any $t > 0$, we take the $\mathcal{L}^2$-inner product of both sides of the differential equation (5.10) with $\mathcal{A}u$. The result is a scalar differential equation,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\mathcal{H}^1}^2 + \|u\|_{\mathcal{H}^2}^2 = (\mathcal{B}u, \mathcal{A}u)_{\mathcal{L}^2}\,, \quad t > 0. \tag{5.12}$$

We estimate the quantity in the right member, for each $t > 0$, as follows.

We start with the trivial estimates

$$|(\mathcal{B}u, \mathcal{A}u)_{\mathcal{L}^2}| \le \|\mathcal{B}u\|_{\mathcal{L}^2}\|\mathcal{A}u\|_{\mathcal{L}^2} \le C\|\mathcal{B}u\|_{\mathcal{L}^2}\|u\|_{\mathcal{H}^2}.$$

We claim that the operator norm of $\mathcal{B} : \mathcal{H}^1 \to \mathcal{L}^2$ is bounded.

The proof of the claim proceeds along the same lines as the corresponding proof in the time-independent case. The expression (5.11) has the same structure as the corresponding expression (5.4); the one additional term is estimated like all the others,

$$\|(\nabla\cdot\boldsymbol{A}_1)\psi + \psi_2(\nabla\cdot\boldsymbol{A})\|_{L^2} \le \|\nabla\cdot\boldsymbol{A}_1\|_{L^3}\|\psi\|_{L^6} + \|\psi_2\|_{L^\infty}\|\nabla\cdot\boldsymbol{A}\|_{L^2}$$

$$\le \|\boldsymbol{A}_1\|_{H^{1+\alpha}}\|\psi\|_{L^6} + \|\psi_2\|_{H^{1+\alpha}}\|\boldsymbol{A}\|_{H^1} \le C\|u\|_{\mathcal{H}^1},$$

for some constant C. Because the orbit of any initial value in $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ has compact closure in $\mathcal{H}^{1+\alpha}$, the constants C can be fixed independently of t.

Consequently,

$$|(\mathcal{B}u, \mathcal{A}u)_{\mathcal{L}^2}| \le C\|u\|_{\mathcal{H}^1}\|u\|_{\mathcal{H}^2} \le \varepsilon\|u\|_{\mathcal{H}^2}^2 + C(\varepsilon)\|u\|_{\mathcal{H}^1}^2, \quad t > 0,$$

for any $\varepsilon > 0$. Fixing ε in the interval $(0,1)$, we absorb the term $\varepsilon\|u\|_{\mathcal{H}^2}^2$ in the left member of Eq. (5.12) and let $\nu = 2(1-\varepsilon)$. The inequality (5.9) follows. ∎

The following theorem shows that, in the limit of large time, the asymptotic behavior of a solution of the gauged TDGL equations is determined completely and exactly by its asymptotic behavior on a sufficiently dense, possibly varying, finite point set.

Theorem 5.2 *Let $(\psi_1, \boldsymbol{A}_1)$ and $(\psi_2, \boldsymbol{A}_2)$ be two solutions of the gauged TDGL equations, such that $u_1 = (\psi_1, \boldsymbol{A}_1 - \boldsymbol{A}_{\mathbf{H}})$ and $u_2 = (\psi_2, \boldsymbol{A}_2 - \boldsymbol{A}_{\mathbf{H}})$ satisfy Eq. (3.20). Let $\{\mathcal{E}(t) : t \geq 0\}$ be a family of finite point sets $\mathcal{E}(t) = \{x_j(t) \in \Omega : j = 1, \ldots, N\}$ in Ω, whose density $d_\Omega(\mathcal{E}(t))$ is defined in Eq.(4.3).*

There exists a positive number δ_1 such that, if $\limsup_{t\to\infty} d_\Omega(\mathcal{E}(t)) \leq \delta_1$ and

$$\lim_{t\to\infty} |(\psi_1, \boldsymbol{A}_1)(x_j(t), t) - (\psi_2, \boldsymbol{A}_2)(x_j(t), t)| = 0, \quad j = 1, \ldots, N, \tag{5.13}$$

then

$$\lim_{t\to\infty} \|(\psi_1, \boldsymbol{A}_1)(t) - (\psi_2, \boldsymbol{A}_2)(t)\|_{\mathcal{H}^1} = 0 \tag{5.14}$$

and

$$\lim_{t\to\infty} [\sup\{|(\psi_1, \boldsymbol{A}_1)(x, t) - (\psi_2, \boldsymbol{A}_2)(x, t)| : x \in \Omega\}] = 0. \tag{5.15}$$

Proof. Let $\psi = \psi_1 - \psi_2$ and $\boldsymbol{A} = \boldsymbol{A}_1 - \boldsymbol{A}_2$, and define $u = (\psi, \boldsymbol{A})$. Then $\lim_{t\to\infty} u(x_j(t), t) = 0$ for $j = 1, \ldots, N$. As shown in Lemma 5.2, $u(t) \in \mathcal{D}(\mathcal{A})$ for all $t > 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{\mathcal{H}^1} + \nu\|u(t)\|^2_{\mathcal{H}^2} \leq C\|u(t)\|^2_{\mathcal{H}^1}, \quad t > 0, \tag{5.16}$$

for some positive constants ν and C.

Taking any $\tau \geq 0$ and $\theta = 1$ in the inequality (4.7), we obtain an estimate for $\|u(t)\|^2_{\mathcal{H}^1}$ in terms of the quality of the coverage of Ω by the sets $\mathcal{E}(t)$ for $t \geq \tau$ and the norm of $u(t)$ in $\mathcal{H}^2$,

$$\|u(t)\|^2_{\mathcal{H}^1} \leq C\left(|u|^{1/2}_{\mathcal{E},\tau}\|u(t)\|^{1/2}_{\mathcal{H}^2} + d^{1/4}_\tau\|u(t)\|_{\mathcal{H}^2}\right)^2 \leq C\left(|u|_{\mathcal{E},\tau}\|u(t)\|_{\mathcal{H}^2} + d^{1/2}_\tau\|u(t)\|^2_{\mathcal{H}^2}\right)$$

$$\leq \varepsilon\|u(t)\|^2_{\mathcal{H}^2} + C(\varepsilon)\left(|u|^2_{\mathcal{E},\tau} + d^{1/2}_\tau\|u(t)\|^2_{\mathcal{H}^2}\right), \quad t \geq \tau,$$

for any $\varepsilon > 0$. (The quantities $|u|_{\mathcal{E},\tau}$ and d_τ are defined after the estimate (4.7).) Fixing ε in the interval $(0, \nu/C)$, we absorb the term $\varepsilon C\|u(t)\|^2_{\mathcal{H}^2}$ in the left member of the differential inequality (5.16). Thus we find that, for any $\tau \geq 0$, there exist a $\mu > 0$ and a positive constant C such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{\mathcal{H}^1} + \mu\|u(t)\|^2_{\mathcal{H}^2} \leq C\left(|u|^2_{\mathcal{E},\tau} + d^{1/2}_\tau\|u(t)\|^2_{\mathcal{H}^2}\right), \quad t > \tau.$$

Given this constant C, we fix $\delta_1 < (\mu/C)^2$, so $\mu - Cd_\tau^{1/2} > 0$ for all $d_\tau \le \delta_1$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{\mathcal{H}^1}^2 + \left(\mu - C\delta_1^{1/2}\right)\|u(t)\|_{\mathcal{H}^2}^2 \le C|u|_{\mathcal{E},\tau}^2, \quad t > \tau,$$

whenever $d_\tau \le \delta_1$. A similar inequality holds with $\|u(t)\|_{\mathcal{H}^2}^2$ in the left member replaced by $\|u(t)\|_{\mathcal{H}^1}^2$, because of the continuous imbedding of $\mathcal{H}^2$ into $\mathcal{H}^1$. Thus, whenever the coverage of Ω by the point sets $\mathcal{E}(t)$, $t \ge \tau$, is uniformly better than δ_1 (that is, whenever $d_\Omega(\mathcal{E}(t)) \le \delta_1$ for all $t \ge \tau$), there exist a $\lambda > 0$ and a positive constant C such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{\mathcal{H}^1}^2 + \lambda\|u(t)\|_{\mathcal{H}^1}^2 \le C|u|_{\mathcal{E},\tau}^2, \quad t > \tau.$$

Given any small positive ε, we fix $\tau \equiv \tau(\varepsilon)$ such that $C|u|_{\mathcal{E},\tau}^2 < \varepsilon$. Such a choice is certainly possible, because $\lim_{t\to\infty} u(x_j(t), t) = 0$ for $j = 1, \ldots, N$. Having thus fixed τ, we find that $\|u(t)\|_{\mathcal{H}^1}^2$ satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{\mathcal{H}^1}^2 + \lambda\|u(t)\|_{\mathcal{H}^1}^2 < \varepsilon, \quad t > \tau.$$

Then Gronwall's lemma yields the estimate

$$\|u(t)\|_{\mathcal{H}^1}^2 < (\varepsilon/\lambda) + \mathrm{e}^{-\lambda(t-\tau)}\|u(\tau)\|_{\mathcal{H}^1}^2, \quad t > \tau.$$

Since ε is arbitrarily small, we conclude that $\lim_{t\to\infty} u(t) = 0$ in the topology of $\mathcal{H}^1$. Convergence in the uniform topology on Ω follows from the fact that the orbit of any initial value in $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ has compact closure in $\mathcal{H}^{1+\alpha}$ and $\mathcal{H}^{1+\alpha}$ is compactly imbedded in the space of continuous functions $\mathcal{C}(\overline{\Omega})$ if $\alpha > \frac{1}{2}$. ∎

5.3 Asymptotically Autonomous Case

If the applied magnetic field is constant or asymptotically stationary in time, we can be more specific about the long-time asymptotic behavior of the solution of the gauged TDGL equations. Since a constant field is a special case of an asymptotically stationary field, we discuss only the latter. Instead of (5.8), we impose the stronger hypotheses

$$\boldsymbol{H} \in L^\infty([0,\infty); [H^{\alpha'}(\Omega)]^n) \quad \text{for some } \alpha' \in (\alpha, 1), \alpha \in (\tfrac{1}{2}, 1), \tag{5.17}$$

$$\partial_t \boldsymbol{H} \in [L^1(0,\infty; [L^2(\Omega)]^n) \cap [L^2(0,\infty; [L^2(\Omega)]^n). \tag{5.18}$$

Then $\boldsymbol{H}$ is asymptotically stationary, and $\lim_{t\to\infty}\boldsymbol{H}(t) = \boldsymbol{H}_\infty$ in $[H^\alpha(\Omega)]^n$. The dynamical process U defined on $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ by the initial-value problem (3.20) is asymptotically autonomous; its large-time asymptotic limit is the dynamical system S_∞ defined on $\mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ by Eq. (3.31).

Lemma 5.3 *Let $(\psi, \boldsymbol{A})$ be a solution of the gauged TDGL equations, such that $u = (\psi, \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}})$ satisfies Eq. (3.20). For every fixed s ($s > 0$), there exist positive constants ν and C such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t) - u(t+s)\|^2_{\mathcal{H}^1} + \nu\|u(t) - u(t+s)\|^2_{\mathcal{H}^2} \le C\left(\|u(t) - u(t+s)\|^2_{\mathcal{H}^1}\right.$$
$$\left. + \|\boldsymbol{H}(t) - \boldsymbol{H}(t+s)\|^2_{H^\alpha} + \|\partial_t\boldsymbol{H}(t) - \partial_t\boldsymbol{H}(t+s)\|^2_{L^2}\right), \quad t > 0. \tag{5.19}$$

Proof. Fix $s > 0$ and define the function v on $[0,\infty)$ by the expression $v(t) = u(t) - u(t+s)$ for $t \ge 0$. Let $\psi_v(t) = \psi(t) - \psi(t+s)$ and $\boldsymbol{A}_v(t) = \boldsymbol{A}(t) - \boldsymbol{A}(t+s)$, so $v = (\psi_v, \boldsymbol{A}_v)$. Then $v(t) \in \mathcal{D}(\mathcal{A})$ for all $t > 0$. A straightforward calculation shows that v satisfies the linear differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} + \mathcal{A}v = \mathcal{F}(t, u(t)) - \mathcal{F}(t+s, u(t+s)) = \mathcal{B}_1(\psi(t), \boldsymbol{A}(t); \psi(t+s), \boldsymbol{A}(t+s))\, v$$
$$+\mathcal{B}_2(\psi(t), \boldsymbol{A}(t); \psi(t+s), \boldsymbol{A}(t+s))(\boldsymbol{A}_{\mathbf{H}}(t) - \boldsymbol{A}_{\mathbf{H}}(t+s))$$
$$+(0, \partial_t\boldsymbol{A}_{\mathbf{H}}(t) - \partial_t\boldsymbol{A}_{\mathbf{H}}(t+s)), \quad t > 0, \tag{5.20}$$

where $\mathcal{B}_1$ has the same structure as $\mathcal{B}$ in the preceding section (cf. Eq. (5.11)),

$$\mathcal{B}_1(\psi_1, \boldsymbol{A}_1; \psi_2, \boldsymbol{A}_2)\, v = \left(\frac{1}{\eta}\psi_v, 0\right)$$
$$+\frac{1}{i\kappa}\left(\frac{2}{\eta}\left[\boldsymbol{A}_1\cdot(\nabla\psi_v) + (\nabla\psi_2)\cdot\boldsymbol{A}_v\right] + \frac{1}{\eta}(1 - \eta\kappa^2\omega)\left[(\nabla\cdot\boldsymbol{A}_1)\psi_v + \psi_2(\nabla\cdot\boldsymbol{A}_v)\right],\right.$$
$$\left.\frac{1}{2}\left[(\nabla\psi_1)\psi_v^* - (\nabla\psi_1^*)\psi_v + \psi_2^*\nabla\psi_v - \psi_2\nabla\psi_v^*\right]\right)$$
$$-\left(\frac{1}{\eta}\left[|\boldsymbol{A}_1|^2\psi_v + \psi_2(\boldsymbol{A}_1 + \boldsymbol{A}_2)\cdot\boldsymbol{A}_v + (|\psi_1|^2 + |\psi_2|^2)\psi_v + \psi_1\psi_2\psi_v^*\right],\right.$$
$$\left.\boldsymbol{A}_1(\psi_1^*\psi_v + \psi_2\psi_v^*) + |\psi_2|^2\boldsymbol{A}_v\right), \tag{5.21}$$

and

$$\mathcal{B}_2(\psi_1, \boldsymbol{A}_1; \psi_2, \boldsymbol{A}_2)(\boldsymbol{A}_{\mathbf{H}}(t) - \boldsymbol{A}_{\mathbf{H}}(t+s)) = \frac{1}{i\kappa}\left(\frac{2}{\eta}(\nabla\psi_2\cdot(\boldsymbol{A}_{\mathbf{H}}(t) - \boldsymbol{A}_{\mathbf{H}}(t+s)), 0\right)$$

$$-\left(\frac{1}{\eta}\psi_2(\boldsymbol{A}_1+\boldsymbol{A}_2)\cdot(\boldsymbol{A}_{\mathbf{H}}(t)-\boldsymbol{A}_{\mathbf{H}}(t+s)),|\psi_2|^2(\boldsymbol{A}_{\mathbf{H}}(t)-\boldsymbol{A}_{\mathbf{H}}(t+s))\right). \tag{5.22}$$

If the applied magnetic field is constant, the right-hand side of Eq. (5.20) reduces to the single term involving $\mathcal{B}_1$.

Choosing any $t>0$, we take the $\mathcal{L}^2$ inner product of both sides of Eq. (5.20) with $\mathcal{A}v$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{\mathcal{H}^1}^2+\|v\|_{\mathcal{H}^2}^2=(\mathcal{B}_1 v,\mathcal{A}v)_{\mathcal{L}^2}$$
$$+(\mathcal{B}_2(\boldsymbol{A}_{\mathbf{H}}(t)-\boldsymbol{A}_{\mathbf{H}}(t+s)),\mathcal{A}v)_{\mathcal{L}^2}+((0,\partial_t\boldsymbol{A}_{\mathbf{H}}(t)-\partial_t\boldsymbol{A}_{\mathbf{H}}(t+s)),\mathcal{A}v)_{\mathcal{L}^2},\quad t>0. \tag{5.23}$$

The linear operator $\mathcal{B}_1$ is continuous from $\mathcal{H}^1$ into $\mathcal{L}^2$, just like $\mathcal{B}$ in the preceding section, so

$$|(\mathcal{B}_1 v,\mathcal{A}v)_{\mathcal{L}^2}|\le C\|v\|_{\mathcal{H}^1}\|v\|_{\mathcal{H}^2}\le\varepsilon\|v\|_{\mathcal{H}^2}^2+C(\varepsilon)\|v\|_{\mathcal{H}^1}^2,$$

for any $\varepsilon>0$.

We claim that $\mathcal{B}_2$ is a bounded linear operator from $[H^{1+\alpha}(\Omega)]^n$ into $\mathcal{L}^2$. The claim is proved with the usual type of estimates; for example,

$$\|(\nabla\psi_2)\cdot\boldsymbol{A}_{\mathbf{H}}\|_{L^2}\le\|\nabla\psi_2\|_{L^3}\|\boldsymbol{A}_{\mathbf{H}}\|_{L^6}\le C\|\psi_2\|_{H^{1+\alpha}}\|\boldsymbol{A}_{\mathbf{H}}\|_{L^6}\le C\|\boldsymbol{A}_{\mathbf{H}}\|_{H^{1+\alpha}}.$$

The constant can be fixed independently of t, because of the compact closure of the orbit in $\mathcal{H}^{1+\alpha}$. Hence, if we also use the fact that the mapping $\boldsymbol{H}(t)\mapsto\boldsymbol{A}_{\mathbf{H}}(t)$ is continuous from $[H^{\alpha}(\Omega)]^n$ into $[H^{1+\alpha}(\Omega)]^n$, we find that

$$|(\mathcal{B}_2(\boldsymbol{A}_{\mathbf{H}}(t)-\boldsymbol{A}_{\mathbf{H}}(t+s)),\mathcal{A}v)_{\mathcal{L}^2}|\le C\|\boldsymbol{A}_{\mathbf{H}}(t)-\boldsymbol{A}_{\mathbf{H}}(t+s)\|_{H^{1+\alpha}}\|v\|_{\mathcal{H}^2}$$
$$\le C\|\boldsymbol{H}(t)-\boldsymbol{H}(t+s)\|_{H^{\alpha}}\|v\|_{\mathcal{H}^2}\le\varepsilon\|v\|_{\mathcal{H}^2}^2+C(\varepsilon)\|\boldsymbol{H}(t)-\boldsymbol{H}(t+s)\|_{H^{\alpha}}^2,$$

for any $\varepsilon>0$.

Finally, we estimate the last term in (5.23). Here we use the fact that, under the hypothesis (5.18), $\partial_t\boldsymbol{H}(t)\in[L^2(\Omega)]^n$ for all $t>0$. Then $\partial_t\boldsymbol{A}_{\mathbf{H}}(t)=\boldsymbol{A}_{\partial_t\mathbf{H}(t)}\in[H^1(\Omega)]^n$ for all $t>0$, so it is certainly true that

$$|((0,\partial_t\boldsymbol{A}_{\mathbf{H}}(t)-\partial_t\boldsymbol{A}_{\mathbf{H}}(t+s)),\mathcal{A}v)_{\mathcal{L}^2}|\le C\|\partial_t\boldsymbol{H}(t)-\partial_t\boldsymbol{H}(t+s))\|_{L^2}\|v\|_{\mathcal{H}^2}$$
$$\le\varepsilon\|v\|_{\mathcal{H}^2}^2+C(\varepsilon)\|\partial_t\boldsymbol{H}(t)-\partial_t\boldsymbol{H}(t+s)\|_{L^2}^2,$$

for any $\varepsilon>0$.

The inequality (5.19) results if we fix each ε in the interval $(0,\frac{1}{3})$ and absorb the three terms $\varepsilon\|v\|_{\mathcal{H}^2}^2$ in the left member of Eq. (5.23). ∎

Assuming that the point sets $\mathcal{E}(t)$, defined in Eq. (4.6), actually converge, in the sense that there exists a point set $\mathcal{E} = \{\xi_j \in \Omega : j = 1, \ldots, N\}$ such that

$$\lim_{t\to\infty} x_j(t) = \xi_j, \quad j = 1, \ldots, N, \tag{5.24}$$

we will show that the limiting values of the solution of the gauged TDGL equations on the point set $\mathcal{E}$ define a unique solution of the gauged time-independent GL equations throughout the entire domain.

Theorem 5.3 *Let $(\psi, \boldsymbol{A})$ be a solution of the gauged TDGL equations such that $u = (\psi, \boldsymbol{A} - \boldsymbol{A}_{\mathbf{H}})$ satisfies Eq. (3.20). Let $\{\mathcal{E}(t) : t \geq 0\}$ be a family of finite point sets $\mathcal{E}(t) = \{x_j(t) \in \Omega : j = 1, \ldots, N\}$ in Ω, whose density $d_\Omega(\mathcal{E}(t))$ is defined in Eq. (4.3), which converges in the sense of Eq. (5.24).*

There exist a positive number δ_2 and a unique solution $(\psi_\infty, \boldsymbol{A}_\infty)$ of the gauged GL equations (2.10)–(2.12) such that, if $\limsup_{t\to\infty} d_\Omega(\mathcal{E}(t)) \leq \delta_2$, and if

$$\lim_{t\to\infty} (\psi, \boldsymbol{A})(x_j(t), t)) = (\phi_j, \boldsymbol{F}_j), \quad j = 1, \ldots, N. \tag{5.25}$$

for some $(\phi_j, \boldsymbol{F}_j) \in \mathbf{C} \times \mathbf{R}^n$, $j = 1, \ldots, N$, then

$$\lim_{t\to\infty} (\psi, \boldsymbol{A})(\cdot\,, t) = (\psi_\infty, \boldsymbol{A}_\infty) \tag{5.26}$$

in the topology of $\mathcal{H}^1$ and in the topology of uniform convergence on Ω. Moreover,

$$(\psi_\infty, \boldsymbol{A}_\infty)(\xi_j) = (\phi_j, \boldsymbol{F}_j), \quad j = 1, \ldots, N. \tag{5.27}$$

Proof. The pointwise-convergence assumption (5.25), together with the limiting condition (5.24), implies that

$$\lim_{t\to\infty} u(x_j(t), t) = (\phi_j, \boldsymbol{F}_j - \boldsymbol{A}_{\mathbf{H}}(\xi_j)), \quad j = 1, \ldots, N. \tag{5.28}$$

Let s be fixed, $s > 0$, and let v be defined on $[0, \infty)$ by the identity $v(t) = u(t+s) - u(t)$, $t \geq 0$. Then $\lim_{t\to\infty} v(x_j(t), t) = 0$ for $j = 1, \ldots, N$. According to Lemma 5.3, there exist positive constants ν and C such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|^2_{\mathcal{H}^1} + \nu\|v(t)\|^2_{\mathcal{H}^2} \leq C\left(\|v(t)\|^2_{\mathcal{H}^1}\right.$$

$$\left. + \|\boldsymbol{H}(t) - \boldsymbol{H}(t+s)\|^2_{H^\alpha} + \|\partial_t \boldsymbol{H}(t) - \partial_t \boldsymbol{H}(t+s)\|^2_{L^2}\right), \quad t > 0. \tag{5.29}$$

As in the proof of Theorem 5.2, we use the estimate (4.5) to relate $\|v(t)\|_{\mathcal{H}^1}^2$ to the quality of the coverage of Ω by the point sets $\mathcal{E}(t)$ and the norm of $v(t)$ in $\mathcal{H}^2$. For any $\tau \geq 0$, we have

$$\|v(t)\|_{\mathcal{H}^1}^2 \leq C\left(|v|_{\mathcal{E},\tau}\|v(t)\|_{\mathcal{H}^2} + d_\tau^{1/2}\|v(t)\|_{\mathcal{H}^2}^2\right)$$

$$\leq \varepsilon\|v(t)\|_{\mathcal{H}^2}^2 + C(\varepsilon)\left(|v|_{\mathcal{E},\tau}^2 + d_\tau^{1/2}\|v(t)\|_{\mathcal{H}^2}^2\right), \quad t \geq \tau,$$

for any $\varepsilon > 0$. (The quantities $|v|_{\mathcal{E},\tau}$ and d_τ are defined after the estimate (4.7).) Fixing ε in the interval $(0, \nu/C)$, we absorb the term $\varepsilon C\|v(t)\|_{\mathcal{H}^2}^2$ in the left member of the differential inequality (5.29). Thus we find that there exist a $\mu > 0$ and a positive constant C such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{\mathcal{H}^1}^2 + \mu\|v(t)\|_{\mathcal{H}^2}^2 \leq C\left(|v|_{\mathcal{E},\tau}^2 + d_\tau^{1/2}\|v(t)\|_{\mathcal{H}^2}^2\right.$$

$$\left.+\|\boldsymbol{H}(t) - \boldsymbol{H}(t+s)\|_{H^\alpha}^2 + \|\partial_t\boldsymbol{H}(t) - \partial_t\boldsymbol{H}(t+s)\|_{L^2}^2\right), \quad t > 0.$$

Given this constant C, we fix $\delta_2 < (\mu/C)^2$, so $\mu - Cd_\tau^{1/2} > 0$ whenever $d_\tau \leq \delta_2$. Then, if $d_\tau \leq \delta_2$, we find that there exists a constant C such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{\mathcal{H}^1}^2 + \left(\mu - C\delta_2^{1/2}\right)\|v(t)\|_{\mathcal{H}^2}^2 \leq C\left(|v|_{\mathcal{E},\tau}^2\right.$$

$$\left.+\|\boldsymbol{H}(t) - \boldsymbol{H}(t+s)\|_{H^\alpha}^2 + \|\partial_t\boldsymbol{H}(t) - \partial_t\boldsymbol{H}(t+s)\|_{L^2}^2\right), \quad t > \tau.$$

Because of the continuous imbedding of $\mathcal{H}^2$ into $\mathcal{H}^1$, we can replace the $\mathcal{H}^2$ norm of $v(t)$ in the left member by its $\mathcal{H}^1$ norm. Thus, whenever the coverage of Ω by the point sets $\mathcal{E}(t)$, $t \geq \tau$, is uniformly better than δ_2, there exist a $\lambda > 0$ and a positive constant C such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{\mathcal{H}^1}^2 + \lambda\|v(t)\|_{\mathcal{H}^1}^2 \leq C\left(|v|_{\mathcal{E},\tau}^2\right.$$

$$\left.+\|\boldsymbol{H}(t) - \boldsymbol{H}(t+s)\|_{H^\alpha}^2 + \|\partial_t\boldsymbol{H}(t) - \partial_t\boldsymbol{H}(t+s)\|_{L^2}^2\right), \quad t > \tau.$$

Given any $\varepsilon > 0$, we fix $\tau \equiv \tau(\varepsilon)$ such that each term in the right member is less than $\frac{1}{3}\varepsilon$. Such a choice is certainly possible because the condition (5.28) implies that $\lim_{\tau\to\infty}|v|_{\mathcal{E},\tau} = 0$, the condition (5.17) implies that $\lim_{t\to\infty}\|\boldsymbol{H}(t) - \boldsymbol{H}(t+s)\|_{H^\alpha} = 0$, and the condition (5.18) implies that $\lim_{t\to\infty}\|\partial_t\boldsymbol{H}(t) - \partial_t\boldsymbol{H}(t+s)\|_{L^2} = 0$.

Having thus fixed τ, we find that $\|v(t)\|_{\mathcal{H}^1}$ satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_{\mathcal{H}^1}^2 + \lambda\|v(t)\|_{\mathcal{H}^1}^2 < \varepsilon, \quad t > \tau.$$

Applying Gronwall's lemma and returning to the original function u, we conclude that

$$\|u(t) - u(t+s)\|_{\mathcal{H}^1}^2 < (\varepsilon/\lambda) + \mathrm{e}^{-\lambda(t-\tau)}\|u(\tau) - u(\tau+s)\|_{\mathcal{H}^1}^2, \quad t > \tau.$$

Hence,

$$\limsup_{t\to\infty} \|u(t) - u(t+s)\|_{\mathcal{H}^1}^2 < \frac{\varepsilon}{\lambda}.$$

Since ε is arbitrarily small, we have shown that the values $\{u(t) : t > 0\}$ of u form Cauchy sequence in $\mathcal{H}^1$ as $t \to \infty$.

There exists therefore an element $u_\infty \in \mathcal{H}^1$ such that

$$\lim_{t\to\infty} \|u(t) - u_\infty\|_{\mathcal{H}^1} = 0.$$

The orbit of any $u_0 \in \mathcal{D}(\mathcal{A}^{(1+\alpha)/2})$ has compact closure in $\mathcal{H}^{1+\alpha}$. Since the injection of $\mathcal{H}^{1+\alpha}$ into $\mathcal{C}^{\alpha-1/2}(\overline{\Omega})$ is compact, the family $\{u(t) : t \geq 0\}$ is compact in $\mathcal{C}^{\alpha-1/2}(\overline{\Omega})$. Hence, we also have

$$\lim_{t\to\infty} [\sup\{|u(x,t) - u_\infty(x)| : x \in \Omega\}] = 0.$$

The element u_∞ is associated with a pair $(\psi_\infty, \boldsymbol{A}'_\infty) \in [H^1(\Omega)]^2 \times [H^1(\Omega)]^n$, which defines, in turn, a pair $(\psi_\infty, \boldsymbol{A}_\infty) \in [H^1(\Omega)]^2 \times [H^1(\Omega)]^n$,

$$(\psi_\infty, \boldsymbol{A}_\infty) = (\psi_\infty, \boldsymbol{A}'_\infty + \boldsymbol{A}_{\mathbf{H}_\infty}).$$

This pair necessarily satisfies the time-independent GL equations (2.10)–(2.12). Moreover, because of (5.28), it must be the case that

$$(\psi_\infty, \boldsymbol{A}_\infty)(\xi_j) = (\phi_j, \boldsymbol{F}_j), \quad j = 1, \ldots, N.$$

It remains to prove that $(\psi_\infty, \boldsymbol{A}_\infty)$ is uniquely determined. But this fact is an immediate consequence of Theorem 5.1; all we need to do is decrease δ_2 if necessary to make sure that $\delta_2 \leq \delta_0$. ∎

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