An Improved Model Equation with Globally Defined Flux for the Vortex Sheet Equation: Analytical Results

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3 September 1997

Abstract

We present an improved model for the vortex sheet equation, that combines some of the features of the models of Beale and Schaeffer, Dhanak, and Baker et al. We regularize the Beale-Schaeffer equation with a second-order viscous regularizing term, and we add a globally defined flux term in conservative form. We obtain $u_t^\nu + iu_x^\nu = [H(u^\nu)u^\nu]_x + [|u_x^\nu|^2 u_x^\nu]_x + \nu u_{xx}^\nu$, where $i^2 = -1$, and $H(u^\nu)$ is the Hilbert transform of u^ν. We derive bounds for the solution of the equation and its first-order spatial derivatives in L^2 and in the maximum norm, independent of ν. We show that the function u_t^ν satisfies an L^2 norm bound that depends linearly on ν; all the other derivatives satisfy bounds that depend on negative powers of ν. We show that, for $\nu > 0$, the solution exists and is unique. We also prove that, for $\nu > 0$, in the limit of $\nu \to 0$, the sequence of functions $(u^\nu)_{\nu>0}$ has a weak limit; the weak limit may not be unique.

1 Introduction

At high Reynold's numbers, a thin shear layer, generated by shedding vortices from a solid boundary, takes the asymptotic form of a vortex sheet: a layer of vorticity distributed as a delta function on a surface [6]. We restrict ourselves to two-dimensional flow where the surface is a curve and the vorticity axis points out of the plane. The equation of motion for the location of the vortex sheet, $\boldsymbol{x}(p,t) = (x(p,t), y(p,t))$, is given by [18]

$$\text{(1a)} \qquad \boldsymbol{x}_t(p,t) = \frac{1}{2\pi} \oint \gamma(p,t) \frac{(y(q,t) - y(p,t), x(p,t) - x(q,t))}{|\boldsymbol{x}(p,t) - \boldsymbol{x}(q,t)|^2} dq,$$

$$\text{(1b)} \qquad \gamma_t(p,t) = 0,$$

where the integral must be evaluated as a principal value. The parametrization variable p is a Lagrangian parameter in that the quantity γ remains constant along the trajectory of a marker on the sheet labeled by p. Alternatively, p may be regarded as a characteristic variable, and (1a) and (1b) are the equations for the characteristics of a partial differential equation that describes the transport of the vorticity along the sheet.

Both analytic [5], [13], and [14] and numerical [10] and [20] evidence suggests that vortex sheets develop curvature singularities in finite time. Studies concentrate on the long-time evolution of unstable modes of a slightly perturbed, initially flat vortex sheet, $(p, \epsilon \sin(kp))$. The linearized motion about a flat sheet with uniform strength $\overline{\gamma}$ indicates that the growth rate for such modes satisfies $\sigma = \overline{\gamma}k/2$; that is, the amplitude grows according to $\epsilon \exp(\sigma t)$. Modes with the largest wave number k grow the fastest. Consequently, the motion is linearly ill-posed [19]. Recent studies have demonstrated that the motion does lead to singularity formation in finite time.

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Various regularizations of vortex sheet motion have been used to assess the behavior beyond the time of singularity formation. Layers of finite thickness [2] show similar behavior to those whose motion is viscous [22]: a small vortex core replaces the point of curvature singularity, and a thin layer spirals around this core. Krasny [10] uses a numerical regularization that keeps the vorticity on the curve but smoothes the velocity of the markers. His results also show the appearance of a spiral, but without a core. When the vortex sheet represents the interface between two immiscible fluids of equal density, surface tension becomes an important physical regularization. Numerical calculations that include surface tension effects reveal a spiral structure [9], but the arms show oscillations that may lead to a breakup of the spiral into detached drops.

So far, the results suggest that some form of spiral may be the weak solution to vortex sheet motion beyond the time of singularity formation. Rigorous theory [7] and [12] establishes the global existence in time for vortex sheet motion in a weak class of functions, but it does not clarify the nature of the vortex sheet. The continuum of model equations with globally defined flux [16], which included the two particular model equations investigated in [1] and [17], shed some light on the behavior of equations with globally defined flux, but it did not give sufficient insight to determine the precise behavior of the vortex sheet in the limit of vanishing viscosity.

We propose the following equation

$$u_t^\nu + iu_x^\nu = [H(u^\nu)u^\nu]_x + [|u_x^\nu|^2 u_x^\nu]_x + \nu u_{xx}^\nu, \tag{2a}$$

$$u^\nu(x,0) = f(x), \tag{2b}$$

as an improved model equation for the vortex sheet equation. To derive the equation, we proceed as in [1]: we take Dhanak's equation

$$\gamma_t + (V\gamma)_s = \nu\gamma_{ss}, \tag{3}$$

where γ is the effective vortex sheet strength assigned to the curve in the center of the layer, s is the arclength along this curve, and V is the tangential component of the velocity generated by the vortex sheet, that is, the tangential component of (1a). Then we replace the diffusive term by $\nu\gamma_{ss} + [|\gamma_s|^2\gamma_s]_s$ and the tangential component of the Biot-Savart integral (1a) by a one dimensional analogue, the Hilbert transform,

$$\frac{dx}{dt} = -\frac{1}{\pi}\,\mathrm{P.V.}\!\int_{-\infty}^{\infty} \frac{u^\nu(\eta)}{\eta - x}\, d\eta, \tag{4}$$

where we have used $u^\nu(x)$ in place of $\gamma(p)$. For (2a), we have 2π-periodic initial and boundary conditions and $i^2 = -1$; the function $H(u^\nu)$ denotes the Hilbert transform of u^ν.

The new model is a better model of vortex sheet motion because it provides two features: (1) the linearization of (2a) about a constant flat sheet leads to the Cauchy-Riemann equation regularized with a second-order viscous term; and (2) the model has a globally defined flux term like the vortex sheet equation. Previous models considered [1], [16], [17] did not have both features. The proposed model (2a) is intended to better characterize the weak limit in the limit of vanishing viscosity.

In Section 2, assuming that the solution of (2a) exists and that the initial condition is of zero-average, we derive bounds in the L^2 and in the maximum norm for the solution of (2a). We first derive bounds for u^ν and u_x^ν in the L^2 norm, independent of ν. We also get bounds for u^ν and u_x^ν in the maximum norm, independent of ν, the bound for u^ν being deduced from the L^2 norm for u^ν and u_x^ν and Sobolev's inequality and the bound for u_x^ν by first obtaining bounds for the L^{2p} norm of u_x^ν and then using Arzela-Ascoli's theorem. We obtain a bound for the time integral of the square of the L^2 norm of u_t^ν; the bound depends linearly on ν. We then show that all the other derivatives of u^ν satisfy bounds, in the L^2 and in the maximum norm, that depend on negative powers of ν, proceeding as in [15].

In Section 3, we prove the long-term existence of the solution of (2a) when $\nu > 0$ by proving the short-term existence of the solution of

$$v_t^\nu + iv_x^\nu = [H(v^\nu)v^\nu]_x + (|v_x^\nu|^2 v_x^\nu)_x + \nu v_{xx}^\nu - \mu v_{xxxx}^\nu, \tag{5a}$$

$$v^\nu(x,0) = f(x). \tag{5b}$$

The short-term existence of the solution of (5a) is deduced from Ball's theorem. Then we derive estimates for the solution of (5a), independent of μ. These and the *a priori* bounds of Section 2 let us prove the long-term existence of the solution (2a).

In Section 4, we show that the sequence of functions $(u^\nu)_{\nu>0}$ has a weak limit in $H^1([0,2\pi]\times[0,t])$ and that for fixed t, u_x is in $L^\infty([0,2\pi])$.

2 Estimates

In this section, assuming existence of a C^∞ solution, we first derive estimates for the solution of (2a) subject to 2π-periodic initial and boundary conditions. We derive estimates independent of ν for the solution of (2a) and its first-order spatial derivative.

Without restriction, from now on, we assume that the initial condition f is of zero-average. The average over a period of the solution of (2a) is conserved.

Before deriving bounds for the solution of (2a), we first recall some of the properties of the Hilbert transform [21]:

- Let g be a C^∞, 2π-periodic function. Then $H(g)$ is a C^∞, 2π-periodic function and

$$[H(g)]_x = H(g_x).$$

- The L^2 norm of $H(g)$ satisfies the bound

$$||H(g)|| \leq ||g||.$$

- $H(e^{ikx}) = i\,\mathrm{sign}(k)e^{ikx}$.

We summarize the bounds for the solution of (2a) and its first-order derivative independent of the viscosity ν as follows.

Lemma 2.1 *Let the initial condition f for (2a) be a C^∞ 2π-periodic function and of zero-average. Then the solution of (2a) satisfies*

$$\text{(6a)} \qquad ||u^\nu(\cdot,t)|| \leq \sqrt{8\pi t + ||f||^2},$$

$$\text{(6b)} \qquad ||u^\nu_x(\cdot,t)|| \leq \sqrt{40\pi t + 4||f||^2 + \left|\left|\frac{df}{dx}\right|\right|^2},$$

$$\text{(6c)} \qquad ||u^\nu(\cdot,t)||_\infty \leq \sqrt{2}\left(8\pi t + ||f||^2\right)^{1/4}\left(40\pi t + 4||f||^2 + \left|\left|\frac{df}{dx}\right|\right|^2\right)^{1/4}$$

$$\text{(6d)} \qquad ||u^\nu_x(\cdot,t)||_{2n} \leq C\left(\frac{n^3}{2n-1}\right)^{\frac{1}{2n}}\left(20t + 4||f||^2_\infty + \left|\left|\frac{df}{dx}\right|\right|^2_\infty\right)^{1-\frac{1}{2n}},$$

$$\text{(6e)} \qquad \frac{n}{(n+1)^2}\int_0^t ||[(u^\nu_x)^{(n+1)}]_x(\cdot,\tau)||^2 d\tau \leq 2C^{2n}\frac{n^3}{2n-1}\left(20t + 4||f||^2_\infty + \left|\left|\frac{df}{dx}\right|\right|^2_\infty\right)^{2n-1},$$

$$\text{(6f)} \qquad ||u^\nu_x(\cdot,t)||_\infty \leq C\left(20t + 4||f||^2_\infty + \left|\left|\frac{df}{dx}\right|\right|^2_\infty\right).$$

The above bounds depend on time and the initial condition but are independent of the viscosity ν. The constant C in (6d), (6e), and (6f) is independent of n and ν.

The bounds (6a), (6b), and (6c) indicate that the square of the L^2 norm of the solution of (2a) and its first-order spatial derivatives as well as the square of the maximum norm of the solution of (2a), grow linearly in time. The estimate (6d) tells us that the L^{2n} norm of the solution of (2a) grows nearly linearly in time; more precisely the $2n$th power of the L^{2n} norm of the solution of (2a) *grows like t^{2n-1}. The last bound, (6f), tells us that the weak maximum norm of the first-order spatial derivative of the solution of (2a) grows linearly in time.*

Also, the solution of (2a) satisfies a bound in $H^1([0,2\pi]\times[0,t])$ that depends on ν:

$$\begin{aligned}\int_0^t \|u_t^\nu(\cdot,\tau)\|^2 d\tau \quad &\leq \quad 8(8\pi t+\|f\|^2)\left(1+4\sqrt{8\pi t+\|f\|^2}\sqrt{40\pi t+4\|f\|^2+\left\|\frac{df}{dx}\right\|^2}\right) \\ &\quad +8\left(20\pi t+4\|f\|^2+\left\|\frac{df}{dx}\right\|^2\right)t+\left\|\frac{df}{dx}\right\|_4^4+2\nu\left\|\frac{df}{dx}\right\|^2. \end{aligned} \tag{6g}$$

The above estimate tells us that the time integral of the square of the L^2 norm of the first-order temporal derivative of the solution of (2a) grows quadratically in time and linearly in ν.

Proof: We first derive (6a). We take the scalar product of u^ν with (2a) and integrate by parts the scalar products (u^ν, u^ν_{xx}), $(u^\nu,[H(u^\nu)u^\nu]_x)$, and $(u^\nu,[|u_x^\nu|^2u_x^\nu]_x)$, decreasing the derivative order of u^ν_{xx}, $[H(u^\nu)u^\nu]_x$, and $[|u_x^\nu|^2u_x^\nu]_x$. We obtain

$$\frac{d}{dt}\|u^\nu\|^2 = -i(u^\nu,u_x^\nu)+i(u_x^\nu,u^\nu)-(u_x^\nu,H(u^\nu)u^\nu)-(H(u^\nu)u^\nu,u_x^\nu)-2\nu\|u_x^\nu\|^2-2\|u_x^\nu\|_4^4, \tag{7}$$

with $\|u^\nu\|$ the L^2 norm of u^ν and $\|u^\nu\|_p$ its L^p norm. We crudely estimate the scalar products (u^ν,u_x^ν) and $(u_x^\nu,H(u^\nu)u^\nu)$, and we use the zero-average property of the solution of (2a) and the properties of the Hilbert transform to obtain

$$|(u^\nu,u_x^\nu)| \quad \leq \quad \|u^\nu\|\,\|u_x^\nu\| \leq \|u_x^\nu\|^2 \leq \frac{\alpha}{2}\|u_x^\nu\|_4^4+\frac{\pi}{\alpha}, \tag{8a}$$

$$|(u_x^\nu,H(u^\nu)u^\nu)| \quad \leq \quad \|u^\nu\|\,\|u^\nu u_x^\nu\| \leq \frac{\beta}{2}\|u^\nu u_x^\nu\|^2+\frac{1}{2\beta}\|u^\nu\|^2 \leq \frac{\beta}{2}\|u_x^\nu\|_4^4+\frac{\gamma}{4\beta}\|u_x^\nu\|_4^4+\frac{\pi}{2\gamma\beta}. \tag{8b}$$

The inequality (8a) is deduced from the zero-average property of the solution of (2a) and Cauchy-Schwartz's inequality. the inequality (8b) is derived similarly. Then (8a) and (8b), $\alpha=\beta=\gamma=1/2$, are used to obtain the differential inequality

$$\frac{d}{dt}\|u^\nu\|^2 \leq 8\pi-\frac{1}{2}\|u_x^\nu\|_4^4-2\nu\|u_x^\nu\|^2. \tag{9}$$

Time integration of (9) gives (6a) and

$$\int_0^t \|u_x^\nu(\cdot,\tau)\|_4^4 d\tau \leq 16\pi t+2\|f\|^2. \tag{10}$$

The function $z=u_x^\nu$ satisfies

$$z_t+iz_x \quad = \quad [H(u^\nu)u^\nu]_{xx}+[|z|^2z]_{xx}+\nu z_{xx}, \tag{11a}$$

$$z(x,0) \quad = \quad \frac{df}{dx}(x). \tag{11b}$$

We take the scalar product of z with (11a) and integrate by parts the scalar products (z,z_{xx}), $(z,[H(u^\nu)u^\nu]_{xx})$, and $(z,[|z|^2z]_{xx})$, decreasing the derivative order of z_{xx}, $[H(u^\nu)u^\nu]_{xx}$, and $[|z|^2z]_{xx}$. We obtain

$$\begin{aligned}\frac{d}{dt}\|z\|^2 \quad &= \quad -i(z,z_x)+i(z_x,z)-(z_x,H(u^\nu)z)-(z_x,H(z)u^\nu)-(H(u^\nu)z,z_x)-(H(z)u^\nu,z_x) \\ &\quad -4\|zz_x\|^2-(z_x,z^2\overline{z}_x)-(z^2\overline{z}_x,z_x)-2\nu\|z_x\|^2. \end{aligned} \tag{12}$$

We crudely estimate the scalar products (z,z_x), $(z_x,H(u^\nu)z)$, and $(z_x,H(z)u^\nu)$, note that $|(z_x,z^2\overline{z}_x)| \leq \|zz_x\|^2$, and use the zero-average property of the solution of (2a) and the properties of the Hilbert transform to obtain

$$(13a)\qquad |(z,z_x)| \le \sqrt{2\pi}\,||zz_x|| \le \frac{\alpha}{2}||zz_x||^2 + \frac{\pi}{\alpha},$$

$$(13b)\qquad |(z_x,H(u^\nu)z)| \le ||zz_x||\,||u^\nu|| \le \frac{\beta}{2}||zz_x||^2 + \frac{1}{2\beta}||u^\nu||^2 \le \frac{\beta}{2}||zz_x||^2 + \frac{\epsilon}{4\beta}||z||_4^4 + \frac{\pi}{2\epsilon\beta},$$

$$(13c)\qquad |(z_x,H(z)u^\nu)| \le ||u^\nu z_x||\,||z|| \le \frac{\gamma}{2}||zz_x||^2 + \frac{1}{2\gamma}||z||^2 \le \frac{\gamma}{2}||zz_x||^2 + \frac{\delta}{4\gamma}||z||_4^4 + \frac{\pi}{2\delta\gamma}.$$

More precisely, we obtain (13a) using Cauchy-Schwartz's inequality and (13b) and (13c) using the properties of the Hilbert transform, the zero-average property of the solution of (2a), and Cauchy-Schwartz. Then (13a), (13b), (13c), and the inequality $|(z_x, z^2\overline{z}_x)| \le ||zz_x||^2$, $\alpha=\beta=\gamma=1/2$, and $\delta=\epsilon=1$, are used to obtain the differential inequality

$$(14)\qquad \frac{d}{dt}||z||^2 \le 8\pi + 2||z||_4^4 - \frac{1}{2}||zz_x||^2 - 2\nu||z_x||^2.$$

Time integration of (14) with the bound (10) gives (6b). In the process, we also obtain (6e) for $n=1$, since $||f|| \le \sqrt{2\pi}||f||_\infty$ and $||df/dx|| \le \sqrt{2\pi}||df/dx||_\infty$. Sobolev's inequality, (6a), and (6b) give (6c).

To derive (6d) for $n \ge 2$, we proceed by induction on n. Using (11a), we obtain

$$\begin{aligned}
\frac{d}{dt}||z^n||^2 \;=\; & -in(z^n\overline{z}^{n-1}, z_x) + in(z_x, z^n\overline{z}^{n-1}) + n(z^n\overline{z}^{n-1}, [H(u^\nu)u^\nu]_{xx}) \\
& + n([H(u^\nu)u^\nu]_{xx}, z^n\overline{z}^{n-1}) + n(z^n\overline{z}^{n-1}, [|z|^2z]_{xx}) + n([|z|^2z]_{xx}, z^n\overline{z}^{n-1}) \\
(15a)\qquad & + \nu n(z^n\overline{z}^{n-1}, z_{xx}) + \nu n(z_{xx}, z^n\overline{z}^{n-1}).
\end{aligned}$$

Integrating by parts the scalar products $(z^n\overline{z}^{n-1}, [H(u^\nu)u^\nu]_{xx})$, $(z^n\overline{z}^{n-1}, [|z|^2z]_{xx})$, and $(z^n\overline{z}^{n-1}, z_{xx})$, decreasing the derivative order of $[H(u^\nu)u^\nu]_{xx}$, $[|z|^2z]_{xx}$, and z_{xx}, and expanding the terms $[H(u^\nu)u^\nu]_x$ and $[|z|^2z]_x$, we obtain

$$\begin{aligned}
(z^n\overline{z}^{n-1}, z_{xx}) &= -n||z^{n-1}z_x||^2 - (n-1)(z^n\overline{z}_x, z^{n-2}z_x), \\
(z^n\overline{z}^{n-1}, [H(u^\nu)u^\nu]_{xx}) &= -n(z^{n-1}\overline{z}z_x, H(u^\nu)z^{n-1}) - (n-1)(z^n\overline{z}_x, H(u^\nu)z^{n-1}) \\
&\quad - n(z^{n-1}\overline{z}z_x, u^\nu H(z)z^{n-2}) - (n-1)(z^n\overline{z}_x, u^\nu H(z)z^{n-2}), \\
(z^n\overline{z}^{n-1}, [|z|^2z]_{xx}) &= -(3n-1)||z^nz_x||^2 - n(z^{n-1}z_x, z^{n+1}\overline{z}_x) - 2(n-1)(z^{n+1}\overline{z}_x, z^{n-1}z_x).
\end{aligned}$$

We crudely estimate the scalar products $(z^n\overline{z}^{n-1}, z_x)$, $(z^{n-1}\overline{z}z_x, H(u^\nu)z^{n-1})$, $(z^n\overline{z}_x, H(u^\nu)z^{n-1})$, $(z^{n-1}\overline{z}z_x, u^\nu H(z)z^{n-2})$, and $(z^n\overline{z}_x, u^\nu H(z)z^{n-2})$ and use Young's inequality [8],

$$(16)\qquad |ab| \le ||a||_p\,||b||_q, \quad \text{with} \quad \frac{1}{p}+\frac{1}{q}=1,$$

to obtain

$$(17a)\qquad |(z^n\overline{z}^{n-1}, z_x)| \le ||z^nz_x||\,||z^{n-1}|| \le \frac{\alpha}{2}||z^nz_x||^2 + \frac{1}{2\alpha}||z^{n-1}||^2,$$

$$\begin{aligned}
|(z^{n-1}\overline{z}z_x, H(u^\nu)z^{n-1})| &\le ||z^nz_x||\,||H(u^\nu)||_\infty||z^{n-1}|| \le \frac{\beta}{2}||z^nz_x||^2 \\
(17b)\qquad &\quad + \frac{1}{2\beta}||H(u^\nu)||_\infty^2||z^{n-1}||^2,
\end{aligned}$$

$$(17c)\qquad |(z^n\overline{z}_x, H(u^\nu)z^{n-1})| \le ||z^nz_x||\,||H(u^\nu)||_\infty||z^{n-1}|| \le \frac{\beta}{2}||z^nz_x||^2 + \frac{1}{2\beta}||H(u^\nu)||_\infty^2||z^{n-1}||^2,$$

$$\begin{aligned}
|(z^{n-1}\overline{z}z_x, u^\nu H(z)z^{n-2})| &\le ||z^nz_x||\,||u^\nu||_\infty||H(z)z^{n-2}|| \le \frac{\gamma}{2}||z^nz_x||^2 + \frac{1}{2\gamma}||u^\nu||_\infty^2||H(z)z^{n-2}||^2 \\
(17d)\qquad &\le \frac{\gamma}{2}||z^nz_x||^2 + \frac{1}{2\gamma}||u^\nu||_\infty^2||H(z)||_{2(n-1)}^2||z||_{2(n-1)}^{2(n-2)},
\end{aligned}$$

$$\begin{aligned}
|(z^n\overline{z}_x, u^\nu H(z)z^{n-2})| &\le ||z^nz_x||\,||u^\nu||_\infty||H(z)z^{n-2}|| \le \frac{\gamma}{2}||z^nz_x||^2 + \frac{1}{2\gamma}||u^\nu||_\infty^2||H(z)z^{n-2}||^2 \\
(17e)\qquad &\le \frac{\gamma}{2}||z^nz_x||^2 + \frac{1}{2\gamma}||u^\nu||_\infty^2||H(z)||_{2(n-1)}^2||z||_{2(n-1)}^{2(n-2)}.
\end{aligned}$$

The inequality (17a) is directly deduced from the Cauchy-Schwartz inequality; (17b) and (17c) from Sobolev's inequality and the Cauchy-Schwartz inequality; (17d) and (17e) from Sobolev's inequality and Young's inequality (16) with $a = H(z)\overline{H(z)}$, $b = z^{n-2}\overline{z}^{n-2}$, $p = n-1$, and $q = (n-1)/(n-2)$.

From the fact that $|(z^{n-1}z_x, z^{n+1}\overline{z}_x)| \leq ||z^n z_x||^2$, $|(z^n\overline{z}_x, z^{n-2}z_x)| \leq ||z^{n-1}z_x||^2$, and the crude bounds given above, the L^{2n} norm of z satisfies the differential inequality

$$\begin{aligned}\frac{d}{dt}||z||_{2n}^{2n} \quad &\leq \quad n(\alpha + (2n-1)\beta + (2n-1)\gamma)||z^n z_x||^2 + \frac{n}{\alpha}||z||_{2(n-1)}^{2(n-1)} \\ &+ \frac{n(2n-1)}{\beta}||H(u^\nu)||_\infty^2 ||z||_{2(n-1)}^{2(n-1)} - 2n||z^n z_x||^2 \\ &+ \frac{n(2n-1)}{\gamma}||u^\nu||_\infty^2 ||H(z)||_{2(n-1)}^2 ||z||_{2(n-1)}^{2(n-2)} - 2\nu n||z^{n-1}z_x||^2. \end{aligned} \tag{18a}$$

In (18a) we take $\alpha = 1/2$, $\beta = \gamma = 1/(4n-2)$, and we use the L^p bound for the Hilbert transform of z derived in [17] to obtain

$$\begin{aligned}\frac{d}{dt}||z||_{2n}^{2n} \quad &\leq \quad n\left[2 + (2n-1)^2||H(u^\nu)||_\infty^2 + 2(2n-1)^2||u^\nu||_\infty^2\right]||z||_{2(n-1)}^{2(n-1)} \\ &- \frac{n}{2}||z^n z_x||^2 - 2\nu n||z^{n-1}z_x||^2. \end{aligned} \tag{18b}$$

The bounds for u^ν and $H(u^\nu)$ in the maximum norm, independent of ν, the induction assumption for the $L^{2(n-1)}$ norm bound of z, and integration of (18b) give us

$$||z||_{2n}^{2n} \quad \leq \quad C^{2n}\frac{n^3}{2n-1}\left(20t + 4||f||_\infty^2 + \left|\left|\frac{df}{dx}\right|\right|_\infty^2\right)^{2n-1}, \tag{19a}$$

$$\frac{n}{2(n+1)^2}\int_0^t ||[(u_x^\nu)^{(n+1)}]_x(\cdot,\tau)||^2 d\tau \quad \leq \quad C^{2n}\frac{n^3}{2n-1}\left(20t + 4||f||_\infty^2 + \left|\left|\frac{df}{dx}\right|\right|_\infty^2\right)^{2n-1}. \tag{19b}$$

In the above inequalities, C is a constant independent of n and ν. The bound (6d) is obtained from (19a) by taking its $2n$th root; (6e) is deduced from (19b), and (6f) is deduced from (6d), Arzela-Ascoli's theorem, and the fact that $\lim_{n\to\infty} n^{1/n} = 1$.

We now derive a bound for the integral over time of the L^2 norm of u_t^ν. Taking the scalar product of u_t^ν with (2a), we obtain

$$\begin{aligned}2||u_t^\nu||^2 \quad &= \quad -i(u_t^\nu, u_x^\nu) + i(u_x^\nu, u_t^\nu) + (u_t^\nu, [H(u^\nu)u^\nu]_x) + ([H(u^\nu)u^\nu]_x, u_t^\nu) + (u_t^\nu, [|u_x^\nu|^2 u_x^\nu]_x) \\ &+ ([|u_x^\nu|^2 u_x^\nu]_x, u_t^\nu) + \nu(u_t^\nu, u_{xx}^\nu) + \nu(u_{xx}^\nu, u_t^\nu). \end{aligned} \tag{20a}$$

We integrate by parts the scalar products $(u_t^\nu, [|u_x^\nu|^2 u_x^\nu]_x)$ and (u_t^ν, u_{xx}^ν), decreasing the derivative order of $[|u_x^\nu|^2 u_x^\nu]_x$ and u_{xx}^ν to obtain

$$(u_t^\nu, [|u_x^\nu|^2 u_x^\nu]_x) + ([|u_x^\nu|^2 u_x^\nu]_x, u_t^\nu) \quad = \quad -(u_x^\nu u_{xt}^\nu, [u_x^\nu]^2) - ([u_x^\nu]^2, u_x^\nu u_{xt}^\nu) = -\frac{1}{2}\frac{d}{dt}||u_x^\nu||_4^4, \tag{21a}$$

$$(u_t^\nu, u_{xx}^\nu) + (u_{xx}^\nu, u_t^\nu) \quad = \quad -(u_{xt}^\nu, u_x^\nu) - (u_x^\nu, u_{xt}^\nu) = -\frac{d}{dt}||u_x^\nu||^2. \tag{21b}$$

We expand the term $[H(u^\nu)u^\nu]_x$ and crudely estimate the scalar products on which no algebraic manipulations have been performed to obtain

$$|(u_t^\nu, u_x^\nu)| \quad \leq \quad ||u_t^\nu||\,||u_x^\nu|| \leq \frac{\alpha}{2}||u_t^\nu||^2 + \frac{1}{2\alpha}||u_x^\nu||^2 \leq \frac{\alpha}{2}||u_t^\nu||^2 + \frac{1}{2\alpha}||u_x^\nu||_4^4 + \frac{\pi}{\alpha}, \tag{21c}$$

$$\begin{aligned}|(u_t^\nu, H(u^\nu)u_x^\nu)| \quad &\leq \quad ||H(u^\nu)||_\infty ||u_t^\nu||\,||u_x^\nu|| \leq \frac{\beta}{2}||u_t^\nu||^2 + \frac{1}{2\beta}||H(u^\nu)||_\infty^2 ||u_x^\nu||^2 \\ &\leq \frac{\beta}{2}||u_t^\nu||^2 + \frac{1}{2\beta}||H(u^\nu)||_\infty^2 ||u_x^\nu||_4^4 + \frac{\pi}{\beta}||H(u^\nu)||_\infty^2, \end{aligned} \tag{21d}$$

$$\begin{aligned}|(u_t^\nu, u^\nu H(u_x^\nu))| \quad &\leq \quad ||u^\nu||_\infty ||u_t^\nu||\,||H(u_x^\nu)|| \leq \frac{\gamma}{2}||u_t^\nu||^2 + \frac{1}{2\gamma}||u^\nu||_\infty^2 ||u_x^\nu||^2 \\ &\leq \frac{\gamma}{2}||u_t^\nu||^2 + \frac{1}{2\gamma}||u^\nu||_\infty^2 ||u_x^\nu||_4^4 + \frac{\pi}{\gamma}||u^\nu||_\infty^2. \end{aligned} \tag{21e}$$

The above inequalities are deduced by using the Cauchy-Schwartz inequality and the zero-average property of the solution of (2a).

Substituting in (20a) for the scalar products their upper bounds or an equivalent expression, we obtain

$$\begin{aligned} 2||u_t^\nu||^2 \quad &\leq \quad (\alpha+\beta+\gamma)||u_t^\nu||^2 + \left(\frac{1}{\alpha} + \frac{||H(u^\nu)||_\infty^2}{\beta} + \frac{||u^\nu||_\infty^2}{\gamma}\right)||u_x^\nu||_4^4 \\ &\quad +2\pi\frac{||H(u^\nu)||_\infty^2}{\beta} + 2\pi\frac{||u^\nu||_\infty^2}{\gamma} - \frac{1}{2}\frac{d}{dt}||u_x^\nu||_4^4 - \nu\frac{d}{dt}||u_x^\nu||^2. \end{aligned} \tag{22a}$$

Then we take $\alpha=\beta=\gamma=1/2$ to obtain

$$\begin{aligned} ||u_t^\nu||^2 \quad &\leq \quad 4\left(1+||H(u^\nu)||_\infty^2 + ||u^\nu||_\infty^2\right)||u_x^\nu||_4^4 + 8\pi(||u^\nu||_\infty^2 + ||H(u^\nu)||_\infty^2) \\ &\quad -\frac{d}{dt}||u_x^\nu||_4^4 - 2\nu\frac{d}{dt}||u_x^\nu||^2. \end{aligned} \tag{22b}$$

Time integration of (22b), the properties of the Hilbert transform, and the bounds (6a), (6b), (6c), and (10) give us (6g). ∎

Now we derive bounds for the solution of (2a) that depend explicitly on negative powers of ν.

Lemma 2.2 *Let the initial condition f for (2a) be a C^∞ 2π-periodic function of zero-average. Then the mixed derivative $\partial^{k+m}u^\nu/\partial x^k\partial t^m$ is bounded; if $m=0$ and $k\geq 2$ or if $m\geq 1$ and $k\geq 0$, the bounds depend on the initial condition, the derivative order, and negative powers of the viscosity ν.*

Proof: The proof proceeds as in [15], since the term $[H(u^\nu)u^\nu]_x$ is a lower-order term and since in Lemma 2.1 bounds were derived for the maximum norm of $H(u^\nu)$ and u^ν. ∎

We have the following theorem

Theorem 2.1 *Let the initial condition f for (2a) be a C^∞ 2π-periodic function of zero-average. Then if the solution of (2a) exists, it is infinitely many times differentiable. Furthermore, the functions u^ν, u_x^ν satisfy bounds in L^2 and the maximum norm independent of ν, the maximum norm bound for u_x^ν being in a weak sense; the function u_t^ν also satisfies a bound in $L^2([0,2\pi]\times[0,t])$, the square of its norm depending linearly on ν. Bounds for all the other derivatives of the solution of (2a) depend explicitly on negative powers of the viscosity ν.*

Proof: The result is a direct consequence of Lemmas 2.1 and 2.2. ∎

3 Existence and Uniqueness

In this section, we proceed as in [15] to obtain existence and uniqueness results for the solution of (2a). More precisely, we use Ball's theorem [3], the hyperregularized equation (5a), and the *a priori* estimates of the preceding section.

Ball's theorem is stated as follows

Theorem 3.1 *Consider the equation*

$$\frac{d}{dt}u = Au + f(u), \tag{23}$$

where A is the generator of a holomorphic semigroup $S(t)$ of bounded operators on a Banach space X. Suppose that $||S(t)||\leq M$ for some constant $M>0$ and all $t\in\mathbb{R}^+$. Under these hypotheses the fractional powers $(-A)^{-\alpha}$ can be defined for $0<\alpha<1$, and $(-A)^{-\alpha}$ is a closed linear operator with domain $X_\alpha = Domain((-A)^{-\alpha})$ dense in X. Let $f(u)$ be locally Lipschitz; that is, for each bounded subset U of X_α there exists a constant C_U such that

$$||f(u)-f(v)|| \leq C_U||u-v|| \qquad \forall u, \quad v\in U.$$

Then, given $u_0 \in X$, there exists a finite time interval $[0,t)$ and a unique solution to (23) with $u(\cdot,0)=u_0$ on that time interval, and the solution can be continued uniquely on a maximal interval of existence $[0,T^\star)$. Moreover, if $T^\star<\infty$, then necessarily

$$\lim_{t\to T^\star} ||u(t)||_\alpha = \infty.$$

We directly apply Theorem 3.1 to (5a) with $A=-\mu\partial^4/\partial x^4$, $X=L^2([0,2\pi])$, and

$$f(u) = -iu_x + [H(u)u]_x + [|u_x|^2 u_x]_x + \nu u_{xx}.$$

Then $X_\alpha = H^3$ and

$$\begin{aligned}
||f(u)-f(v)|| &\le \big(1+||v||_\infty + ||H(u)||_\infty + [||u_x+v_x||_\infty + 2||u_x||_\infty \\
&\qquad +2||v_x||_\infty]||v_{xx}||_\infty\big)||u_x-v_x|| + (||H(u_x)||_\infty + ||v_x||_\infty)||u-v|| \\
&\qquad +(3||u_x||^2_\infty + \nu)||u_{xx}-v_{xx}||, \\
&\le C\left(1+||u||_{H^2}+||v||_{H^2}+||u||^2_{H^3}+||v||^2_{H^3}\right)||u-v||_{H^2}.
\end{aligned}$$

The above bounds are a direct consequence of the properties of the Hilbert transform and of Sobolev's inequalities. So, f is locally Lipschitz on H^3. Theorem 3.1 implies that a solution exists on any time interval in which the H^3 norm of the solution is controlled.

Then one derives bounds for the solution of (5a) independent of μ; the *a priori* bounds of the preceding section, the bounds for the solution of (5a) independent of μ, together with the short-term existence of this section, imply that the solution of (2a) exists for all time when $\nu>0$.

Theorem 3.2 *Let the initial condition be C^∞ and 2π-periodic, and let $\nu>0$. Equation (2a) has a unique, 2π-periodic solution u on $[0,\infty)$, which is infinitely many time differentiable.*

Proof: Existence follows from the short-term existence result, *a priori* bounds for (5a), *a priori* bounds for (2a), and the arguments in Theorem 4.2.2 in [11]. We have only to show uniqueness. Let u and v be solutions of (2a) that satisfy the same initial condition. Their difference $w=u-v$ satisfies

$$\begin{aligned}
w_t &= -iw_x + H(u)w_x + vH(w_x) + H(u_x)w + v_xH(w) + 2|u_x|^2 w_{xx} + [u_x]^2\overline{w}_{xx} \\
&\qquad +2u_xv_{xx}\overline{w}_x + (2\overline{v}_xv_{xx} + u_x\overline{v}_{xx} + v_x\overline{v}_{xx})w_x + \nu w_{xx}, \qquad \text{(24a)}\\
w(x,0) &= 0. \qquad \text{(24b)}
\end{aligned}$$

Integration by parts of the inner products $(w,|u_x|^2w_{xx})$ and $(w,[u_x]^2\overline{w}_{xx})$, decreasing the derivative order of w_{xx} and $\overline{w}_{xx}$, leads us to $-||u_xw_x||^2 - (w,u_x\overline{u}_{xx}w_x) - (w,u_{xx}\overline{u}_xw_x)$ and $-(w_x,[u_x]^2\overline{w}_x) - 2(w,u_xu_{xx}\overline{w}_x)$. The inner products $(w,u_x\overline{u}_{xx}w_x)$, $(w,u_{xx}\overline{u}_xw_x)$, and $(w,u_xu_{xx}\overline{w}_x)$ are crudely estimated by

$$\begin{aligned}
|(w,u_xu_{xx}\overline{w}_x)| &\le ||u_x||_\infty||u_{xx}||_\infty||w||\,||w_x|| \le \frac{\nu}{15}||w_x||^2 + \frac{15}{4\nu}||u_x||^2_\infty||u_{xx}||^2_\infty||w||^2,\\
|(w,u_x\overline{u}_{xx}w_x)| &\le ||u_x||_\infty||u_{xx}||_\infty||w||\,||w_x|| \le \frac{\nu}{15}||w_x||^2 + \frac{15}{4\nu}||u_x||^2_\infty||u_{xx}||^2_\infty||w||^2,\\
|(w,\overline{u}_xu_{xx}w_x)| &\le ||u_x||_\infty||u_{xx}||_\infty||w||\,||w_x|| \le \frac{\nu}{15}||w_x||^2 + \frac{15}{4\nu}||u_x||^2_\infty||u_{xx}||^2_\infty||w||^2.
\end{aligned}$$

The above bounds are directly deduced from Sobolev's and the Cauchy-Schwarz inequality. Note that $|(w_x,[u_x]^2\overline{w}_x)| \le ||u_xw_x||^2$. We also estimate crudely the contributions from the low-order terms in (24a):

$$\begin{aligned}
|(w,w_x)| &\le ||w||\,||w_x|| \le \frac{\nu}{15}||w_x||^2 + \frac{15}{4\nu}||w||^2,\\
|(w,H(u)w_x)| &\le ||H(u)||_\infty||w||\,||w_x|| \le \frac{\nu}{15}||w_x||^2 + \frac{15}{4\nu}||H(u)||^2_\infty||w||^2,
\end{aligned}$$

$$
\begin{aligned}
|(w, vH(w_x))| &\leq ||v||_\infty ||w||\, ||w_x|| \leq \frac{\nu}{15}||w_x||^2 + \frac{15}{4\nu}||v||_\infty^2 ||w||^2, \\
|(w, u_x v_{xx} \overline{w}_x)| &\leq ||u_x||_\infty ||v_{xx}||_\infty ||w||\, ||w_x|| \leq \frac{\nu}{15}||w_x||^2 + \frac{15}{4\nu}||u_x||_\infty^2 ||v_{xx}||_\infty^2 ||w||^2, \\
|(w, \overline{v}_x v_{xx} w_x)| &\leq ||v_x||_\infty ||v_{xx}||_\infty ||w||\, ||w_x|| \leq \frac{\nu}{15}||w_x||^2 + \frac{15}{4\nu}||v_x||_\infty^2 ||v_{xx}||_\infty^2 ||w||^2, \\
|(w, v_x \overline{v}_{xx} w_x)| &\leq ||v_x||_\infty ||v_{xx}||_\infty ||w||\, ||w_x|| \leq \frac{\nu}{15}||w_x||^2 + \frac{15}{4\nu}||v_x||_\infty^2 ||v_{xx}||_\infty^2 ||w||^2, \\
|(w, u_x \overline{v}_{xx} w_x)| &\leq ||u_x||_\infty ||v_{xx}||_\infty ||w||\, ||w_x|| \leq \frac{\nu}{15}||w_x||^2 + \frac{15}{4\nu}||v_x||_\infty^2 ||v_{xx}||_\infty^2 ||w||^2.
\end{aligned}
$$

The above bounds are directly deduced from Sobolev's and the Cauchy-Schwarz inequality.

The above estimates may be used to obtain the differential inequality

$$
\begin{aligned}
\frac{d}{dt}||w||^2 \leq & \Big[\frac{15}{2\nu}\left(6||u_x||_\infty^2 ||u_{xx}||_\infty^2 + 1 + ||H(u)||_\infty^2 + ||v||_\infty^2 + 3||v_x||_\infty^2 ||v_{xx}||_\infty^2 \right. \\
& \left. + 3||u_x||_\infty^2 ||v_{xx}||_\infty^2\right) + 2\left(||H(u_x)||_\infty + ||v_x||_\infty\right)\Big] ||w||^2.
\end{aligned}
$$

Gronwall-Bellman's inequality implies that $w = 0$. ∎

4 Weak Limit

We have shown in Section 2 that the solution of (2a) and its first-order spatial derivative satisfy bounds independent of ν in L^2 and in the maximum norms; we also have derived a bound for u_t^ν in $L^2([0, 2\pi] \times [0, t])$. In this section, we prove that, in the limit of $\nu \to 0$, $\nu > 0$, the sequence of functions $(u^\nu)_{\nu > 0}$, u^ν solution of (2a), has a limit u in a weak sense. To do so, we use the *a priori* bounds of Lemma 2.1 independent of ν or depending linearly on ν, and we show that $[H(u^\nu)u^\nu]_x$ satisfies a bound independent of ν in the L^2 norm.

Lemma 4.1 *Let u^ν be the solution of (2a). Then the L^2 norm $[H(u^\nu)u^\nu]_x$ is bounded by*

$$
2\sqrt{2}\left(8\pi t + ||f||^2\right)^{1/4}\left(40\pi t + 4||f||^2 + \left|\left|\frac{df}{dx}\right|\right|^2\right)^{3/4}.
$$

The above bound tells us that the square of the L^2 norm of $[H(u^\nu)u^\nu]_x$ grows linearly in time.

Proof: The bound for the L^2 norm of $[H(u^\nu)u^\nu]_x$ is immediately deduced from (6a), (6b), the properties of the Hilbert transform, and Sobolev's inequality. ∎

We now have to show that if u is a limit of the sequence $(u^\nu)_{\nu>0}$, in the limit of $\nu \to 0$, then $w^0 = \lim_{\nu\to 0} H(u^\nu)u^\nu$ is equal to $H(u)u$ and $w^1 = \lim_{\nu \to 0} |u_x^\nu|^2 u_x^\nu$ is equal to $|u_x|^2 u_x$, in $L^2([0, 2\pi] \times [0, t])$, in a weak sense. To do so, we proceed as in Section 5 of [15]. We show only that $w^0 = H(u)u$, since it has been shown in [15] that $w^1 = |u_x|^2 u_x$.

Let ϕ be a test function in $L^2([0, 2\pi] \times [0, t])$. Then

$$
\begin{aligned}
\int_0^t \int_0^{2\pi} (w^0 - H(u)u)\phi\, dx d\tau = & \int_0^t \int_0^{2\pi} (w^0 - H(u^\nu)u^\nu)\phi\, dx d\tau + \int_0^t \int_0^{2\pi} H(u^\nu)[u^\nu - u]\phi\, dx d\tau \\
& + \int_0^t \int_0^{2\pi} u[H(u^\nu) - H(u)]\phi\, dx d\tau.
\end{aligned}
$$

The first integral in the above expression can be made smaller than $\epsilon/3$ because w^0 is a weak limit of $H(u^\nu)u^\nu$; the second integral can also be made smaller than $\epsilon/3$ because $H(u^\nu)$ is bounded in the maximum norm and because u is a weak limit of u^ν; similarly, the third integral can also be made smaller than $\epsilon/3$ because u is bounded in the maximum norm and because $H(u)$ is a weak limit of $H(u^\nu)$. (To prove that, we use the fact that $H(u^\nu)$ is weakly convergent and has a weak limit g.

The weak limit g is equal to $H(u)$ because $||g - H(u)|| = \lim_{\nu\to 0} ||H(u^\nu - u)|| \leq \lim_{\nu\to 0} ||u^\nu - u||$). So $w^0 = H(u)u$ in a weak sense.

With the estimates of Lemmas 2.1 and 4.1 and the fact that $w^0 = H(u)u$ and $w^1 = |u_x|^2 u_x$, we have shown that a weak limit u satisfies

$$u_t + iu_x = [H(u)u]_x + [|u_x|^2 u_x]_x, \tag{25a}$$
$$u(x,0) = f(x), \tag{25b}$$

in a weak sense.

Theorem 4.1 *A weak limit of the sequence of functions $(u^\nu)_{\nu>0}$ satisfies (25a). The function u belongs to $H^1([0,2\pi]\times[0,t])$, and for each fixed t, u_x belongs to $L^\infty([0,2\pi])$. The function u may not be unique.*

5 Conclusions and Open Questions

In this paper, we have derived a new model for the vortex sheet equation that seems to be better than those studied in [1], [4], and [6] in the sense that it has a globally defined flux and its linearization about a constant state gives us the Cauchy-Riemann equation. In our previous work [1] and [16], we showed that the globally defined flux cannot always be controlled by a linear second-order viscous regularization and that Burgers' equation was not a good enough model because when the local flux is replaced by a global flux, for a certain class of initial conditions, delta function singularities form in finite time [1]. The solution of our new model seems to have similar properties to the solution of the vortex sheet equation: without a second-order viscous regularizing term, the solution ceases to exist in finite time, and if a singularity forms, the second-order spatial derivative and higher-order spatial derivatives of the solution become infinite. We were able to show that a weak limit, in the limit of zero viscosity, exists but it may not be unique.

Several open questions remain to be addressed:

- What is the effect of the nonlinear term $[H(u^\nu)u^\nu]_x$ on the numerical solution of the equation? How do the numerical solutions of (2a) and of

  $$y_t + iy_x = [|y_x|^2 y_x]_x + \nu y_{xx}, \tag{26a}$$
  $$y(x,0) = f(x). \tag{26b}$$

  differ?
- Can we characterize a weak limit of (2a), in the limit of $\nu \to 0$ using asymptotics?
- Can we show that a weak limit is an infinite spiral?

The result presented here could be generalized to the case where the term $[H(u^\nu)u^\nu]_x$ in (2a) is replaced by the term $[f(u^\nu)u^\nu]_x$ with $f(u^\nu)$ satisfying the bound

$$||f(u^\nu)|| \leq K||u^\nu|| + K_0,$$

where K and K_0 are constant independent of ν. Then the estimates of Section 2 hold. That is, the square of the L^2 norm of the solution of the new equation and of its first-order spatial derivative and of the square of the maximum norm of the solution of the modified equation grow linearly in time; the L^{2n} norm of the first-order spatial derivative of the solution of the modified equation is nearly a linear function of time; the weak maximum norm bound for the first-order spatial derivative of the solution of the modified equation is a quadratic function in time; the bound for the time integral of the square of the L^2 norm of the solution of the modified equation is a quadratic function of time. The solution of the modified equation would have a weak limit in the limit of zero viscosity because the equivalent of Lemma 4.1 holds.

Acknowledgments

I thank Drs. G. R. Baker and G. K. Leaf for several stimulating discussions and suggestions. The preparation of this work was supported in part by the Mathematical, Information, and Computational Sciences Division subprogram of the Office of Computational and Technology Research, U.S. Department of Energy, under contract W-31-109-Eng-38.

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