Time-Stepping for Three-Dimensional Rigid Body Dynamics

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Abstract

Traditional methods for simulating rigid body dynamics involve determining the current contact arrangement (e.g., each contact is either a "rolling" or "sliding" contact). this approach is most clearly seen in the work of and Pfeiffer and Glocker. However, there has been a controversy about the status of rigid body problems in the area that do not famous, if not the earliest example is due Recently, a number of time-stepping methods have been developed to overcome this difficulty. These time-stepping methods use *integrals* of the forces over forces. separate newest of these methods are developed in terms of *complementarity problems.* time-stepping procedure simulating rigid body dynamics with friction. Proof of the existence of solutions to the continuous problem can be *measure differential inclusions* methods. There are, however, a number of limitations with the methods. In this paper several variants will be discussed and their essential properties proven.

1 Introduction

Recently, a number of new methods have been developed for solving problems in rigid body dynamics that are explicitly based on a time-stepping formulation of the problem [4, 33, 36, 38, 39, 55, 56].

These methods contrast with the more traditional approaches, which involve formulating a system of equations, or complementarity problem, at each time-step to be solved for the forces, which are then used as input for a differential equations or differential-algebraic equations (DAE's) solver [22, 23, 30, 31, 32, 47]. The problem with the traditional approaches is that the systems of equations or complementarity problems to be solved at each instant in time may not have a solution. This difficulty relates directly to problems that do not appear to have solutions, discovered by P. Painlevé [41] in 1895. The controversy begun by this discovery has continued ever since [6, 7, 12, 13, 15, 28, 30, 31, 32, 33, 34, 36, 37, 38, 47].

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The point of the new time-stepping techniques is that they avoid this problem by implicitly allowing impulsive forces at any time during contact, not just at the instant of impact.

Painlevé's problems resulted in researchers focusing their efforts on finding computational techniques. Relatively little effort has focused on the questions of *convergence* of these techniques or on the nature of the resulting limiting trajectories. Notable exceptions are the work of Monteiro-Marques [33] and Stewart [52, 53]. There is, however, much to be done in this area. In the above works, the continuous problem of rigid body dynamics is understood in terms of *measure differential inclusions* (see, e.g., [40]) and *measure differential equations*. These are problems of the kind

$$M(q)\,\frac{dv}{dt} \in K(q(t)), \qquad \frac{dq}{dt} = v, \tag{1}$$

where $K(\cdot)$ is a set-valued function with closed and convex, but not necessarily bounded, values. The unboundedness of $K(q)$ allows for *impulsive* forces and accelerations associated with directions in which $K(q)$ is unbounded. Impulsive forces give rise to discontinuities in the velocity function $v(\cdot)$. The velocity function, however, is not completely irregular: not only is it bounded on bounded intervals in time, but it is a function of *bounded variation*. The sum of the sizes of all the discontinuities is therefore finite, and completely erratic behavior is not allowed.

A number of other conditions are imposed on the solutions of rigid body dynamics problems as well as the measure differential inclusion condition. These conditions include the maximal dissipation property of Coulomb's friction law and the conditions describing the elasticity of impact (ranging from completely inelastic to perfectly elastic).

This paper considers a wide range of time-stepping methods and discusses their implications for convergence theory and the nature of the limiting solutions.

The variations in the time-stepping methods considered include the following:

1. Friction cones (the set of generalized contact forces for a given configuration) are approximated by polyhedral cones. New modifications have been developed recently that avoid this approximation, for which existence results for the time-stepping scheme can be proven [45].

2. All collisions are perfectly inelastic (i.e. no bounce). One scheme that incorporates elastic and partly elastic collisions while preserving the dissipativity property is reported in [4]. This scheme splits the step into "compression" and "expansion" phases. Because this scheme is developed in terms of the generalized forces, it is a "Poisson"-style scheme. Other schemes can be developed in terms of the velocities without splitting the step; these are "Newton"-style schemes.

3. It is assumed that the generalized coordinate vectors and the generalized velocity vectors have the same dimension. This can be done for three-dimensional problems by using Euler angles, for example, to specify the orientation of the rigid body. But any such parameterization must have singularities. On the other hand, using unit quaternions or 3×3 orthogonal matrices to represent the orientation of a body means that the angular velocity vector must have a different dimension to the vector containing the orientation information. Also, care must be taken to preserve the normalization properties of the vector containing the orientation information.

Each of these variations will be described, and the implications for computational practice and for convergence theory will be described. In Section 2, the basic time-stepping scheme will be described. In Section 3, the modifications for general nonpolyhedral friction cones will be explained. In Section 4, both "Newton" and "Poisson" approaches to partly elastic contact will be explained and compared. In Section 5, modifications for different dimensions of the coordinate q and velocity v vectors will be explained.

The model can be modified to handle equality constraints (joints) in a way that does not affect either the solvability of the mixed linear complementarity problem or the upper bounds on the norm of the resulting velocities [4]. Since the challenges of the problem we consider here reside in the description and properties of the contact and friction constraints, we will confine our development to the case in which no joints are involved.

2 Time-Stepping Methods and Convergence Theory

2.1 Rigid Body Dynamics

The study of rigid body dynamics seeks to understand and simulate systems of rigid bodies that may or may not be in contact. These systems are good approximations to many situations in the world around us, such as walking, using a bicycle, playing a ball game, or picking up a pen. All of these activities involve contact between solid, fairly rigid bodies. Implicit in many of these problems is the presence of dry, or Coulomb, friction.

When two bodies are not in contact, there are no contact forces between them (although there may be small and subtle effects due to the motion of the fluid between them, for example). On the other hand, when two bodies make contact, if they are rigid they cannot interpenetrate. Unless there is adhesion (which depends on the physics of the situation), the normal component of the contact force at a point on a body must be acting away from the body against the other.

Friction may be present, in which case equal and opposite tangential forces act on the two bodies in contact. If the bodies are sliding against each other, the friction forces must oppose the slip, although the forces may not be in the opposite direction to the slippage if the friction is not isotropic. The magnitude of the frictional forces is bounded by μN, where μ is the coefficient of friction and N is the normal component of the contact force. If there is slippage, the friction forces must have exactly this magnitude (for isotropic friction); if there is no slippage, any equal and opposite friction forces within this bound are admissible.

To obtain a mathematical formulation of these problems, we need to begin with a formulation of rigid body dynamics *without contact.* Such a formulation can be obtained by using a Lagrangian framework with generalized coordinates q and corresponding generalized velocities v. (Usually $v = dq/dt$, although this will be relaxed in Section 5.)

Let $T(q, v) = \frac{1}{2} v^T M(q)\, v$ be the kinetic energy for configuration q and velocity v; $M(q)$ will be called a *mass matrix*, although if q contains angles, and thus v contains angular velocities, some components of $M(q)$ will be moments of inertia, rather than masses. Let $V(q)$ denote the potential energy of configuration q. Without external forces or contacts, and provided $v = dq/dt$, the equations of motion can be written as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad i = 1, \ldots, n,$$

where $L(q, v) = T(q, v) - V(q)$ is the Lagrangian of the system. With a suitable rearrangement of the equations, this can be put in more explicit form:

$$M(q)\frac{dv}{dt} = k(q, v) - \nabla V(q) \tag{2}$$

where

$$k_i(q, v) = -\frac{1}{2}\sum_{r,s}\left[\frac{\partial m_{ir}}{\partial q_s}(q) + \frac{\partial m_{is}}{\partial q_r}(q) - \frac{\partial m_{rs}}{\partial q_i}(q)\right] v_r\, v_s.$$

In the case of frictionless contact, the only contact force is the normal contact force N. Contact is assumed to be represented by a function $f(q)$ so that if $f(q) > 0$, there is no contact; if $f(q) = 0$, there is contact; and if $f(q) < 0$, then q is an inadmissible configuration. The set of admissible configurations is $C = \{\, q \mid f(q) \geq 0 \,\}$. Assuming that f is a smooth function, where $\nabla f(q) \neq 0$ whenever $f(q) = 0$, the equations of motion for frictionless contact can be written as

$$M(q)\frac{dv}{dt} = k(q, v) - \nabla V(q) + N\, \nabla f(q). \tag{3}$$

Usually we write $n(q) = \nabla f(q)$ as the normal vector into the region of admissible configurations. The contact conditions can be represented as a *complementarity condition*:

$$f(q) \geq 0, \qquad N \geq 0, \qquad f(q)\, N = 0. \tag{4}$$

Frictional problems include a frictional force F that belongs to a suitable linear subspace of the generalized coordinates. It is further restricted by the normal contact force N. More precisely, we require that $F \in N\, FC_0(q)$, where $FC_0(q)$ is a convex, balanced ($-FC_0(q) = FC_0(q)$) set in the space of generalized forces. The vector space span $FC_0(q)$ generated by $FC_0(q)$ represents the plane in which the friction forces act. This plane must *not* contain the normal direction vector $n(q)$; otherwise, it would not be possible to separate normal contact forces from frictional forces. In the case that $n(q)$ is in span $FC_0(q)$, either the component of $FC_0(q)$ in that direction is redundant, or friction forces could be used to "glue" the bodies together.

For a single contact between three-dimensional bodies, this plane is typically two dimensional, representing the tangent plane to the bodies in contact at the contact point. Multiple contacts and "soft finger contact" [24] will change the number of dimensions spanned by $FC_0(q)$. The set of possible contact forces is the *friction cone*:

$$FC(q) = \{\, F + N\, n(q) \mid F \in N\, FC_0(q) \,\}, \tag{5}$$

which is a closed, convex cone for each q. If there is no contact, then the total contact clearly must be zero: $FC(q) = \{0\}$.

Given the friction cone $FC(q)$, Coulomb's friction law can be cast in terms of the maximal dissipation principle [20], which states that the frictional force maximizes the energy dissipation rate over all possible friction forces F, given the normal contact force N. This formulation can be used to describe nonisotropic friction, as occurs in ice-skating, for example.

2.2 Time-Stepping Methods

Time-stepping methods for rigid body dynamics are based on the idea of using integrals of the forces over each time-step, rather than trying to find the instantaneous forces at each instant, and using the results to drive an ODE solver of some kind. Since the friction cone $FC(q)$ is scale invariant, the short-time integrals of the contact forces also belong to approximations of the friction cone. The maximal dissipation formulation of Coulomb's law can also be applied to the short-time integrals, provided the velocity *at the end of the time-step* is used to determine the direction of slip. The contact condition (no contact implies no contact forces) can be easily represented, although the way that this is done has implications for the representation of the impact type (inelastic, partly elastic, or perfectly elastic).

The basic formulation presented in this section is for *inelastic* impact. It is based on the formulations of Stewart and Trinkle [55, 56] and Anitescu and Potra [4]. Work on the convergence theory of related formulations of rigid body dynamics can be found in Stewart [52, 53].

The basic formulation uses a *polygonal approximation* to the friction cone:

$$\widehat{FC}(q) = \operatorname{cone}\{\, n(q) + \mu d_i(q) \mid i = 1, \ldots, r\,\}. \tag{6}$$

(Note that $\operatorname{cone} X$ is the smallest cone containing X; it is the set of all linear combinations $\alpha_1 x_1 + \ldots + \alpha_k x_k$ where $x_i \in X$ and $\alpha_i \geq 0$ for all i.) Let $D(q) = [d_1(q),\, d_2(q),\, \ldots,\, d_r(q)]$.

Let $h > 0$ be the time-step used. The time-stepping formulation, given the configuration q^l and velocity v^l for time $t_l = l\,h$, provides a way of computing the configuration q^{l+1} and velocity v^{l+1} for time $t_{l+1} = (l+1)\,h$. In the process, additional quantities are computed, such as the integrated contact forces $c_n^{l+1}\, n(q^l) + D(q^l)\,\beta^{l+1}$. The time-stepping formulation is presented as a mixed complementarity problem: If $f(q^l + h\,v^l) \leq 0$, then

$$\begin{array}{rcl} M(q^{l+1})\,(v^{l+1} - v^l) = n(q^l)\,c_n^{l+1} + D(q^l)\,\beta^{l+1} + h\,[k(q^l, v^l) - \nabla V(q^l)] & & \\ 0 \leq n(q^l)^T v^{l+1} \qquad\qquad & \perp & c_n^{l+1} \geq 0 \\ 0 \leq \lambda^{l+1} e + D(q^l)^T v^{l+1} \qquad\qquad & \perp & \beta^{l+1} \geq 0 \\ 0 \leq \mu c_n^{l+1} - e^T \beta^{l+1}. \qquad\qquad & \perp & \lambda^{l+1} \geq 0 \end{array} \tag{7}$$

Note that e is a vector of all ones $e = [1,\, 1,\, \ldots,\, 1]^T$ of the appropriate size. Also note that "$a \perp b$" means that $a^T b = 0$, or if a and b are both scalars, that $a\,b = 0$. The quantity λ^{l+1} is not in itself a physical quantity, although it is usually equal to $\|D^T v^{l+1}\|_\infty$, which represents the sliding velocity at the contact.

Note that if $M(q)$ is constant, then (7) can be reduced to a linear complementarity problem (LCP) [11] in $(c_n^{l+1}, \beta^{l+1}, \lambda^{l+1})$ by substituting for v^{l+1} in terms of c_n^{l+1} and β^{l+1}. It can be shown in the case of constant $M(q)$ that the matrix for the reduced LCP is copositive [11, pp. 176–184], and from this, that solutions for (7) exist and can be found using Lemke's algorithm [11, pp. 265–288].

The conditions of (7) can be interpreted physically. The first line is simply a discrete approximation to the equations of motion. The second line is the contact condition: no contact force unless there is contact at the end of the time-interval. The right-hand side of line 3 and the left-hand side of line 4 of (7) imply that the contact force $n(q)c_n^{l+1} + D(q)\beta^{l+1}$ lies in the approximate friction cone $\widehat{FC}(q^l)$. Unless $\beta^{l+1} = 0$ (i.e., no friction force), the complementarity condition in line 3 implies that $\beta_i^{l+1} > 0$ only if $d_i(q^l)^T v^{l+1}$ minimizes $d_j(q^l)^T v^{l+1}$ over all j. That is, the friction force β^{l+1} maximizes the dissipation $-(\beta^{l+1})^T D(q^l)^T v^{l+1}$ over all permissible β^{l+1}. Finally, the complementarity condition on line 4 implies that the contact force $n(q)c_n^{l+1} + D(q)\beta^{l+1}$ must lie on the boundary of the approximate friction cone $\widehat{FC}(q^l)$, unless $D(q^l)^T v^{l+1} = 0$; that is, the total contact force can only lie strictly inside the friction cone if the sliding velocity is zero.

2.3 Convergence Theory and the Continuous Problem

A proper mathematical framework to describe rigid body dynamics must handle inequalities, impulsive forces and discontinuous relationships, such as arise in Coulomb friction. In order to incorporate impulses, the theory should be grounded in measure theory on the real line. A *measure* μ is a function of sets E in the real line $\mathbf{R}$: $\mu(E)$ is a real number, or perhaps a vector. For rigid body dynamics, $\mu(E)$ is usually understood as defining the total impulse, the integral of the force, acting over the time represented by the set $E \subset \mathbf{R}$. If there are no impulses in the set E, and the (finite) force at time t is $F(t)$, then

$$\mu(E) = \int_E F(t)\, dt$$

is the total impulse over the time period represented by E.

Discontinuities from Coulomb friction provide another challenge. Even with known normal contact forces that do not have impulses, the equations of motion for one object sliding against another are discontinuous. As such, the equations of motion do not necessarily have solutions [17, 18, 19]. What is needed is to change the problem from a discontinuous differential *equation* to a differential *inclusion*: Replace

$$m\frac{dv}{dt} = F(q, v) \qquad \text{with} \qquad m\frac{dv}{dt} \in \mathcal{F}(q, v)$$

where $\mathcal{F}(q, v)$ is a set valued function of (q, v). Where F is continuous, $\mathcal{F}$ is single-valued and continuous; where F is discontinuous, $\mathcal{F}$ is set-valued. The set $\mathcal{F}(q, v)$ is the convex hull of the limits of $F(\widehat{q}, \widehat{v})$ for points $(\widehat{q}, \widehat{v})$ converging to (q, v).

The mathematical theory that can best deal with these two issues is that of *measure differential inclusions* invented by J. J. Moreau [36, 38, 40] and used by Monteiro-Marques These are differential inclusions having the form

$$\frac{dv}{dt} \in K(q) + f(q, v), \qquad \frac{dq}{dt} = g(q, v),$$

where $f(q, v)$ and $g(q, v)$ are ordinary continuous functions of (q, v). Note that $K(q)$ is a set-valued function that has closed, convex values and whose graph $\{\, (u, q) \mid u \in K(q) \,\}$ is a closed set. The equations of motion with the friction cone in place of the contact forces is a suitable measure differential inclusion:

$$M(q)\frac{dv}{dt} \in FC(q) + k(q, v) - \nabla V(q), \qquad \frac{dq}{dt} = v. \tag{8}$$

Alternatively, it can be described as a measure differential equation:

$$M(q)\frac{dv}{dt} = n(q)\, c_n + D(q)\, \beta + k(q, v) - \nabla V(q), \qquad \frac{dq}{dt} = v, \tag{9}$$

where c_n is the normal contact force measure, and $D(q)\,\beta$ is the frictional force measure. Using the measure differential equation representation, we need to add conditions to the measures c_n and β to ensure that "$n(q)\, c_n + D(q)\, \beta \in FC(q)$" for all time.

Other conditions can be described directly in terms of complementarity conditions, and other relationships involving measures and functions. For example, the contact condition can be simply described by the conditions

$$\begin{aligned} f(q(t)) &\geq 0 \qquad \text{for all } t, \\ c_n &\geq 0 \qquad \text{as a measure,} \\ \int f(q(t))\, c_n(dt) &= 0. \end{aligned} \tag{10}$$

The condition that "$c_n \geq 0$ as a measure" simply amounts to requiring that $c_n(E) \geq 0$ for all Borel measurable sets E; or, alternatively, that

$$\int \phi(t)\, c_n(dt) \geq 0 \tag{11}$$

for all continuous functions ϕ where $\phi(t) \geq 0$ for all t. The requirement that the contact force lie inside the approximate friction cone $\widehat{FC}(q(t))$ becomes the requirement that

$$\mu c_n - \sum_{r=1}^{m} \beta_i \geq 0 \qquad \text{as a measure.} \tag{12}$$

The maximal dissipation property can be represented as

$$\int \left[\|D(q(t))^T v^+(t)\|_\infty + d_i(q(t))^T v^+(t) \right] \beta_i(dt) = 0 \tag{13}$$

for all i.

The main problem for the convergence theory is to show that the numerical trajectories $(q^h(\cdot), v^h(\cdot))$ produced by a time-stepping scheme such as (7) converge (in some suitable sense) to a limit as the step size $h \downarrow 0$ and that these limits satisfy all the conditions required in the continuous problem. This effort usually amounts to showing that for a subsequence of $(q^h(\cdot), v^h(\cdot))$, $q^h(\cdot) \to q(\cdot)$ *uniformly*, while $v^h(\cdot) \to v(\cdot)$ *pointwise*. The impulses c_n^{l+1} and β^{l+1} are used to construct measures $c_n^h = \sum_{l=0}^{\lceil t/h \rceil} c_n^{l+1}$ and $\beta^h = \sum_{l=0}^{\lceil t/h \rceil} \beta^{l+1}$, for which there are subsequences that converge *weak** to measures $c_n(\cdot)$ and $\beta(\cdot)$.

The strongest convergence proof for problems of this kind can be found in Stewart [52]. A summary of the results and a sketch of the proof can be found in [53].

2.4 Multiple contacts

Multiple contacts do not require completely new ways of reformulating contact problems. Rather, it is a matter of adding contact forces and contact constraints appropriate for each contact. So, for the jth contact, there is a contact normal $n^{(j)}(q)$ with a normal contact force $c_n^{(j)}$, a matrix of direction vectors $D^{(j)}(q)$ that define the frictional forces $D^{(j)}\beta^{(j)}$, and a constraint $f^{(j)}(q) \geq 0$ that represents the "no-interpenetration" condition for the jth contact. Let p be the number of contacts. Then the corresponding formulation to (7) for multiple contacts is

$$\begin{aligned}
M(q^{l+1})\,(v^{l+1} - v^l) = \sum_{j \in J(q,v)} & \left[n^{(j)}(q^l)\, c_n^{(j)\,l+1} + D^{(j)}(q^l)\, \beta^{(j)\,l+1} \right] \\
& + h\,[k(q^l, v^l) - \nabla V(q^l)], \\
0 \leq n^{(j)}(q^l)^T v^{l+1} \qquad & \perp \qquad c_n^{(j)\,l+1} \geq 0 \qquad \text{for } j \in J(q,v), \\
0 \leq \lambda^{(j)\,l+1} e + D^{(j)}(q^l)^T v^{l+1} \qquad & \perp \qquad \beta^{(j)\,l+1} \geq 0 \qquad \text{for } j \in J(q,v), \\
0 \leq \mu c_n^{(j)\,l+1} - e^T \beta^{(j)\,l+1} \qquad & \perp \qquad \lambda^{(j)\,l+1} \geq 0 \qquad \text{for } j \in J(q,v),
\end{aligned} \tag{14}$$

where $J(q,v) = \{\, j \mid f^{(j)}(q^l + h\, v^l) < 0 \,\}$.

Note that the set of possible (total) contact forces is the sum of the friction cones for each of the contacts in generalized coordinates. (Note that adding bodies to a system means adding to the dimensionality of the q and v vectors, as well as adding to the set of possible contacts.)

Large systems of particles appear to produce large complementarity problems. Since even solving linear equations for n unknowns requires $O(n^3)$ time using conventional algorithms, these problems seem to be extremely expensive. However, independent subsystems can be solved independently. Hence, a pair of bodies that are not in contact, and cannot be connected by pairs of bodies that are in contact, can be dealt with independently. Typically, no more than three to four bodies will be connected to each other in this way. Thus, instead of having to solve for tens, hundreds, or thousands of bodies in a single system, one can solve a large number of small systems, which will take $O(n)$ time instead of $O(n^3)$ time.

One of the most time-intensive aspects of multibody dynamics problems with large numbers of bodies is collision detection. While this topic is beyond the scope of this article, it is an important issue and has been extensively discussed in the robotics, graphics, and computational geometry literature. (See, for example, [10, 25, 35].) These algorithms can be used with the above formulation to produce fast algorithms for rigid body simulations.

2.5 Equality Constraints

Equality constraints can also be incorporated into the above formulation (7). This can be done by using Lagrange multipliers as (generalized) forces. Given a single *unilateral* contact, and a number of other *equality constraints* $g_i(q) = 0$, $i = 1, 2, \ldots, q$, we introduce a vector $c_\nu = [\, c_{\nu,1},\, c_{\nu,2}, \ldots,\, c_{\nu,q} \,]^T$ of Lagrange multipliers. Then (7) is replaced with the following: if $f(q^l +$

$h\,v^l) \leq 0$, then

$$\begin{array}{rcl} M(q^{l+1})\,(v^{l+1} - v^l) = n(q^l)\,c_n^{l+1} + D(q^l)\,\beta^{l+1} + (\nabla g(q^l))^T c_\nu \\ + h\,[k(q^l, v^l) - \nabla V(q^l)] \\ 0 \leq n(q^l)^T v^{l+1} & \perp & c_n^{l+1} \geq 0 \\ 0 \leq \lambda^{l+1} e + D(q^l)^T v^{l+1} & \perp & \beta^{l+1} \geq 0 \\ 0 \leq \mu c_n^{l+1} - e^T \beta^{l+1} & \perp & \lambda^{l+1} \geq 0 \\ 0 = \nabla g(q^l)\, v^{l+1}. \end{array} \tag{15}$$

Provided that

$$\{\, \nabla g_i(q) \mid i = 1, 2, \ldots, q \,\} \quad \begin{array}{l} \text{is linear independent for all } q \\ \text{satisfying } g(q) = 0, \end{array} \tag{16}$$

this complementarity problem can be solved. (This condition is known as a *constraint qualification* in optimization and is often necessary for Lagrange multipliers to exist.)

A practical problem with this version of the method is that the solution can "drift" from $g(q) = 0$. That is, the distance from the computed q^l and the manifold $\{\, q \mid g(q) = 0 \,\}$ can increase until the constraint does not hold in any practical sense. There are ways of correcting this problem, of which projection is the simplest. This technique involves periodically solving the Newton equations $\nabla g(q^{l+1})\delta q^{l+1} = -g(q^{l+1})$ — in fact, finding the smallest such δq^{l+1} — and then setting $q^{l+1} \leftarrow q^{l+1} + \delta q^{l+1}$. If this is done often enough, then the computed solution will not drift far from the constraint manifold $\{\, q \mid g(q) = 0 \,\}$.

If there are no unilateral contacts (i.e., the only constraints are equality constraints), then the system can be regarded as a differential algebraic equation (DAE) of index three. Constrained mechanical systems have been often discussed from the point of view of differential algebraic equations [5, 8, 9, 14, 46, 48, 58]. If all constraints are equality constraints, and the constraint qualification (16) holds, there can be no unbounded forces, and more conventional DAE techniques can be used.

3 General (Nonpolyhedral) Friction Cones

The polyhedral approximations

$$\widehat{FC}(q) = \text{cone}\,\{\, n(q) + \mu d_i(q) \mid i = 1, 2, \ldots, p \,\} \tag{17}$$

give a versatile and general technique for dealing with a wide range of friction phenomena (e.g., "soft finger" contact, anisotropic friction). However, these approximations have a number of drawbacks:

1. They are approximate for contact between three-dimensional bodies.
2. They tend to favor particular directions.
3. They tend to require a large number of variables (β_i^{l+1}, for $i = 1, 2, \ldots, p$) to accurately represent the friction force.

Recently, some new techniques of Pang and Stewart [45] allow the use of general convex friction cones. Pang and Stewart [45] consider general *convex* friction cones of the form

$$FC(q) = \{\, n(q)c_n + \mu \widehat{D}(q)\widehat{\beta} \mid \phi_i(c_n, \widehat{\beta}, q) \leq 0,\ i = 1, 2, \ldots, n_{fc} \,\} \tag{18}$$

where each function $\phi_i(c_n, \widehat{\beta}, q)$ satisfies the following conditions:

1. $\phi_i(c_n, \widehat{\beta}, q)$ is convex in $\widehat{\beta}$.
2. $\phi_i(c_n, 0, u) \leq 0$ with equality holding if and only if $c_n = 0$.
3. $\phi_i(0, \widehat{\beta}, u) \leq 0$ implies $\widehat{\beta} = 0$
4. For all $i = 1, \ldots, n_{fc}$, there exists a positive scalar $\gamma \geq 1$ such that for all u, $\phi_i(c_n, 0, u)$ is positively homogeneous of degree γ for $c_n \geq 0$; that is, for $c_n \geq 0$,

$$\phi_i(\tau c_n, 0, u) = \tau^{\gamma}\, \phi_i(c_n, 0, u), \quad \text{for all } \tau \geq 0.$$

Here we also require that $\phi_i(c_n, \widehat{\beta}, q)$ is homogeneous in $\widehat{\beta}$ with exponent γ in order that $FC(q)$ is a true cone; that is, if $z \in FC(q)$, so is $\alpha z \in FC(q)$ for any $\alpha \geq 0$. Note that the formulation (18) includes the polyhedral approximations to the friction cone (with $\gamma = 1$). It also includes the standard quadratic friction models for contact between three-dimensional bodies:

$$FC(q) = \{\, n(q)c_n + \mu \widehat{D}(q)\widehat{\beta} \mid \|\widehat{\beta}\|_2 \leq c_n \,\}, \tag{19}$$

in which case $\phi(c_n, \widehat{\beta}, q) = (\mu c_n)^2 - \|\widehat{\beta}\|_2^2$.

The maximal dissipation principle applied to the time-stepping formulation says that $\widehat{\beta}^{l+1} = \widehat{\beta}$ is chosen to maximize

$$-(v^{l+1})^T \widehat{D}(q^l)^T \widehat{\beta} \tag{20}$$

over all $\widehat{\beta}$ satisfying $\phi_i(c_n, \widehat{\beta}, q) \leq 0$ for $i = 1, 2, \ldots, n_{fc}$. This can be expressed in complementarity form through the Kuhn–Tucker conditions, *except for the case when* $c_n = 0$, $\widehat{\beta} = 0$. At $c_n = 0$, $\widehat{\beta} = 0$ constraint qualifications fail. The simplest applicable constraint qualification in this case is Slater's which requires an interior point in the feasible set for convex constraint functions. For the case $c_n = 0$, though, this fails. To handle this, a Fritz John condition [27] is used with a Fritz John parameter that is related to the value of c_n and to the homogeneity constants γ:

$$\begin{aligned} (c_n)^{\gamma} \widehat{D}(q)^T v^{l+1} - \sum_{i=1}^{n_{fc}} \lambda_i \nabla_{\widehat{\beta}} \phi_i(c_n, \widehat{\beta}, q) &= 0, \qquad i = 1, 2, \ldots, n_{fc} \\ 0 \leq \lambda_i, \qquad \lambda_i \phi_i(c_n, \widehat{\beta}, q) &= 0. \end{aligned} \tag{21}$$

The time-stepping scheme with this formulation of the Coulomb law for general friction cones becomes the following: If $f(q^l + h\, v^l) \leq 0$, then

$$\begin{aligned} M(q^{l+1})\,(v^{l+1} - v^l) = n(q^l)\, c_n^{l+1} + D(q^l)\, \beta^{l+1} + h\,[k(q^l, v^l) - \nabla V(q^l)] \\ 0 \leq n(q^l)^T v^{l+1} \quad \perp \quad c_n^{l+1} \geq 0 \\ 0 \leq \lambda \quad \perp \quad \phi(c_n, \widehat{\beta}, q) \leq 0, \\ (c_n)^{\gamma} \widehat{D}(q)^T v^{l+1} - \sum_{i=1}^{n_{fc}} \lambda_i \nabla_{\widehat{\beta}} \phi_i(c_n, \widehat{\beta}, q) = 0. \end{aligned} \tag{22}$$

While the original time-stepping scheme (7) is a linear complementarity problem (LCP) for constant $M(q)$, the new formulation is a highly nonlinear mixed complementarity problem.

3.1 Theoretical Issues

The nonlinearity of the mixed complementarity problem (22) means that linear complementarity theory [11] cannot be applied. Instead, a homotopy argument is used in [45] to show that solutions exist for problems of the same type as (22).

One of us (Anitescu) proposed an alternative to this approach using a *infinite, but continuously indexed*, family of direction vectors $d(s,q)$ for $s \in [-1,+1]$ with $d(s,q) = -d(s+1,q)$ for $-1 \leq s \leq 0$, instead of a finite matrix $D(q) = [\, d_1(q),\, d_2(q),\, \ldots,\, d_p(q)\,]$. For example, one could take $d(s,q) = \cos(\pi s)\widehat{d}_1(q) + \sin(\pi s)\widehat{d}_2(q)$ to describe two-dimensional isotropic friction. Take finite approximations

$$D_N(q) = [\, d(s_1,q),\, d(s_2,q),\, \ldots,\, d(s_{2N},q)\,]$$

for sequences $0 \leq s_1 < s_2 < \cdots < s_{2N} \leq 1$, where $s_{i+N} = s_i + 1$ so that $d_{i+N}(q) = -d_i(q)$. Then previous existence theory can be applied (see Stewart [52, Lemma 2], or Stewart and Trinkle [56, §3.2] for the linear complementarity formulation). Index the finite sequences $s_1 < s_2 < \cdots < s_{2N}$ by N, and let $q^{(N);l+1}$, $v^{(N);l+1}$, $c_n^{(N);l+1}$ and $\beta^{(N);l+1}$ be the solutions for $D(q) = D_N(q)$ in (7). Then by compactness, there are subsequences $\mathcal{N}$ where $q^{(N);l+1} \to q^{l+1}$, $v^{(N);l+1} \to v^{l+1}$, and $c_n^{(N);l+1} \to c_n^{l+1}$ for $N \to \infty$ in $\mathcal{N}$. While the $\beta^{(N);l+1}$ cannot converge as $N \to \infty$, the limit $D_N(q^l)\beta^{(N);l+1} \to F^{l+1}$ can be found. In fact, the limit β^{l+1} of the measures $\beta^{(N);l+1}(s) = \sum_{i=1}^{2N} \beta_i^{(N);l+1}\delta(s - s_i)$ is a solution of the complementarity problem

$$\begin{array}{lcll}
M(q^{l+1})\,(v^{l+1} - v^l) = n(q^l)\,c_n^{l+1} + \int_{[-1,+1]} d(q^l,s)\,\beta^{l+1}(ds) & & & \\
\qquad\qquad + h\,[k(q^l,v^l) - \nabla V(q^l)] & & & \\
0 \leq n(q^l)^T v^{l+1} & \perp & c_n^{l+1} \geq 0 & \\
0 \leq \lambda^{l+1} + d(s,q^l)^T v^{l+1} & \perp & \beta^{l+1}(s) \geq 0 & \text{for all } s \\
0 \leq \mu c_n^{l+1} - \int_{[-1,+1]} \beta^{l+1}(ds) & \perp & \lambda^{l+1} \geq 0 &
\end{array} \tag{23}$$

where β^{l+1} is understood to be a measure on $[-1,+1]$. Note that the middle complementarity condition should be understood as saying that $\lambda^{l+1} + d(s,q^l)^T v^{l+1} \geq 0$ for all s; that $\beta^{l+1} \geq 0$ as a measure; and that $\int_{[-1,+1]}(\lambda^{l+1} + d(s,q^l)^T v^{l+1})\,\beta^{l+1}(ds) = 0$.

Another issue can arise with nonpolyhedral friction cones in the context of multiple contacts. In generalized coordinates, each contact has an associated friction cone $FC_j(q)$. The combined friction cone when all contact are considered is $FC(q) = FC_1(q) + FC_2(q) + \cdots + FC_{n_c}(q)$, where n_c is the number of contacts. Theoretical difficulties can arise because the total friction cone $FC(q)$ *is not necessarily closed.* If each of the $FC_i(q)$ is polyhedral, however, the total friction cone is also polyhedral and closed. This is not so for more general closed, convex cones.

Although having an open reaction cone is unlikely, it is possibile, as proved by the following three-dimensional example. The example consists of a particle in a 90-degree wedge, as shown in Figure 1. The friction coefficients on both sides of the wedge are 1, which generate two 45-degree angle cones on each side of the particle. The particle is assumed to have radius 0; therefore no torque and inertia appear. Let K_1, K_2 be the cone with normal n_1 and n_2, respectively.

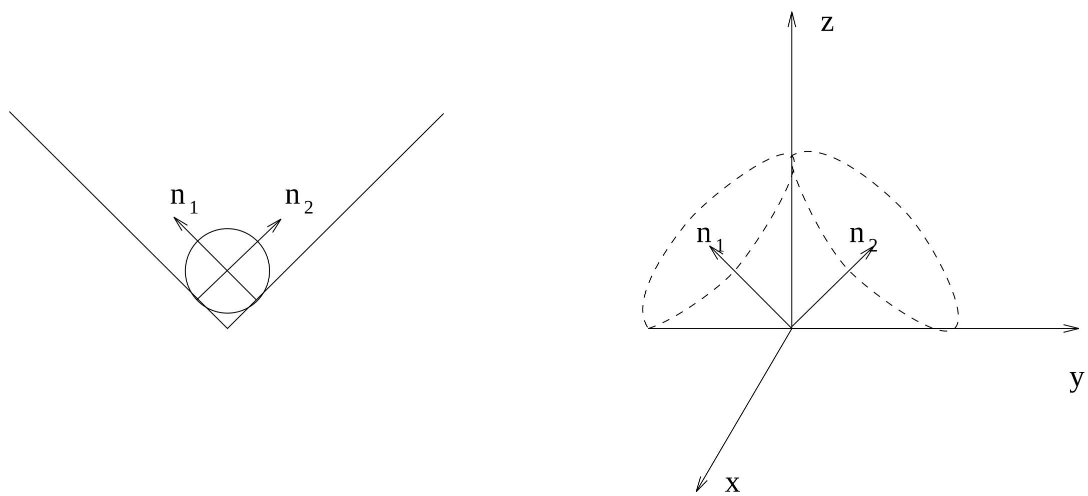

Figure 1: A configuration with a friction cone that is not closed.

Then $(0,-n,0)^T \in K_1$, for any $n \in \mathbf{N}$. But also $(1,n,\frac{1}{2n})^T \in K_2$, since

$$\frac{[0,1,1]\,[1,n,1/2n]^T}{\sqrt{2}\sqrt{n^2+1+(1/4n^2)}} = \frac{n+(1/2n)}{\sqrt{2}\sqrt{(n+(1/2n))^2}} = \frac{1}{\sqrt{2}}, \tag{24}$$

which implies that the angle between $[0,1,1]^T$ and $[1,n,1/2n]^T$ is exactly 45 degrees. This means that

$$[1,n,1/2n]^T + [0,-n,0]^T = [1,0,1/2n]^T \in K_1+K_2,\ \forall n \in \mathbf{N}. \tag{25}$$

But, $[1,0,0]^T \notin K_1+K_2$. Since

$$\lim_{n\to\infty}[1,0,\frac{1}{2n}]^T = [1,0,0]^T \tag{26}$$

and the entire sequence from the left side is in K_1+K_2, it follows that the sum cone is not closed.

This problem can be avoided if the closure of the sum of the cones is *pointed*; that is,

$$(\overline{K_1+K_2})\bigcap -(\overline{K_1+K_2}) = \{0\}.$$

It should be noted that pointedness of the friction cone $FC(q)$ is an essential part of the convergence proof in [52, Lemma 6]. Note that $FC(q)$ needs to be pointed as well as closed to prove that weak* limits of solutions of a measure differential inclusions are also solutions [54], which is also used in the convergence proof in [52].

The main lesson that should be drawn is that a number of theoretical issues become problematic if the closure of the friction cone $FC(q)$ fails to be pointed. There are two main issues here: (1) the variation of the computed velocities may not remain bounded, because of the numerical velocity trajectories "chattering" — rapidly jumping between several different values; and (2) attempting to find an approximate cone containing $FC(\widehat{q})$ for any $\widehat{q}$ sufficiently close to q will fail. To see why a useful outer approximation doesn't exist, suppose $0 \neq z \in FC(q) \cap -FC(q)$. A suitable outer approximation cone K should contain a neighborhood of $+z$ and $-z$. Since zero lies on the line between $+z$ and $-z$, K would also contain a neighborhood of zero, which implies that the cone K contains the entire space $\mathbf{R}^n$.

3.2 Computational Issues

The Pang and Stewart formulation (22) has some immediate computational implications:

- Linear Complementarity Problem solvers are no longer appropriate for finding solutions because of the highly nonlinear complementarity conditions.
- On the other hand, many fewer variables need to be solved for.

Solving highly nonlinear complementarity problems can be difficult. Complementarity problems can be reformulated as systems of equations that are, however, nonlinear and nonsmooth. There has also been a great deal of recent work on Newton-type methods for nonsmooth systems of equations [21, 26, 42, 43, 44, 49, 50, 51].

While Newton-type methods have fast local convergence, they may fail to converge when started far from the solution. Fortunately for rigid body simulations, the solution from the previous step will often give a suitable (close) starting point. However, if there is a change in the contact state (due to a collision, for example), the complementarity problem to be solved will be radically different, at least for the bodies in contact with the colliding bodies. In order to ensure that the nonlinear complementarity problem is solved, global equation solving methods such as homotopy methods need to be used (see, for example, [1, 2, 29, 57]). Care will be needed near the nonsmooth points on the homotopy path. Care will also be needed to deal with the degenerate cases that inevitably occur in simulations of rigid body dynamics, such as the case of a wheel rolling without slip where are no friction force and no sliding velocity.

4 Partly Elastic Contact

The foundation for our partly elastic contact model is the *Poisson hypothesis*. Under this hypothesis, the impact has two phases. In the first phase, *compression*, interpenetration is prevented by normal compression contact impulses. In the second phase, *decompression*, a fraction ϵ_N of each normal compression contact impulse is restituted to the system (Poisson hypothesis, [47]). The quantity ϵ_N is called (normal) *restitution coefficient* and may be different for any contact active during a certain collision, if multiple contacts are involved. Each of these phases is considered instantaneous, and only the velocity of the system but not its position, can change during the collision. Since (7) is formulated in impulses and velocities, it can be used to model both the compression and decompression. In each phase, there will be no impulse due to the external forces ($h = 0$), only intrinsic initial velocity and restitution impulses. Formally, $hk(q^l, v^l)$ will be changed with the appropriate external impulse for each phase.

Let q be the position at which the collision occurs, v^-, v^c, and v^+ be, respectively, the initial, postcompression and postcollision velocities of the system. The superscripts ^{c} and ^{d} refer to data from the compression and decompression phases, respectively. For one contact (resulted from a collision), the compression phase can be set up as

$$\begin{array}{lcl} M(q)\,(v^c - v^-) = n(q)\,c_n^c + D(q^l)\,\beta^c & & \\ 0 \le n(q)^T v^c & \perp & c_n^c \ge 0 \\ 0 \le \lambda^c e + D(q)^T v^c & \perp & \beta^c \ge 0 \\ 0 \le \mu c_n^c - e^T \beta^c & \perp & \lambda^c \ge 0. \end{array} \tag{27}$$

If the collision is proper ($n(q)^T v^- < 0$, [16]), then v^- will not be a solution of (27) and $v^c \ne v^-$ will be determined. As a result, $c_n^c \ne 0$.

Based on the Poisson hypothesis, in the decompression phase there will be an external restitution impulse I_r acting on the system. The postcollision velocity v^+ is a solution of the following mixed linear complementarity problem:

$$\begin{array}{lcl} M(q)\,(v^+ - v^c) = n(q)\,c_n^d + D(q)\,\beta^d + I_r & & \\ 0 \le n(q)^T v^+ & \perp & c_n^d \ge 0 \\ 0 \le \lambda^d e + D(q)^T v^d & \perp & \beta^d \ge 0 \\ 0 \le \mu c_n^d - e^T \beta^d & \perp & \lambda^d \ge 0 \end{array} \tag{28}$$

With our assumptions, $I_r = \epsilon_N c_n^c$. A more sophisticated model could assume that the tangential impulse is partly reversible [47] and the restituted impulse is $I_r = \epsilon_N c_n^c + \epsilon_T D(q)\,\beta^c$, where ϵ_T is the tangential restitution coefficient. However, reversibility of the tangential impulse is usually insignificant, unless the materials involved are highly elastic (as is the case for the so-called superballs, [47]).

Solvability of the mixed linear complementarity problems is guaranteed for (27) and (28) [4, 55]. Therefore, a v^+ will be available at the end of the collision resolution. However, simple examples show that the model (27–28) will not necessarily produce the right energy balance when several contacts are involved. The fact that the kinetic energy will not increase after the collision, $\frac{1}{2}v^+ M(q) v^+ \le \frac{1}{2} v^- M v^-$, can be proven only for very special cases [4, 47]. In particular, if friction is present, then the kinetic energy has been proved to be nonincreasing only if $\epsilon_N = \epsilon_T$, which is a very unlikely particular case. Nevertheless, an interesting common denominator for all these particular cases is that $v^+ = (1+\epsilon_N)v^c - \epsilon_N v^-$. Under the assumption that the collision is exactly detected when multiple contacts are involved it can be proved that $(1+\epsilon_N)v^c - \epsilon_N v^-$ is a part of a feasible point of (28) that satisfies the energy balance [4]. Therefore, for an expedient collision resolution, one could solve only (28) and choose $v^+ = (1+\epsilon_N)v^c - \epsilon_N v^-$.

5 Differing Representations of q and v

Often it is convenient to have different forms of representation for q and v so that "$dq/dt = v$" is no longer true. For example, if q represents the orientation of an object and v the angular velocity, then "$dq/dt = v$" cannot be true globally as q represents the group of 3×3 orthogonal matrices with determinant $+1$: $SO(3)$. We can represent q by directly by 3×3 orthogonal matrices, by unit quaternions, by Euler angles, or by Rodrigues vectors (see, e.g., Angeles [3, §§2.3–2.4]). Note that while the angular velocity is a three-dimensional vector, there are nine parameters in a 3×3 matrix, four in a quaternion, and three for either Euler angles or Rodrigues vectors.

In any case, the general form relating q and v is given by

$$\frac{dq}{dt} = G(q)\, v \tag{29}$$

for some matrix function $G(q)$. For the case of representation by a 3×3 matrix, the equation relating $q = \operatorname{vec} Q$ to $v = \omega$ is

$$\frac{dQ}{dt} = \omega^*\, Q, \tag{30}$$

where

$$\omega^* = \begin{bmatrix} 0 & -\omega_3 & +\omega 2 \\ +\omega_3 & 0 & -\omega_1 \\ -\omega_2 & +\omega_1 & 0 \end{bmatrix}.$$

This gives $dq/dt = \operatorname{vec}(\omega^*\, Q) = G(q)\, v$ where $G(q)$ is linear in q. Similar equations can be developed for the other representations, except for Euler angles, which necessarily introduce singularities into the representation. (Rodrigues vectors have a similar problem, but since there are two Rodrigues vectors that represent any orientation except for the reference orientation, the singularity can be avoided by switching from one representation to the other when the singularity is approached.)

We need the following assumption:

$$G(q) \quad \text{is a full column rank matrix for all } q. \tag{31}$$

This is satisfied by all of the above orientation representation techniques excepting again the Euler angles at the singularity. This implies that the pseudo-inverse $G(q)^+$ is a continuous function of q, and $G(q)^+\, G(q) = I$ for all q.

Several changes must be made in the formulation. One involves the Lagrangian equation. The action is

$$S = \int L(q, v)\, dt = \int \left[\frac{1}{2} v^T M(q) v - V(q)\right] dt. \tag{32}$$

We can write $v = G(q)^+(dq/dt)$. Then

$$S = \int L(q, v)\, dt = \int \left[\frac{1}{2}(\frac{dq}{dt})^T (G(q)^+)^T M(q) G(q)^+ \frac{dq}{dt} - V(q)\right] dt. \tag{33}$$

The equations of motion without contact can then be derived as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad \text{for all } i. \tag{34}$$

If we write $\widetilde{M}(q) = (G(q)^+)^T M(q) G(q)^+$, we obtain

$$\widetilde{M}(q)\, \frac{d^2 q}{dt^2} = \widetilde{k}(q, v) - \nabla V(q) \tag{35}$$

where $\widetilde{k}(q,v)$ is quadratic in v and contains partial derivatives of $\widetilde{M}(q)$. After some algebra, this can be reduced to an equation of the form

$$M(q)\frac{dv}{dt} = \widehat{k}(q,v) - G(q)^T \nabla V(q) \tag{36}$$

where $\widehat{k}(q,v)$ is quadratic in v and contains derivatives of $M(q)$ and $G(q)$, as well as the values $M(q)$, $G(q)$ and $G(q)^+$.

Since the formulation (7) uses constraints written in terms of q, but is used to update v, some additional modifications need to be made. Note that if $f(q(t^*)) = 0$, then $(d/dt)f(q(t))|_{t=t^*} \geq 0$ for feasibility of the trajectory. But $(d/dt)f(q(t))|_{t=t^*} = \nabla f(q(t^*))^T (dq/dt)(t^*) = \nabla f(q(t^*))^T G(q(t^*))\, v(t^*)$. Instead of using the normal direction vector $n(q) = \nabla f(q)$, we should use $\widehat{n}(q) = G(q)^T \nabla f(q)$ for the contact condition.

To preserve the symmetry of the formulation, as well as the correct physics, we need to note that the direction vector for the normal contact force in the velocity co-ordinates is $\widehat{n}(q) = G(q)^T \nabla f(q)$. Thus, the formulation then becomes the following: If $f(q^l + h\, G(q^l)\, v^l) \leq 0$, then

$$\begin{array}{lcl} M(q^{l+1})\,(v^{l+1} - v^l) = \widehat{n}(q^l)\, c_n^{l+1} + D(q^l)\,\beta^{l+1} & & \\ \qquad\qquad + h\,[\widehat{k}(q^l, v^l) - G(q^l)^T \nabla V(q^l)] & & \\ 0 \leq \widehat{n}(q^l)^T v^{l+1} & \perp & c_n^{l+1} \geq 0 \\ 0 \leq \lambda^{l+1} e + D(q^l)^T v^{l+1} & \perp & \beta^{l+1} \geq 0 \\ 0 \leq \mu c_n^{l+1} - e^T \beta^{l+1} & \perp & \lambda^{l+1} \geq 0. \end{array} \tag{37}$$

If we consider the situation where $M(q)$ and $G(q)$ are constant, and substitute for q^{l+1}, v^{l+1} in terms of c_n^{l+1} and β^{l+1} we obtain a pure linear complementarity problem (LCP) for c_n^{l+1}, β^{l+1} and λ^{l+1}. As for (7), the matrix of this LCP is copositive, and solutions can be found by using Lemke's algorithm [11, p. 176ff, p. 265ff].

The theoretical properties of this generalization follow those of (7). The only additional practical considerations for using, for example, quaternions, to represent orientation is that drift may lead to $|\mathbf{q}|$ significantly different from one. This can be remedied by projection and other techniques as discussed in Section 2.5.

References

[1] Eugene Allgower and Kurt Georg. Simplicial and continuation methods for approximating fixed points and solutions to systems of equations. *SIAM Rev.*, 22:28–85, 1980.

[2] Eugene Allgower and Kurt Georg. *Numerical Continuation Methods: An introduction.* Springer–Verlag, Berlin, 1990.

[3] Jorge Angeles. *Fundamentals of Robotic Mechanical Systems.* Mech. Eng. Series. Springer-Verlag, New York, Berlin, Heidelberg, 1997.

[4] M. Anitescu and F. A. Potra. Formulating dynamic multi-rigid-body contact problems with friction as solvable linear complementarity problems. *ASME Nonlinear Dynamics*, 1997. To appear.

[5] Uri M. Ascher, Hongsheng Chin, Linda R. Petzold, and Sebastian Reich. Stabilization of constrained mechanical systems with DAEs and invariant manifolds. *Mech. Structures Mach.*, 23(2):135–157, 1995.

[6] D. Baraff. Fast contact force computation for non-penetrating rigid bodies. In *Proceedings, SIGGRAPH '94*, pages 23–34. ACM, ACM, 1994.

[7] H. Béghin. Sur certain problemes de frottement. *Nouvelles Annales (série 2)*, 5:305–312, 1923-1924.

[8] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations.* Number 14 in Classics in Applied Mathematics. SIAM Publ., Philadelphia, 1996.

[9] A. Cardona and M. Géradin. Numerical integration of second order differential-algebraic systems in flexible mechanism dynamics. In *Computer-aided analysis of rigid and flexible mechanical systems (Tróia, 1993)*, volume 268 of *NATO Adv. Sci. Inst. Ser. E Appl. Sci.*, pages 501–529. Kluwer, Dordrecht, 1994.

[10] J. D. Cohen, Ming C. Lin, D. Manocha, and M. K. Ponamgi. I-COLLIDE: An interactive and exact collison detection system for large-scaled environments. In *Symposium on Interactive 3D Graphics.* ACM SIGGRAPH, ACM Publ., April 1995.

[11] R. W. Cottle, J.-S. Pang, and R. E. Stone. *The Linear Complementarity Problem.* Academic Press, Boston, San Diego, New York, 1992. Series on Computer Science and Scientific Computing.

[12] E. Delassus. Considérations sur le frottement de glissement. *Nouv. Ann. de Math. (4ème série)*, 20:485–496, 1920.

[13] E. Delassus. Sur les lois du frottement de glissement. *Bull. Soc. Math. France*, 51:22–33, 1923.

[14] E. Eich and M. Hanke. Regularization methods for constrained mechanical multibody systems. *Z. Angew. Math. Mech.*, 75(10):761–773, 1995.

[15] M. Erdmann. On a representation of friction in configuration space. *Int. J. Robotics Research*, 13(3):240–271, 1994.

[16] R. Featherstone. *Robotics Dynamics Algorithms.* Kluwer, Boston, 1987.

[17] A. F. Filippov. Differential equations with discontinuous right-hand side. *Amer. Math. Soc. Tranlations*, 42:199–231, 1964. Original in Russian in *Math. Sbornik*, **5**, pp. 99–127, (1960).

[18] A. F. Filippov. Classical solutions of differential equations with multivalued right-hand side. *SIAM J. Control Optim.*, 5:609–621, 1967.

[19] A. F. Filippov. *Differential Equations with Discontinuous Right-Hand Side.* Kluver, 1988.

[20] S. Goyal. *Planar sliding of a rigid body with dry friction: Limit surfaces and dynamics of motion.* Ph.D. thesis, Mechanical Eng., Cornell University, Jan. 1989.

[21] S.-P. Han, J.-S. Pang, and N. Rangaraj. Globally convergent Newton methods for nonsmooth equations. *Math. Oper. Res.*, 17(3):586–607, 1992.

[22] E. J. Haug, S. C. Wu, and S. M. Yang. Dynamics of mechanical systems with Coulomb friction, stiction, impact and constraint addition, deletion I, II and III. *Mechanism and Machine Theory*, 21:401–425, 1986.

[23] Edward J. Haug and Roderic C. Deyo, editors. *Real-time integration methods for mechanical system simulation*, volume 69 of *NATO ASI series F*, Berlin 1991. Springer-Verlag. NATO Advanced Research Workshop on Real-Time Integration Methods for Mechanical System Simulation (1989 : Snowbird, Utah).

[24] R. D. Howe and M. R. Cutkosky. Practical force-motion models for sliding manipulation. *Internat. J. Robotics Research*, 15:557–572, 1996.

[25] P. M. Hubbard. Approximating polyhedra with spheres for time-critical collision detection. *ACM Trans. Graphics*, 15(3):179–210, 1996.

[26] Houyuan Jiang and Liqun Qi. A new nonsmooth equations approach to nonlinear complementarity problems. *SIAM J. Control Optim.*, 35(1):178–193, 1997.

[27] Fritz John. Extremum problems with inequalities as subsidiary conditions. In *Studies and Essays*, pages 187–204. Interscience Inc., New York, 1948.

[28] F. Klein. Zu Painleves Kritik der Coulombschen Reibungsgesetze. *Zeit. Math. Physik*, 58:186–191, 1909.

[29] M. Kojima, N. Megiddo, and T. Noma. Homotopy continuation methods for nonlinear complementarity problems. *Math. Oper. Res.*, 16(4):754–774, 1991.

[30] P. Lötstedt. Coulomb friction in two-dimensional rigid-body systems. *Z. Angewandte Math. Mech.*, 61:605–615, 1981.

[31] P. Lötstedt. Mechanical systems of rigid bodies subject to unilateral constraints. *SIAM J. Appl. Math.*, 42(2):281–296, 1982.

[32] P. Lötstedt. Time-dependent contact problems in rigid-body mechanics. *Math. Prog. Study*, 17:103–110, 1982.

[33] M. D. P. Monteiro Marques. *Differential Inclusions in Nonsmooth Mechanical Problems: Shocks and Dry Friction*, volume 9 of *Progress in Nonlinear Differential Equations and Their Applications*. Birkhäuser Verlag, Basel, 1993.

[34] M. T. Mason and Y. Wang. On the inconsistency of rigid-body frictional planar mechanics. In *Proceedings IEEE Int. Conf. Robotics Automation*, pages 524–528, 1988.

[35] Brian V. Mirtich. *Impulse-based Dynamic Simulation of Rigid Body Systems*. Ph.D. thesis, University of California, Berkeley, 1997. Available via `http://www.merl.com/people/mirtich/papers/thesis/thesis.html`.

[36] J. J. Moreau. Standard inelastic shocks and the dynamics of unilateral constraints. In G. del Piero and F. Maceri, editors, *Unilateral Problems in Structural Mechanics*, volume 288 of *C.I.S.M. Courses and Lectures*, pages 173–221, Vienna, 1985. Springer-Verlag.

[37] J. J. Moreau. Frottement, adhésion, lubrification. *Comptes Rendus Acad. Sci. Paris, Série II*, 302:799–801, 1986.

[38] J. J. Moreau. Unilateral contact and dry friction in finite freedom dynamics. In J. J. Moreau and P. D. Panagiotopoulos, editors, *Nonsmooth Mechanics and Applications*, pages 1–82. Springer–Verlag, Vienna, New York, 1988. International Centre for Mechanical Sciences, Courses and Lectures #302.

[39] J. J. Moreau. Numerical experiments in granular dynamics: Vibration-induced size segregation. In M. Raous, M. Jean, and J. J. Moreau, editors, *Contact Mechanics*, pages 347–158, New York, 1995. Plenum Press.

[40] Jean-Jacques Moreau. Bounded variation in time. In *Topics in nonsmooth mechanics*, pages 1–74. Birkhäuser, Boston, 1988.

[41] P. Painlevé. Sur le lois du frottement de glissemment. *Comptes Rendus Acad. Sci. Paris*, 121:112–115, 1895. Subsequent articles under the same title appeared in this journal, vol. 141, pp. 401–405 and 546–552 (1905).

[42] J.-S. Pang. Newton's method for B-differentiable equations. *Math. Oper. Res.*, 15(2):311–341, 1990.

[43] J.-S. Pang. A B-differentiable equation-based, globally and locally quadratically convergent algorithm for nonlinear programs, complementarity and variational inequality problems. *Math. Prog.*, 51:101–131, 1991.

[44] J.-S. Pang and L. Qi. Nonsmooth equations: Motivation and algorithms. *SIAM J. Optim.*, 3(3):443–465, 1993.

[45] J.-S. Pang and D. E. Stewart. A unified approach to frictional contact problems. Preprint,1997.

[46] Linda R. Petzold. Computational challenges in mechanical systems simulation. In *Computer-aided analysis of rigid and flexible mechanical systems (Tróia, 1993)*, volume 268 of *NATO Adv. Sci. Inst. Ser. E Appl. Sci.*, pages 483–499. Kluwer, Dordrecht, 1994.

[47] F. Pfeiffer and Ch. Glocker. *Multibody Dynamics with Unilateral Contacts*. Wiley Series in Nonlinear Science. J. Wiley and Sons, New York, 1996.

[48] F. A. Potra. Numerical methods for differential-algebraic equations with applications to real-time simulation of mechanical systems. *Z. Angew. Math. Mech.*, 74(3):177–187, 1994.

[49] Liqun Qi. Trust region algorithms for solving nonsmooth equations. *SIAM J. Optim.*, 5(1):219–230, 1995.

[50] Liqun Qi. On superlinear convergence of quasi-Newton methods for nonsmooth equations. *Oper. Res. Lett.*, 20(5):223–228, 1997.

[51] Daniel Ralph. Global convergence of damped Newton's method for nonsmooth equations via the path search. *Math. Oper. Res.*, 19(2):352–389, 1994.

[52] D. E. Stewart. Convergence of a time-stepping scheme for rigid body dynamics and resolution of Painlevé's problems. *Archive for Rational Mechanics and Analysis*, 1997. To appear.

[53] D. E. Stewart. Existence of solutions to rigid body dynamics and the paradoxes of Painlevé. *Comptes Rendus de l'Acad. Sci., Sér. I*, 325:689–693, 1997.

[54] D. E. Stewart. Reformulations of measure differential inclusions and their closed graph property. Preprint, 1997.

[55] D. E. Stewart and J. C. Trinkle. Dynamics, friction and complementarity problems. In M. C. Ferris and J.-S. Pang, editors, *Proceedings Internat. Conf. Complementarity Problems*, Philadelphia, 1996. SIAM Publ.

[56] D. E. Stewart and J. C. Trinkle. An implicit time-stepping scheme for rigid body dynamics with inelastic collisions and Coulomb friction. *Internat. J. Numer. Methods Engineering*, 39(15):2673–2691, 1996.

[57] L. T. Watson, S. C. Billups, and A. P. Morgan. Algorithm 652: HOMPACK: A suite of codes for globally convergent homotopy algorithms. *ACM Trans. on Math. Software*, 13:281–310, 1987.

[58] Jeng Yen, Edward J. Haug, and T. O. Tak. Numerical methods for constrained equations of motion in mechanical system dynamics. *Mech. Structures Mach.*, 19(1):41–76, 1991.