BIFURCATING VORTEX SOLUTIONS OF THE COMPLEX GINZBURG-LANDAU EQUATION

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Abstract. It is shown that the complex Ginzburg-Landau (CGL) equation on the real line admits nontrivial 2π-periodic vortex solutions that have $2n$ simple zeros ("vortices") per period. The vortex solutions bifurcate from the trivial solution and inherit their zeros from the solution of the linearized equation. This result rules out the possibility that the vortices are determining nodes for vortex solutions of the CGL equation.

Key words. Complex Ginzburg-Landau equation, bifurcation, vortex solutions, determining nodes

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Proposed running head: Vortex solutions of the CGL equation

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1 Vortex Solutions and Determining Nodes

In this article we investigate the bifurcation of 2π-periodic vortex solutions of the complex Ginzburg-Landau (CGL) equation on the real line,

$$u_t = (1 + i\nu)u_{xx} + (R - (1 + i\mu)|u|^2)u, \quad x \in \mathbf{R},\ t > 0. \tag{1.1}$$

The unknown function u is complex-valued; R, μ, and ν are given real constants. Vortex solutions are nontrivial solutions whose zero set consists of isolated points. (The term "vortex" for a zero of u, which is rather meaningless in the present context, is borrowed from the theory of the Ginzburg-Landau equations of superconductivity in two dimensions. There, a zero of the complex order parameter identifies a vortex of magnetic flux.) The vortex solutions we are interested in are classical solutions of the following type:

$$u(x, t) = U(nx)\mathrm{e}^{-i\omega t}, \quad x \in \mathbf{R},\ t > 0, \tag{1.2}$$

where ω is a suitable real constant that depends on R, μ, and ν, n is a fixed positive integer, and U is a 2π-periodic complex-valued C^2-function that has two simple zeros per period. (Thus u, which is also 2π-periodic, has $2n$ simple zeros per period.)

The investigation is motivated by the observation that the solution of a dissipative partial differential equation such as the CGL equation is determined uniquely and completely by its *nodal values*—that is, by its values at a set of determining nodes. The concept of determining nodes was first introduced by Foias and Temam in the context of the Navier-Stokes equations for viscous incompressible fluids [3]. These authors showed that the solution of the two-dimensional Navier-Stokes equations is determined uniquely and completely by its values at a finite set of isolated points (determining nodes). The existence of a set of determining nodes has since been shown for various equations, including the CGL equation [6], the Kuramoto-Sivashinsky equation [2], and the Ginzburg-Landau equations of superconductivity [5]. These existence results all require that, in some sense, the set of determining nodes be "sufficiently dense" in the domain, although the cardinality of the set is unknown. For the Navier-Stokes equation, an upper bound of the cardinality has been given in terms of the physical parameters [4], but it has been conjectured on the basis of the Takens imbedding theorem [8] that, for dissipative partial differential equations, the cardinality is in fact independent of the parameters and determined entirely by the dimensionality of the spatial domain.

By definition, if two solutions of the CGL equation coincide at the determining nodes, they coincide everywhere in the domain. Since the CGL equation admits the trivial solution, and any vortex solution coincides with the trivial solution at the vortices, the existence of vortex solutions would rule out the possibility that a solution of the CGL equation is determined uniquely and completely by its vortices. Indeed, an example of such a solution satisfying the Neumann boundary conditions on the interval $(0, 1)$ was constructed by Takáč [7, Corollary 3.2]. In the present work, the boundary conditions are replaced by a condition fixing the vortices.

If u is to be a vortex solution of the type (1.2) with $2n$ vortices per period, then U must satisfy the nonlinear differential equation

$$-U'' - U = \rho(r - |U|^2)U, \quad x \in \mathbf{R}, \tag{1.3}$$

where the complex constants ρ and r are defined in terms of R, μ, ν, and n,

$$\rho = \frac{1 + i\mu}{(1 + i\nu)n^2}, \quad r = \frac{R + i\omega - (1 + i\nu)n^2}{1 + i\mu}. \tag{1.4}$$

The problem is thus defined as a bifurcation problem, where ρ is the bifurcation parameter, and we are interested in solutions of Eq. (1.3) that bifurcate from the trivial solution $(r, U) = (0, 0)$.

We show the following results. First, there exist vortex solutions of the CGL equation that have $2n$ simple zeros per period and bifurcate from the trivial solution. This result rules out the possibility that the vortices are determining nodes for vortex solutions of the CGL equation. Second, the bifurcating vortex solutions inherit their zeros from the solution of the linearized equation. The vortices that are introduced at bifurcation are pinned as the bifurcation parameter increases. Moreover, numerical computations indicate that no other zeros arise after a bifurcation.

The first result may seem to contradict a result of Kukavica [6], who showed that the solution of the CGL equation is completely determined by the values at two nodes, provided these nodes are sufficiently close. After all, by choosing n sufficiently large, we can bring the zeros of the bifurcating solution arbitrarily close together. However, there is no contradiction because the upper bound on the distance between the two determining nodes depends on the parameters and decreases as n increases.

The linearized problem is analyzed in Section 2, the bifurcation analysis is given in Section 3, and numerical results are presented in Section 4.

2 Linearized Problem

If Eq. (1.3) is linearized about the trivial solution, it reduces to

$$-U'' - U = 0, \quad x \in \mathbf{R}. \tag{2.1}$$

This equation admits 2π-periodic solutions that have two simple zeros per period. The zeros are uniformly distributed and separated by a distance π.

Now consider the inhomogeneous equation

$$-U'' - U = f, \quad x \in \mathbf{R}, \tag{2.2}$$

where $f : \mathbf{R} \to \mathbf{C}$ is continuous. We claim that, under suitable conditions on f, this equation admits solutions whose zeros coincide with the zeros of the solution of the homogeneous equation. We make this claim precise in the following lemma for the case where the zeros of the two solutions coincide with the zeros of the cosine function. Other cases are treated similarly.

Lemma 2.1 *Equation (2.2) admits a classical solution that has simple zeros at all odd multiples of* $\frac{1}{2}\pi$ *if and only if*

$$\int_{(k-\frac{1}{2})\pi}^{(k+\frac{1}{2})\pi} f(y) \cos y \, \mathrm{d}y = 0, \quad k \in \mathbf{Z}. \tag{2.3}$$

If f satisfies the condition (2.3), then

$$U(x) = v(x) \cos x, \quad x \in \mathbf{R}, \tag{2.4}$$

where $v \in C^2(\mathbf{R})$ *is given locally on each interval* $[(k - \frac{1}{2})\pi, (\mathrm{k} + \frac{1}{2})\pi]$, $k \in \mathbf{Z}$, *by the expression*

$$v(x) = v((k - \tfrac{1}{2})\pi) + \int_{(k-\frac{1}{2})\pi}^{(k+\frac{1}{2})\pi} f(y) \frac{g(x, y)}{\cos x} \, \mathrm{d}y. \tag{2.5}$$

The kernel g is independent of k,

$$g(x, y) = \begin{cases} \cos x \sin y & \text{if } y \leq x, \\ \sin x \cos y & \text{if } y \geq x. \end{cases} \tag{2.6}$$

Proof. Let $f : \mathbf{R} \to \mathbf{C}$ be a given continuous function. If we look for a solution U of Eq. (2.2) of the form (2.4), then v must satisfy the degenerate differential equation

$$-v'' \cos x + 2v' \sin x = f \tag{2.7}$$

for all $x \neq (k+\frac{1}{2})\pi$; moreover, v must remain bounded near the points $(k+\frac{1}{2})\pi$, for all $k \in \mathbf{Z}$.

Equation (2.7) can be integrated locally on any interval $((k-\frac{1}{2})\pi, (\mathrm{k}+\frac{1}{2})\pi)$. In fact, after multiplying both sides of the equation by $\cos x$, we have

$$-(v' \cos^2 x)' = f(x)\cos x. \tag{2.8}$$

If v_k is the local representation of v on $((k-\frac{1}{2})\pi, (\mathrm{k}+\frac{1}{2})\pi)$, then the integration yields

$$v_k'(x)\cos^2 x = v_k'(k\pi) - \int_{k\pi}^{x} f(y)\cos y \,\mathrm{d}y, \quad (k-\tfrac{1}{2})\pi < x < (k+\tfrac{1}{2})\pi.$$

For v_k' to remain bounded near the endpoints $(k \pm \frac{1}{2})\pi$, it is necessary and sufficient that

$$v_k'(k\pi) = \int_{k\pi}^{(k\pm\frac{1}{2})\pi} f(y)\cos y \,\mathrm{d}y,$$

so f must satisfy the solvability condition (2.3).

If f satisfies the condition (2.3), then

$$v_k'(x) = -\frac{1}{\cos^2 x}\int_{(k\pm\frac{1}{2})\pi}^{x} f(y)\cos y \,\mathrm{d}y, \quad (k-\tfrac{1}{2})\pi < x < (k+\tfrac{1}{2})\pi,$$

and $v_k'((k \pm \frac{1}{2})\pi) = \mp\frac{1}{2}\mathrm{f}((\mathrm{k}\pm\frac{1}{2})\pi)$. The expression (2.5) follows upon integration. ∎

While Eq. (2.5) gives a local representation of v on each interval $[(k-\frac{1}{2})\pi, (\mathrm{k}+\frac{1}{2})\pi]$, there also exists a global representation that is valid on the entire real line. First, observe that

$$v((k+\tfrac{1}{2})\pi) = v((k-\tfrac{1}{2})\pi) + \int_{(k-\frac{1}{2})\pi}^{(k+\frac{1}{2})\pi} f(y)\sin y \,\mathrm{d}y, \quad k \in \mathbf{Z}. \tag{2.9}$$

Repeated application of this recurrence relation yields an expression for $v((k-\frac{1}{2})\pi)$ in terms of $v(-\frac{1}{2}\pi)$,

$$v((k-\tfrac{1}{2})\pi) = v(-\tfrac{1}{2}\pi) + \int_{-\frac{1}{2}\pi}^{(k-\frac{1}{2})\pi} f(y)\sin y \,\mathrm{d}y, \quad k \in \mathbf{Z}.$$

Furthermore, because f satisfies (2.3),

$$\int_{x}^{(k+\frac{1}{2})\pi} f(y)\cos y \,\mathrm{d}y = \int_{x}^{\frac{1}{2}\pi} f(y)\cos y \,\mathrm{d}y, \quad k \in \mathbf{Z}.$$

Thus, v is represented globally by the expression

$$v(x) = v(-\tfrac{1}{2}\pi) + \int_{-\frac{1}{2}\pi}^{x} f(y)\sin y \,\mathrm{d}y + \frac{\sin x}{\cos x}\int_{x}^{\frac{1}{2}\pi} f(y)\cos y \,\mathrm{d}y, \quad x \in \mathbf{R}. \tag{2.10}$$

3 Bifurcation Analysis

We now proceed to the bifurcation analysis. We recall that we wish to find solutions of Eq. (1.3) that are 2π-periodic and have two simple zeros per period. In fact, we will try to find solutions whose zeros coincide with the zeros of $\cos x$—the solution of the linearized equation.

We use the results of the preceding section, substituting for f the expression in the right member of Eq. (1.3). Taking U to be of the form (cf. [7, Eq. (3.19)])

$$U(x) = v(x)\cos x, \quad x \in \mathbf{R}, \tag{3.1}$$

we replace the original problem by a bifurcation problem for (r, v) in a neighborhood of $(r, v) = (0, 0) \in \mathbf{C} \times C^2(\mathbf{R})$.

We infer from Lemma 2.1 that the bifurcation analysis can be performed locally on any of the intervals $[(k - \frac{1}{2})\pi, (\mathrm{k} + \frac{1}{2})\pi]$, $k \in \mathbf{Z}$. Hence, it suffices to consider the function v on the interval $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, which we denote by J from now on. According to Eq. (2.5), v must satisfy the following integral equation on J:

$$v(x) = v(-\tfrac{1}{2}\pi) + \int_J f(y)\frac{g(x, y)}{\cos x}\,\mathrm{d}y, \quad x \in J, \tag{3.2}$$

where g is defined in Eq. (2.6) and f is given in terms of v,

$$f(x) = \rho(r - |v|^2\cos^2 x)v\cos x, \quad v \equiv v(x), \quad x \in J. \tag{3.3}$$

The function f must satisfy the condition (2.3) for $k = 0$. With f given by Eq. (3.3), the latter translates into a relation between r and v,

$$r\int_J v(y)\cos^2 y\,\mathrm{d}y = \int_J |v(y)|^2 v(y)\cos^4 y\,\mathrm{d}y. \tag{3.4}$$

If we take this as the definition of r, then we have reduced the bifurcation problem to a problem for v in the neighborhood of $v = 0 \in C(J)$.

We employ the Lyapunov-Schmidt reduction method in much the same way as in [7, Proof of Theorem 3.1]. Let the projection $P : C(J) \to C(J)$ be defined by

$$Pu(x) = \frac{2}{\pi}\int_J u(y)\cos^2 y\,\mathrm{d}y, \quad u \in C(J), \quad x \in J, \tag{3.5}$$

and its complement $P' : C(J) \to C(J)$ by $P' = I - P$. (I is the identity operator in $C(J)$.) The pair (P, P') decomposes the space $C(J)$. Note that Pu is a complex constant-valued function, so we may identify $PC(J)$ with the complex plane. Note also that $P1 = 1$.

Let $C_0(J)$ denote the closed subspace of $C(J)$ consisting of all elements $f \in C(J)$ that satisfy the condition (2.3) for $k = 0$. For any $f \in C_0(I)$, we define $v \in C(I)$ by the relation (3.2); its projection Pv is

$$Pv(x) = v(-\tfrac{1}{2}\pi) + \int_J \left(\frac{2}{\pi} \int_J g(z,y) \cos z \,\mathrm{d}z \right) f(y) \,\mathrm{d}y, \quad x \in J. \tag{3.6}$$

We set $Pv = \varepsilon$ and scale $P'v$ by ε, putting $P'v = \varepsilon w$. Thus,

$$v = \varepsilon(1 + w), \quad \varepsilon \in \mathbf{C}, \quad w \in P'C(J). \tag{3.7}$$

The mapping $f \mapsto \varepsilon w$ defines a linear operator L from $C_0(J)$ into $P'C(J)$,

$$Lf = \varepsilon w, \quad f \in C_0(J). \tag{3.8}$$

Since $\varepsilon w = v - Pv$, the expression for Lf is readily found from Eqs. (3.2) and (3.6),

$$(Lf)(x) = \int_J \left(\frac{g(x,y)}{\cos x} - \frac{2}{\pi} \int_J g(z,y) \cos z \,\mathrm{d}z \right) f(y) \,\mathrm{d}y, \quad x \in J. \tag{3.9}$$

Lemma 3.1 *The linear operator $L : C_0(J) \to C(J)$ defined in Eq. (3.8) is bounded,*

$$\|Lf\|_\infty \le 3\pi \|f\|_\infty, \quad f \in C_0(I). \tag{3.10}$$

Proof. Since $|g(x,y)| \le 1$, it is certainly true that

$$\left| \int_J \frac{2}{\pi} \int_J g(z,y) \cos z \,\mathrm{d}z \;\, f(y) \,\mathrm{d}y \right| \le \frac{2}{\pi} |J|^2 \|f\|_\infty = 2\pi \|f\|_\infty, \quad x \in J. \tag{3.11}$$

To estimate the remaining integral in Eq. (3.9), we distinguish between $x \ge 0$ and $x \le 0$.

Suppose $x \ge 0$. Then

$$\int_J \frac{g(x,y)}{\cos x} f(y) \,\mathrm{d}y = \int_{-\frac{1}{2}\pi}^{x} f(y) \sin y \,\mathrm{d}y + \sin x \int_x^{\frac{1}{2}\pi} f(y) \frac{\cos y}{\cos x} \,\mathrm{d}y.$$

The first term is estimated trivially; its modulus is less than or equal to $(x + \frac{1}{2}\pi)\|\mathrm{f}\|_\infty$. In the second term, we use the fact that $0 \le \cos y / \cos x \le 1$ for all $0 \le x < y \le \frac{1}{2}\pi$;

the modulus of this term is less than $(\frac{1}{2}\pi - \mathrm{x})\|\mathrm{f}\|_\infty$. Together, these two inequalities give the estimate

$$\left|\int_J \frac{g(x,y)}{\cos x} f(y)\,\mathrm{d}y\right| \leq \pi \|f\|_\infty, \quad x \in J, \quad x \geq 0. \tag{3.12}$$

Now suppose $x \leq 0$. Then we start from the expression

$$\int_J \frac{g(x,y)}{\cos x} f(y)\,\mathrm{d}y = \int_{-\frac{1}{2}\pi}^{x} f(y) \sin y\,\mathrm{d}y + \sin x \int_{-\frac{1}{2}\pi}^{x} f(y) \frac{\cos y}{\cos x}\,\mathrm{d}y$$

and find, similarly,

$$\left|\int_J \frac{g(x,y)}{\cos x} f(y)\,\mathrm{d}y\right| \leq \pi \|f\|_\infty, \quad x \in J, \quad x \leq 0. \tag{3.13}$$

Together, the inequalities (3.12) and (3.13) give the estimate

$$\left|\int_J \frac{g(x,y)}{\cos x} f(y)\,\mathrm{d}y\right| \leq \pi \|f\|_\infty, \quad x \in J. \tag{3.14}$$

The statement of the lemma follows from Eqs. (3.9), (3.11), and (3.14). ∎

The integral in the left member of Eq. (3.4) is equal to $\frac{1}{2}\pi \mathrm{Pv}$, where $Pv = \varepsilon$, so the condition (3.4), which we use to define r in terms of v, reduces to

$$r = \frac{2}{\varepsilon\pi} \int_J |v(y)|^2 v(y) \cos^4 y\,\mathrm{d}y. \tag{3.15}$$

When we insert this expression into Eq. (3.3) and make the substitution $v = \varepsilon(1+w)$, we obtain a relation between f and w,

$$f = \varepsilon|\varepsilon|^2 F(w), \quad w \in P'C(J), \tag{3.16}$$

where $F : P'C(J) \to C_0(J)$ is the following nonlinear map:

$$\begin{aligned}[F(w)](x) = \rho \left(\frac{2}{\pi} \int_J |1 + w(y)|^2 (1 + w(y)) \cos^4 y\,\mathrm{d}y - |1 + w(x)|^2 \cos^2 x\right) \\ \times (1 + w(x)) \cos x, \quad x \in J,\, w \in P'C(J).\end{aligned} \tag{3.17}$$

Combining Eqs. (3.8) and (3.16), we obtain an equation for w in $P'C(J)$,

$$w = T_\varepsilon(w) = |\varepsilon|^2 L(F(w)). \tag{3.18}$$

We wish to solve this equation using the Banach contraction principle [1, Theorem 7.1]. We already know that L is bounded from $C_0(J)$ into $P'C(J)$; the following lemma gives the necessary estimates for F.

Let $\mathcal{B}_\sigma$ denote the closed ball of radius σ ($\sigma > 0$) centered at the origin in $P'C(J)$,

$$\mathcal{B}_\sigma = \{w \in P'C(J) : \|w\|_\infty \leq \sigma\}. \tag{3.19}$$

Lemma 3.2 *The nonlinear map $F : P'C(J) \to C_0(J)$ defined in Eq. (3.16) is bounded and Lipschitz continuous,*

$$\|F(w)\|_\infty \leq |\rho|(2+\sigma)(1+\sigma)^3, \quad w \in \mathcal{B}_\sigma, \tag{3.20}$$

$$\|F(w_1) - F(w_2)\|_\infty \leq 3|\rho|(2+\sigma)(1+\sigma)^2\|w_1 - w_2\|_\infty, \quad w_1, w_2 \in \mathcal{B}_\sigma. \tag{3.21}$$

Proof. If $w \in \mathcal{B}_\sigma$, then

$$|[F(w)](x)| \leq |\rho| \left(\frac{2}{\pi}(1+\sigma)^3 \int_J \cos^4 y \, \mathrm{d}y + (1+\sigma)^2 \right)(1+\sigma), \quad x \in J.$$

Because $(2/\pi)\int_J \cos^4 y \, \mathrm{d}y = \frac{3}{4} < 1$, the estimate (3.20) follows.

If $w_1, w_2 \in \mathcal{B}_\sigma$, then

$$\begin{gathered} |[F(w_1)](x) - [F(w_2)](x)| \leq |\rho| \, |s_1(1 + w_1(x)) - s_2(1 + w_2(x))| \\ +|\rho| \, \left| |1 + w_1(x)|^2(1 + w_1(x)) - |1 + w_2(x)|^2(1 + w_2(x)) \right|, \end{gathered}$$

where we have used the abbreviations

$$s_j = \frac{2}{\pi} \int_J |1 + w_j(y)|^2 (1 + w_j(y)) \cos^4 y \, \mathrm{d}y, \quad j = 1, 2.$$

Adding and subtracting terms, we see that

$$\begin{gathered} \left| |1 + w_1|^2(1 + w_1) - |1 + w_2|^2(1 + w_2) \right| \\ = |(|1 + w_1|^2 + |1 + w_2|^2)(w_1 - w_2) + (1 + w_1)(1 + w_2)(\overline{w}_1 - \overline{w}_2)| \\ \leq 3(1+\sigma)^2\|w_1 - w_2\|_\infty. \end{gathered}$$

Furthermore,

$$\begin{gathered} |s_1(1 + w_1) - s_2(1 + w_2)| = |(1 + w_1)(s_1 - s_2) + s_2(w_1 - w_2)| \\ \leq (1+\sigma)|s_1 - s_2| + |s_2||w_1 - w_2|. \end{gathered}$$

One readily verifies that

$$|s_1 - s_2| \le \frac{6}{\pi}(1+\sigma)^2 \left(\int_J \cos^4 y \, \mathrm{d}y \right) \|w_1 - w_2\|_\infty = \frac{9}{4}(1+\sigma)^2 \|w_1 - w_2\|_\infty$$

and

$$|s_2| \le \frac{2}{\pi}(1+\sigma)^3 \left(\int_J \cos^4 y \, \mathrm{d}y \right) = \frac{3}{4}(1+\sigma)^3,$$

so

$$|s_1(1+w_1) - s_2(1+w_2)| \le 3(1+\sigma)^3 \|w_1 - w_2\|_\infty.$$

The inequality (3.21) follows. ∎

We are ready to prove the desired bifurcation result. Let the set Γ be defined by

$$\Gamma = \{(r,U) \in \mathbf{C} \times C^2(\mathbf{R}) : (r,U) \text{ satisfies Eq. (1.3);}$$

$$U(x) = v(x)\cos x,\ x \in \mathbf{R};\ \ v \in C(\mathbf{R})\ v \text{ bounded}\} \tag{3.22}$$

Theorem 3.1 *The point* $(0,0) \in \mathbf{C} \times C^2(\mathbf{R})$ *is a bifurcation point for Eq. (1.3). There exists an open neighborhood* $\mathcal{O}$ *of* $(0,0)$ *in* $\mathbf{C} \times C^2(\mathbf{R})$ *and a positive constant* δ *such that the set* $\Gamma \cap \mathcal{O}$ *coincides with the set of all* $(r,U) \in \mathbf{C} \times C^2(\mathbf{R})$ *having the following representation:*

$$r = \tfrac{3}{4}|\varepsilon|^2 \left(1 + |\varepsilon|^2 \varphi(|\varepsilon|^2) \right), \tag{3.23}$$

$$U(x) = \varepsilon(1 + |\varepsilon|^2 \Phi(|\varepsilon|^2, x)) \cos x, \quad x \in \mathbf{R}, \tag{3.24}$$

where ε *is an arbitrary complex parameter with* $0 < |\varepsilon|^2 < \delta$*, and* $\varphi : [0,\delta) \to \mathbf{C}$ *and* $\Phi : [0,\delta) \times \mathbf{R} \to \mathbf{C}$ *are continuous functions satisfying the following conditions:*

(i) $(r,U) \in \Gamma$,

(ii) $\Phi(s,\cdot) \in C^2(\mathbf{R})$ *for every* $s \in (0,\delta)$ *and* $\int_{\mathbf{R}} \Phi(s,x) \cos^2 x \, \mathrm{d}x = 0$*, and*

(iii) *the real and imaginary parts of* φ *and* Φ *are real-analytic functions of their arguments.*

Proof. Following the steps outlined in the preceding analysis, we reduce the bifurcation problem to a problem for w in the neighborhood of $w = 0 \in P'C(J)$. This function w must be a fixed point of the operator T_ε defined in Eq. (3.18). Once w has been found, we define v in terms of w by means of Eq. (3.7) and (r,U) in terms of v by means of Eqs. (3.15) and (3.1).

From Lemmas 3.1 and 3.2 we obtain

$$\|L(F(w))\|_\infty \leq 3\pi\|F(w)\|_\infty \leq 3\pi|\rho|(2+\sigma)(1+\sigma)^3, \quad w \in \mathcal{B}_\sigma,$$

so $T_\varepsilon = |\varepsilon|^2 LF$ maps $\mathcal{B}_\sigma$ into itself whenever

$$|\varepsilon|^2 < \frac{\sigma}{3\pi|\rho|(2+\sigma)(1+\sigma)^3}.$$

Furthermore,

$$\|T_\varepsilon(w_1 - w_2)\|_\infty \leq 9\pi|\varepsilon|^2|\rho|(2+\sigma)(1+\sigma)^2\|w_1 - w_2\|_\infty, \quad w_1, w_2 \in \mathcal{B}_\sigma,$$

so T_ε is a contraction if

$$|\varepsilon|^2 < \frac{1}{9\pi|\rho|(2+\sigma)(1+\sigma)^2}. \tag{3.25}$$

Hence, if we define

$$\delta \equiv \delta(\sigma) = \frac{1}{3\pi|\rho|(2+\sigma)(1+\sigma)^2}\min\left\{\frac{\sigma}{1+\sigma}, \frac{1}{3}\right\}, \tag{3.26}$$

then T_ε is a contractive mapping of $\mathcal{B}_\sigma$ into itself for every $\varepsilon \in \mathbf{C}$ satisfying $0 < |\varepsilon|^2 < \delta$. Consequently, T_ε has a unique fixed point in $\mathcal{B}_\sigma$, which can be found by iteration. The lowest-order approximation $w = 0$, which corresponds to $v = \varepsilon$, gives $r = \frac{3}{4}|\varepsilon|^2$ and $U(x) = \varepsilon \cos x$.

The statements of the theorem follow from the implicit function theorems [1, Theorems 15.1 and 15.3]. ∎

Theorem 3.1 implies that the CGL equation admits 2π-periodic vortex solutions u, which bifurcate from the trivial solution; these vortex solutions have $2n$ zeros ("vortices") per period; and the vortices are located at the zeros of the cosine function, which is the solution of the linearized equation in the neighborhood of the bifurcation point.

The conditions (i)–(iii), together with the representations (3.23) and (3.24), determine ε, φ, and Φ uniquely.

The representations (3.23) and (3.24) show that we have a supercritical pitchfork bifurcation from $(0, 0)$. Further terms in the representations (3.23) and (3.24) can be computed in a standard manner,

$$r = \tfrac{3}{4}|\varepsilon|^2\left(1 - \tfrac{1}{32}\rho|\varepsilon|^2 + O(|\varepsilon|^4)\right), \tag{3.27}$$

$$U(x) = \varepsilon \cos x \left(1 - \frac{\rho|\varepsilon|^2}{32}\frac{\cos 3x}{\cos x} + \left(\frac{\rho|\varepsilon|^2}{32}\right)^2 \frac{3\cos 3x + \cos 5x}{\cos x} + O(|\varepsilon|^6)\right). \quad (3.28)$$

Furthermore,

$$R = n^2 + \tfrac{3}{4}|\varepsilon|^2 \left(1 - \frac{1 - \mu^2 + 2\mu\nu}{32n^2(1+\nu^2)}|\varepsilon|^2 + O(|\varepsilon|^4)\right), \quad (3.29)$$

and

$$\omega = \nu n^2 + \tfrac{3}{4}|\varepsilon|^2 \left(\mu - \frac{\mu^2\nu + 2\mu - \nu}{32n^2(1+\nu^2)}|\varepsilon|^2 + O(|\varepsilon|^4)\right). \quad (3.30)$$

In particular, $\omega = \mu R + (\nu - \mu)n^2 + O((R - n^2)^2)$.

4 Numerical Results

The results of the preceding bifurcation analysis are supported by the results of numerical computations. (These computations were performed by Michael Levine, participant in the 1998 Energy Research Undergraduate Laboratory Fellowship program at Argonne National Laboratory.)

Three numerical methods were applied. The first method was a fixed-point iteration based on Eq. (3.18). (Observe that the only parameter in Eq. (3.18) is $\rho|\varepsilon|^2$; without loss of generality, we may take $\varepsilon = 1$.) The method converged for ρ in the rectangle $[-3.5, 3.5] \times [0, 1.5]$. The bifurcating solutions were found to be very close to the solutions of the linearized equation. Next, a shooting method was applied to Eq. (1.3). The method yielded bifurcating solutions for ρ in discs centered at the origin with radii up to 9. In a third method, a finite-difference method was applied to Eq. (1.3), and the resulting system of linear equations was solved directly. This method gave results for ρ in discs centered at the origin with radii up to 200.

None of the bifurcating solutions had any additional zeros. The bifurcating solutions were all symmetric with respect to the origin. For values of ρ close to the imaginary axis, additional asymmetric solutions were found that bifurcated from the symmetric ones. These bifurcations occurred multiple times as $|\rho|$ was increased along rays emanating from the origin, and we conjecture that they occur infinitely often.

The properties of the bifurcating solutions are summarized in Figs. 1 and 2.

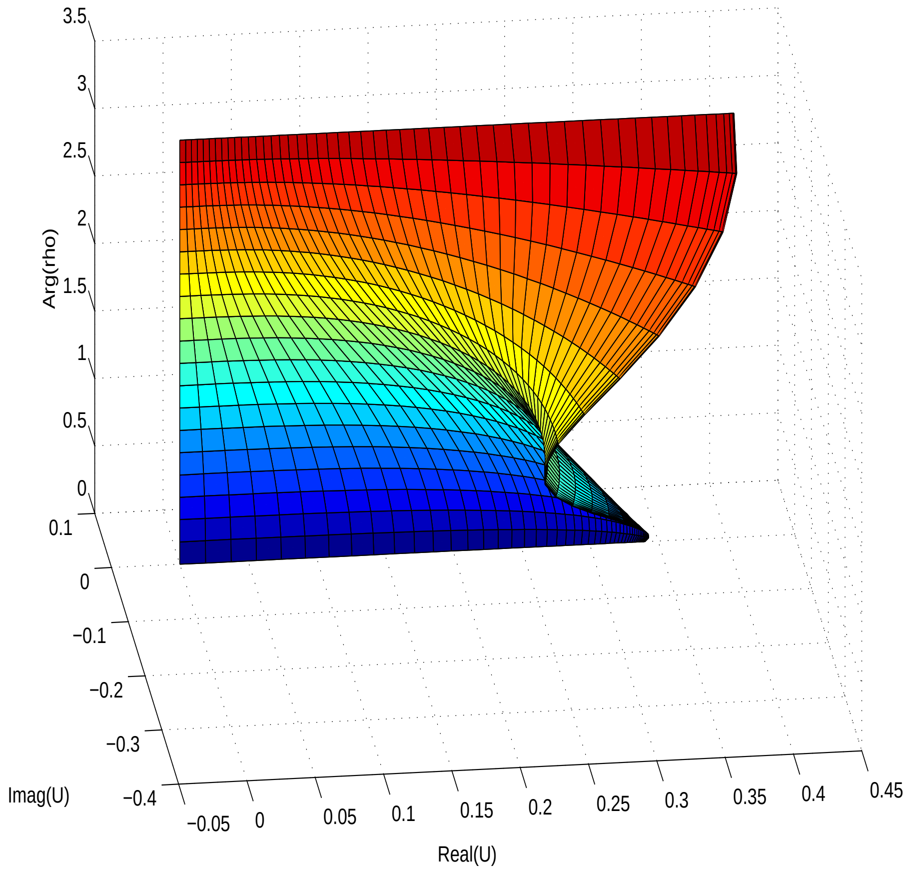

Figure 1: Bifurcating solutions U of the CGL equation as a function of $\arg(\rho)$ for a fixed value of $|\rho|$.

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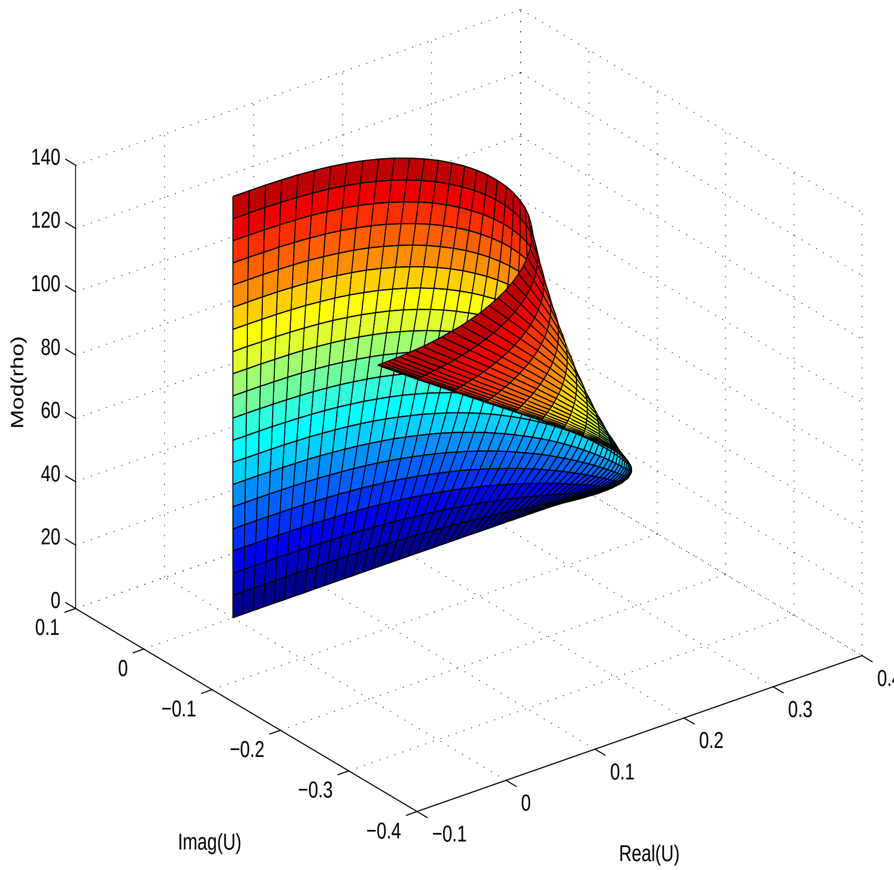

Figure 2: Bifurcating solutions U of the CGL equation as a function of $|\rho|$ for a fixed value of $\arg(\rho)$.

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