A Comparison Result and Elliptic Equations Involving Subcritical Exponents *

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Abstract

It is well known that good bounds for solutions of nonlinear differential equations are difficult to obtain. In this paper, we establish a theorem comparing non-negative solutions (having identical initial values) of the equations $u''(t) + q(t)u^p(t) + r(t)u(t) = 0$ and $v''(t) + k(t)q(t)v^p(t) + r(t)u(t) = 0$, respectively. If $q(t), r(t) \geq 0$, $k(t) \geq 1$, $k(t)$ is non-decreasing, and the first equation satisfies a certain uniqueness criterion, our result asserts that $u(t) \geq v(t)$. Both the uniqueness assumption on the equation and the monotonicity requirement on $k(t)$ are necessary. A particular case of this theorem plays a central role in a recent paper of Atkinson and Peletier in the study of asymptotic behavior of nonlinear elliptic equations involving a critical exponent. A simple corollary of our result provides information on the same type of equations with subcritical exponents.

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1 Introduction

The celebrated Sturm comparison theorem is a useful tool for obtaining bounds for linear second-order ordinary differential equations. Suppose that $u(t)$ and $v(t)$ are, respectively, non-negative solutions of the equations

$$u''(t) + q(t)u(t) = 0, \tag{1.1}$$

and

$$v''(t) + Q(t)v(t) = 0, \tag{1.2}$$

on (a, b), and they satisfy the same initial conditions

$$u(a) = v(a) \geq 0, \quad u'(a) = v'(a). \tag{1.3}$$

Furthermore, suppose that the coefficients satisfy the comparison condition

$$q(t) \leq Q(t) \quad \text{for all } t \in (a, b). \tag{1.4}$$

Sturm's theorem then asserts that

$$u'(t)/u(t) \geq v'(t)/v(t) \quad \text{for all } t \in (a, b). \tag{1.5}$$

Consequently,

$$u(t) \geq v(t) \quad \text{for all } t \in (a, b). \tag{1.6}$$

Hence if one of the two equations mentioned can be solved explicitly, then the computed solutions can be used as bounds for solutions of the other equation.

Let $f : [0, \infty) \to [0, \infty)$ be a continuous non-negative function. A nonlinear equation of the form

$$u''(t) + q(t)f(u(t)) = 0 \tag{1.7}$$

is said to be superlinear (sublinear) if

$$\frac{f(u)}{u} \text{ is an increasing (decreasing) function of } u > 0. \tag{1.8}$$

It is easy to see that an analog of the above linear result remains valid when (1.1) and (1.2) are, respectively, replaced by sublinear equations of the form (1.7) and $v''(t) + Q(t)f(v(t)) = 0$, both having the same nonlinear function $f(u)$.

For superlinear equations, the same is no longer true, as the following example shows. Let

$$f(u) = u^3, \tag{1.9}$$

$$q(t) = \begin{cases} 1 & t \leq 1 \\ 1000 & t > 1 \end{cases}, \tag{1.10}$$

and

$$Q(t) = (1.1)^2 q(t). \tag{1.11}$$

With the initial conditions

$$u(0) = v(0) = 1 \quad \text{and} \quad u'(0) = v'(0) = 0, \tag{1.12}$$

numerical results show that at first $u(t)$ stays above $v(t)$, but dips below $v(t)$ after approximately $t = 1.08$.

In their paper [2], Atkinson and Peletier studied the location of the largest zero of certain solutions of the nonlinear equation

$$v'' + \frac{1}{t^4}\left(v^5 + v^q\right) = 0, \quad 1 \leq q < 5, \quad t > 0. \tag{1.13}$$

This equation arises in the study of radially symmetric solutions of the semilinear elliptic equation

$$\Delta u + u^5 + u^q = 0 \qquad \text{in } R^3. \tag{1.14}$$

Playing a key role in the work is the following lemma.

Let $v(t)$ be the solution of

$$v''(t) + \frac{1}{t^k} f(v(t)) = 0 \tag{1.15}$$

with the property that $\lim_{t\to\infty} v(t) = \gamma > 0$. If the nonlinear function $f(u)$ satisfies

$$uf'(u) \leq (2k-3)f(u), \tag{1.16}$$

then for $t > T$, the largest zero of $v(t)$,

$$v(t) \leq \left[\gamma^{2-k} + \frac{1}{k-1} t^{2-k} \gamma^{1-k} f(\gamma)\right]^{-1/(k-2)}. \tag{1.17}$$

This is in fact a comparison theorem because the righthand side of (1.17) is the solution of the equation

$$u''(t) + \frac{f(\gamma)}{t^k} u^{2k-3} = 0 \tag{1.18}$$

with the asymptotic condition

$$\lim_{t\to\infty} u(t) = \gamma. \tag{1.19}$$

These are the so-called Emden solutions of (1.18). Almost all of the other solutions of (1.18) do not have a simple closed form. Equation (1.18) is a special case of the classical Emden-Fowler equation made famous by Bellman's book [4] and Wong's survey paper [14]. More recently, the work of Brezis and Nirenberg [5] generated extensive interest in the perturbed equation (1.13) and (1.14). Results from their paper will be discussed in more detail in Section 3.

Inequality (1.17) was derived from the fact that the function $t^{k-1}v^{1-k}(t)v'(t)$ is decreasing in t, but this latter assertion had a tricky proof involving some clever use of a Pohozaev-type energy function.

I am grateful to Professor Atkinson and Professor Peletier for drawing my attention to their result. The present work is a direct response to their suggestion to investigate whether a more general comparison theorem is possible. In this paper the following main theorem is established. The concept of the "*uniqueness condition*" will be made precise in the next section. A reflection (replacing t by $-t$) has been executed, so that instead of requiring $u(t)$ and $v(t)$ to agree at some terminal "point", namely ∞, we make them agree at an initial point $t = a$.

Main Theorem *Let $u(t)$ be a positive solution of*

$$u''(t) + q(t)u^p(t) + r(t)u(t) = 0, \quad t \in (a, b) \tag{1.20}$$

where $q(t), r(t) \geq 0$, and $p > 1$. Furthermore, suppose that (1.20) *satisfies a* "uniqueness condition" *for boundary value problems on subintervals of (a, b). Let $k(t) \geq 1$ be any increasing function of t. Then the solution $v(t)$ of the equation*

$$v''(t) + k(t)q(t)v^p(t) + r(t)v(t) = 0, \tag{1.21}$$

satisfying the same initial conditions as those of $u(t)$ at $t = a$, is smaller than $u(t)$, before $v(t)$ vanishes for the first time.

This theorem includes the lemma of Atkinson and Peletier. The approach adopted is completely different from theirs. There are two major extensions. First, a linear term is included, and more general coefficients for the nonlinear term other than $1/t^k$ are allowed. Even in the particular case when this coefficient is a power of t, this power does not have to be tied to one special exponent of u as in the lemma in [2]. Second, the two solutions, $u(t)$ and $v(t)$, can be compared starting from any initial point $t = a$, finite or not.

The exponent $(2k - 3)$ that appears in (1.18) is the well-known (Sobolev) critical exponent for the Emden-Fowler equation. The dynamical behavior of the solutions changes radically as the exponent increases from the subcritical to the supercritical case. There is therefore much interest in attempting to extend the work of Atkinson and Peletier to include both the subcritical and supercritical cases. Our main theorem confirms that in these noncritical cases a lemma analogous to that of Atkinson and Peletier holds.

For equations with non-critical exponents, the solutions satisfying the asymptotic condition (1.19) (or even solutions of the critical exponent case satisfying other initial conditions) no longer have a simple closed form. As the proof of Atkinson and Peletier relies implicitly on such formulas, there seems to be no easy way to extend it to such cases.

In this paper, the main theorem is deduced from a special case, in which the function $k(t)$ is a constant. Our method makes extensive use of the Sturm comparison theorem and is closed related to a method first used by Coffman [6,7] to obtain uniqueness results for boundary value problems. His ideas have been successfully applied by Ni [11], Ni and Nussbaum [13], McLeod and Serrin [10], and Kwong [8]. For a survey of the method and known results, see the survey articles [9,12].

In Section 3 we show how the lemma of Atkinson and Peletier and its generalizations to non-critical exponents follow from the preceding theorem. An application to nonlinear elliptic equations involving subcritical exponents gives results analogous to those of Brezis and Nirenberg for the critical exponent case. These results have been obtained previously using the variational approach, see for example, Ambrosetti and Rabinowitz [1]. We expect that our main theorem will also play an important role in the detailed analysis of the structure of the solution space in the supercritical exponent case.

2 Main Results

We are interested in comparing the positive solutions of the Emden-Fowler equation

$$u''(t) + q(t)u^p(t) + r(t)u(t) = 0, \quad t \in (a,b) \tag{2.1}$$

with those of a similar one having a larger coefficient. We assume that $q(t)$ and $r(t)$ are piecewise continuous and

$$q(t) \geq 0,\ \ r(t) \geq 0,\ \ p > 1. \tag{2.2}$$

The interval (a,b) can be compact or otherwise, i.e., $-\infty \leq a < b \leq \infty$. However, for most of our discussion, it is assumed to be compact.

We may assume without loss of generality that $q(t)$ is not identically zero in any right neighborhood of the left endpoint $t = a$. In the contrary case, we can simply bypass such a neighborhood by shifting the left endpoint over it without affecting the validity of our main result. The requirement that $p > 1$ puts the equation (2.1) in the superlinear category. For such equations it is known that if the solution has either a sufficiently large initial height or a sufficiently large initial slope, it must have a zero close to the initial point a. This is one of the important facts used in the shooting method; see, for example, Bandle and Kwong [3].

As the example comprising (1.10) and (1.11) shows, the expected result is not true unless some extra conditions on the coefficient are imposed. The following simple lemma reveals the connection between the existence of a comparison result and the uniqueness of certain positive boundary value problems.

Lemma 1 *Suppose that $U(t)$ is a solution of (2.1) such that $U(b) = 0$. Let $v(t;\lambda)$, $\lambda \geq 1$, be the solution of*

$$v''(t;\lambda) + \lambda q(t)v^p(t;\lambda) + r(t)v(t;\lambda) = 0 \tag{2.3}$$

with the same initial conditions as those of $U(t)$,

$$v(a;\lambda) = U(a), \quad v'(a;\lambda) = U'(a). \tag{2.4}$$

Suppose it is true that for all $\lambda > \mu$,

$$v(t;\lambda) < v(t;\mu) \tag{2.5}$$

in the subinterval (a,B) in which both solutions are positive. Then for each $B \in (a,b)$, the boundary value problem (2.1) with boundary conditions

$$u'(a)/u(a) = U'(a)/U(a), \quad u(B) = 0, \tag{2.6}$$

and positivity requirement

$$u(t) > 0, \quad t \in (a, B), \tag{2.7}$$

has a unique solution.

Proof. To simplify the matter, let $\alpha = \lambda^{1/(p-1)}$. The functions $u(t;\alpha) = \alpha v(t;\lambda)$ are the only solutions of (2.1) that also satisfy the first boundary condition in (2.6). If the first zero of $v(t;\lambda)$ is at $t = B$, then $u(t;\alpha)$ is a solution of the boundary value problem. By assumption, the first zero of $v(t;\lambda)$ is a strictly decreasing function of λ. Hence no two of them can give rise to solutions of the boundary value problem with the same endpoint B. ∎

The first boundary condition in (2.6) is interpreted as the usual Dirichlet condition $u(a) = 0$ if it happens that $U(a) = 0$. Coffman [5] initiated the study of the uniqueness of ground-state boundary value problems via the first variational equation. The central question is to establish uniqueness of solutions to boundary value problems for the more general equation (1.7), under suitable conditions on the coefficient $q(t)$. We sketch the ideas below. A more detailed account of this method can be found in Ni [11] or Kwong [9].

Let $w(t;\alpha)$ be defined as

$$w(t;\alpha) = \frac{\partial u}{\partial \alpha}(t;\alpha), \tag{2.8}$$

and let $B = B(\alpha)$ denote the first zero of $u(t;\alpha)$. It is not difficult to see that the uniqueness of the boundary value problems (2.1) with (2.6) and (2.7), is implied by (indeed, is almost equivalent to) the fact that

$$w(B(\alpha), \alpha) < 0 \quad \text{for all } \alpha > 0. \tag{2.9}$$

The method continues with a careful analysis of the oscillatory behavior of the function $w(t;\alpha)$ for a fixed α. It satisfies the first variational equation

$$w''(t;\alpha) + pq(t)u^{p-1}(t;\alpha)w(t;\alpha) + r(t)w(t;\alpha) = 0, \tag{2.10}$$

and the initial condition given by the first identity of (2.6). The equation (2.10) can be viewed as a "linear" equation with $pq(t)u^{p-1}(t;\alpha)$ as its coefficient, and Sturm comparison techniques can be applied. It follows easily from the superlinearity nature of (2.1) that the equation (2.10) oscillates faster than (2.1). Hence $w(t;\alpha)$ has a zero before $u(t;\alpha)$ does. In other words, $w(t;\alpha)$ changes from positive to negative at some point prior to reaching $B(\alpha)$. The final step, which is the hardest in the whole method, is to show that under the given hypotheses on $q(t)$, $w(t;\alpha)$ does not change sign for a second time before $B(\alpha)$. This is done by constructing a suitable comparison function which can be shown, via the Sturm theorem again, to oscillate faster than $w(t;\alpha)$ but to have no zero at all in the interval in question.

Many useful uniqueness criteria have been established in this way. Although all of these have been proved only in the cases where the boundary condition at the left endpoint is either of the Dirchlet or of the Neumann type, the proofs work in the more general situation. In fact, any criterion established this way for the boundary value problems with two Dirichlet conditions works when the condition at $t = a$ is replaced by any boundary condition. Likewise, any criterion established for a Neumann condition at $t = a$ and a Dirichlet condition at $t = b$ works for any boundary condition at $t = a$ such that $u'(a)/u(a) \leq 0$.

Although the proof of our main result does not make use of the first variational equation, it is rooted in the Coffman method. For instance, the assertion in Lemma 3 that distinct solutions of (2.1) cannot intersect more than once is the global manifestation of the fact that $w(t;\alpha)$ does not change sign more than once in the appropriate interval.

Given a number $s \in (-\infty, \infty]$, we say that the equation (2.1) satisfies the "uniqueness condition" (U) with respect to s in the interval (a, b) if the following holds.

(U) Suppose first that b is finite. For any two points $A < B \in (a, b]$, and any number $\sigma \leq s$, there exists at most one nontrivial non-negative solution of (2.1) such that $u'(A)/u(A) = \sigma$ and $u(B) = 0$. In addition, for any $\beta > 0$, there exist at most two solutions of (2.1) such that $u'(A)/u(A) = \sigma$ and $u(b) = \beta$. If $b = \infty$, we require that for each finite subinterval the above condition is satisfied.

In practice, it is easier to establish property (U) via the first variational equation as in Coffman's method. Thus (U) is implied by the following property:

(W) For any point $A \in (a, b)$ and any solution of (2.1) such that $u'(A)/u(A) \leq s$, the corresponding solution $w(t)$ of the first variational equation (2.10) such that $w'(A)/w(A) = u'(A)/u(A)$, can change sign at most once before the first zero of $u(t)$, or b in case $u(t)$ does not vanish in (a, b).

Although the statement of conditions (U) and (W) involves solutions with an arbitrary initial point A and an arbitrary initial condition less than s, known uniqueness criteria that work for a particular s and the whole interval (a, b) usually work automatically for smaller s and subintervals. Thus, in practice, reference to these arbitrary numbers is not necessary.

Most known uniqueness criteria deal with the case $r(t) = 0$. When $q(t)$ is any power of t, positive or otherwise, boundary value problems of (2.1), with Dirichlet conditions at both endpoints, or with Dirichlet condition at one and Neumann condition at the other, are unique. This fact was first established by Coffman even though the idea can be traced back to earlier work of Fowler. It is not hard to see that the same proof shows that (2.1) satisfies (U) for any s and any interval. The only other uniqueness criterion available in the literature is a generic one that works for any superlinear equation, given in [9] as an improvement of an earlier result of Coffman. In the next lemma we summarize the known uniqueness criterion and a couple of new ones, the proof of which will appear elsewhere. It is interesting to find more classes of admissible coefficients made possible by the special structure of the nonlinear term $f(u) = u^p$.

LEMMA 2 *With $r(t) = 0$, the equation* (2.1) *satisfies condition* (U) *with respect to any s on (a, b), if one of the following conditions holds:*

1. *$a \geq 0$, and $q(t)$ is a power of t.*
2. $$(t-a)^2 q(t) \text{ is non-decreasing in } t \in (a,b), \tag{2.11}$$
   and
   $$(b-t)^2 q(t) \text{ is non-increasing in } t \in (a,b). \tag{2.12}$$
3. *Suppose $a \geq 0$. Then*
   $$t^{p+1} q(t) \text{ is non-increasing} \tag{2.13}$$
   and
   $$t^{p+3} q(t) \text{ is non-decreasing ;} \tag{2.14}$$
   in particular, if $q(t)$ is a non-negative linear combination of $1/t^k$ with $p+1 \leq k \leq p+3$:
   $$q(t) = \sum c_i t^{-k_i}, \quad c_i > 0, p+1 \leq k_i \leq p+3. \tag{2.15}$$
4. *$q(t)$ is a non-negative linear combination of $1/t^k$ with $p+3 \leq k \leq p+5$:*
   $$q(t) = \sum c_i t^{-k_i}, \quad c_i > 0, p+3 \leq k_i \leq p+5. \tag{2.16}$$
5. *For some number $\delta \geq 0$, $q(t)$ is a non-negative linear combination of powers t^k with $\delta \leq k \leq (p+1)(\delta+2)/p - 1$:*
   $$q(t) = \sum c_i t^{k_i}, \quad c_i > 0, \delta \leq k \leq (p+1)(\delta+2)/p - 1. \tag{2.17}$$

If only (2.12) *holds, then* (2.1) *satisfies* (U) *with respect to any $s \leq 0$.*

A crucial step in obtaining our main theorem is to prove a converse of Lemma 1. We first establish a consequence of condition (U).

LEMMA 3 *Suppose that* (2.1) *satisfies* (U). *Let* $u_1(t)$ *and* $u_2(t)$ *be two distinct positive solutions of* (2.1) *on* $(a, B) \subset (a, b)$, *satisfying the same initial condition*

$$u_1'(a)/u_1(a) = u_2'(a)/u_2(a) \le s. \tag{2.18}$$

They cannot intersect more than once in $(a, B]$.

Proof. The proof consists of a continuity argument making use of elementary topological properties of the plane. We do not insist on absolute rigor while presenting the proof. Theoretical details can be easily filled in.

Let us first look at the case in which B is a zero of one of the solutions, say $u_1(B) = 0$. Then $u_2(B) > 0$ because by (U) there cannot be two solutions that have the same boundary condition at B. At $t = a$, either $u_2(a) > u_1(a)$ or $u_2(a) < u_1(a)$. We use a shooting method argument to show that the first case is vacuous. By keeping the ratio $u'(a)/u(a) = s$, and increasing $u(a)$ starting from $u_2(a)$, we can shoot out various solutions. In case the ratio $u_1(a) = u_2(a) = 0$ is zero, the solutions $u(t)$ starts out with $u(a) = 0$, but with progressively increasing initial slope $u'(a)$. In the following we will not point out this modification explicitly. The value $u(B)$ depends continuously on the initial height. By (U), $u(B)$ cannot vanish, so it must remain positive for all initial height $u(a)$. On the other hand, superlinearity implies that for $u(a)$ large enough, the solution must have a zero in (a, B). Pick the first initial height at which this happens. Because $u(B) > 0$, the solution can be tangent to the t-axis only at this zero, but this is impossible. It follows that the first case, $u_2(a) > u_1(a)$ is empty, as claimed.

In the second case, the two solutions must intersect. Suppose they do so more than once. We shoot out solutions as before but with progressively decreasing initial height. Since $u(B)$ remains positive and two solutions of (2.1) cannot be tangential at any point, the number of points of intersection of $u(t)$ with $u_1(t)$ has to be a constant; in particular, it is greater than one. It is geometrically obvious that as $u(a)$ decreases towards 0, the first intersection point of $u(t)$ with $u_1(t)$ approaches the endpoint B. In other words, if $u(a)$ is sufficiently small, all the intersections occur within a very small right neighborhood of B. Let $W(t) = u(t) - u_1(t)$. Then $W(t)$ changes sign (oscillates) more than once in this neighborhood. The function satisfies the second-order "linear" differential equation

$$W''(t) + \left[\frac{q(t)[u^p(t) - u_1^p(t)]}{u(t) - u_1(t)} + r(t)\right] W(t) = 0. \tag{2.19}$$

Observe that the "coefficient" in this equation, the expression enclosed in large square brackets, is a bounded function in this neighborhood, and it is well known that solutions of such equations cannot oscillate in an arbitrarily small interval. We therefore have a contradiction.

Now suppose that $B = b$ and that both $u_1(b)$ and $u_2(b)$ are positive. We may assume that $u_1(a) < u_2(a)$. We first consider the case where $u_1(b) \leq u_2(b)$. If the two solutions do not intersect, we have nothing to prove. So suppose they do. As a consequence, part of the graph of $u_2(t)$ lies below that of $u_1(t)$. We will show that there are two other solutions that satisfy the same boundary conditions as those of $u_1(t)$, thus contradicting (U). If $u_1(b) = u_2(b)$, the first solution is simply $u_2(t)$. In the contrary case, the first solution is obtained by shooting with initial height above $u_2(a)$. As pointed out before, if $u(a)$ is large enough, the solution must have a zero near the left endpoint. So as $u(a)$ increases from $u_2(a)$, the other end of the curve $u(b)$ must eventually come down and pass through $u_1(b)$, giving the first solution. Next, since both $u_1(t)$ and $u_2(t)$ are bounded away from zero, we see that if we shoot with sufficiently small initial height, the solution will remain small and so will not cross either function. As we gradually increase the initial height, there must be a first time when the solution $u(t)$ intersects one of these given solutions. By the choice of this critical case, the graphs of $u_1(t)$ and $u_2(t)$ must lie entirely above that of $u(t)$. This rules out the possibility that $u(t)$ coincides with $u_1(t)$ since, by assumption, part of the graph of $u_2(t)$ lies below that of the $u_1(t)$. If a point of intersection of $u(t)$ with the other two functions is an interior point of the interval (a, b), then $u(t)$ must be tangential to the solution that it intersects at this point. This contradicts the uniqueness of initial value problems. So the only possibility left is that $u(t)$ intersects $u_1(t)$ at b, giving the second solution we need.

Now consider the case $u_1(b) > u_2(b)$. If the two solutions intersect more than once, they must do so at least three times (recall that solutions cannot be tangent to each other). Let us shoot out solutions as before with increasing initial heights starting from $u_2(a)$, and follow their terminal values $u(b)$. We know that eventually $u(b)$ must hit the t-axis, but before it does so, it may or may not pass through the point $u_1(b)$. Suppose it does. At the moment when this first happens, we are back to the previous case. Note that the number of times $u(t)$ intersects $u_1(t)$ remains a constant during this continuous deformation of $u(t)$ (as $u(a)$ is increased). Suppose that $u(b)$ does not pass through $u_1(b)$ before it hits the t-axis. At the moment when $u(b)$ first becomes zero, we have the very first situation at the beginning of the proof. In all cases, we have a contradiction, and so the proof is complete. ∎

We are now ready to show that the converse of Lemma 1 holds.

LEMMA 4 *Suppose that* (2.1) *satisfies* (U) *with respect to* s. *Let* $U(t)$ *be a positive solution of* (2.1) *in* (a, b) *such that* $U'(a)/U(a) = s$. *Then for any* $\lambda > 1$, *the solution* $v(t;\lambda)$ *of* (2.3) *and* (2.4) *satisfies*

$$v(t;\lambda) \leq U(t) \tag{2.20}$$

for all t *before the first zero of* $v(t)$.

Proof. As in the proof of Lemma 1, we define $u(t;\alpha) = \alpha v(t;\lambda)$, $\alpha = \lambda^{1/(p-1)}$, by scaling the function $v(t;\lambda)$. Let B be the first zero of $v(t;\lambda)$ or b if $v(t;\lambda)$ does not vanish. If $u(t;\alpha)$ intersects $U(t)$ in $(a, B]$, it does so at a unique point, by Lemma 3. Denote this point by C. If the two graphs do not intersect, we take $C = B$. Since $\lambda > 1$, the scaling is a stretching and $u(a;\alpha) > U(a)$. It follows that in (C, B), $v(t;\lambda) < u(t;\alpha) < U(t)$. It remains to show that $v(t;\lambda) < U(t)$ in (a, C) as well.

Let $z(t) = U(t) - v(t;\lambda)$. It satisfies the differential equation

$$z''(t) + q(t)\left[\frac{U^p(t) - v^p(t;\lambda)}{U(t) - v(t;\lambda)}\right] z(t) + r(t)z(t) = (\lambda - 1)q(t)v(t,\lambda). \tag{2.21}$$

It also satisfies the initial conditions

$$z(a) = z'(a) = 0. \tag{2.22}$$

Note that the righthand side of equation (2.21) is positive in (a, C). Using the variation of parameter formula it is easy to see that in some neighborhood of the endpoint $t = a$, $z(t)$ is positive. If positivity prevails throughout (a, C), the proof is complete. So suppose the contrary, and let $D < C$ be the first point at which $z(D) = 0$. We compare equation (2.21) with the one satisfied by $w(t) = u(t;\alpha) - U(t)$,

$$w''(t) + q(t)\left[\frac{u^p(t;\alpha) - U^p(t)}{u(t;\alpha) - U(t)}\right] w(t) + r(t)w(t) = 0. \tag{2.23}$$

In (a, D), $v(t;\lambda) \leq U(t) \leq u(t;\alpha)$. This implies that the expression inside the square brackets in (2.23) is larger than the corresponding expression in (2.21). Rewriting (2.21) in homogeneous form:

$$z''(t) + q(t)\left[\frac{U^p(t) - v^p(t;\lambda)}{U(t) - v(t;\lambda)} - \frac{(\lambda - 1)q(t)v(t,\lambda)}{z(t)}\right] z(t) + r(t)z(t) = 0, \tag{2.24}$$

we see that it has a coefficient smaller than that of (2.23). Hence $w(t)$ oscillates faster than $z(t)$ in $[a, D]$. This contradicts the fact that $w(t)$ has no zero in

(a, D) whereas $z(t)$ vanishes at both endpoints a and D. The proof of the lemma is now complete. ∎

So far we have been considering only a finite left endpoint $a \geq -\infty$. When $a = -\infty$, some technical details have to be added. Since we are comparing only essentially positive solutions, we can include only those equations (2.1) that admit a nonoscillatory solution near $-\infty$. Because of the concavity of $u(t)$, $u'(t)$ approaches a finite non-negative limit as $t \to -\infty$. The value $\lim_{t\to-\infty} u(t)$ may or may not be bounded. In any case, the boundary conditions (2.4) have to be interpreted as asymptotic relations. We can get around this difficulty by requiring that the solution v of (2.3), or of (2.25) below, can be approximated by solutions satisfying (2.4) for finite values of a as we let $a \to -\infty$.

THEOREM 1 *Let $u(t)$ be a positive solution of* (2.1) *in (a, b), $-\infty < a < b \leq \infty$, where $q(t) \geq 0$, and $p > 1$, and suppose that condition* (U) *with respect to $s = u'(a)/u(a)$ is satisfied. Let $k(t) \geq 1$ be any increasing function of t. Then the solution $v(t)$ of the equation*

$$v''(t) + k(t)q(t)v^p(t) + r(t)v(t) = 0, \tag{2.25}$$

such that

$$v(a) = u(a), \quad v'(a) = u'(a), \tag{2.26}$$

satisfies

$$v(t) \leq u(t), \quad t \in (a, B), \tag{2.27}$$

where B is the first zero of $v(t)$ or b if $v(t)$ does not vanish.

Now if $u(t)$ is a positive solution of (2.1) *on $(-\infty, b)$, and if the solution $v_a(t)$ of* (2.25) *satisfying* (2.26) *for some finite $a < b$ converges uniformly to a solution $v(t)$ of* (2.25) *as $a \to -\infty$, then* (2.27) *holds in $(-\infty, B)$.*

Proof. The result for an unbounded interval follows from that for compact intervals using a continuity argument. It also suffices to assume that $k(t)$ is a step function. The general case is obtained by taking limits. The proof of the theorem in the reduced case is done most easily by induction on the number of steps of $k(t)$. By leveling the last step of $k(t)$ with the previous one, we obtain a function $k_n(t)$ with one less step. The original $k(t)$ can easily be recovered as the product of $k_n(t)$ and a two-step function. Let $v_n(t)$ be the solution of the equation of the form (2.25) with $k(t)$ replaced by $k_n(t)$. By applying Lemma 4 to the interval of the last step of $k(t)$, we see that $v(t) \leq v_n(t)$. By the induction hypothesis, $v_n(t) \leq u(t)$. So the conclusion of the theorem follows. ∎

We remark that Theorem 1 is no longer true if the monotonicity hypothesis on $k(t)$ is simply removed. Numerical experimentation quickly yields the following example:

$$k(t) = \begin{cases} 2 & t \leq 1 \\ 1 & t > 1 \end{cases}, \tag{2.28}$$

$$u''(t) + u^3(t) = 0, \quad t \in (0,4), \tag{2.29}$$

$$v''(t) + k(t)v^3(t) = 0, \quad t \in (0,4), \tag{2.30}$$

$$u(0) = v(0) = 0, \quad u'(0) = v'(0) = 0.5. \tag{2.31}$$

The solution $v(t)$ intersects $u(t)$ at approximately $t = 3.143$ and remains larger than $u(t)$ after that.

3 The Atkinson and Peletier Lemma and Subcritical Equations

The following simple consequence of Theorem 1 covers the lemma of Atkinson and Peletier. To see this, we first have to make a reflection, replacing t by $-t$, and then choose $a = -\infty$, $q(t) = t^{-k}$ and $r(t) = 0$. Their condition (1.16) on the function $f(u)$ is restated in an equivalent form which makes the proof more transparent. A differentiation shows that (1.16) holds if and only if $f(u)/u^{2k-3}$ is non-increasing.

THEOREM 2 *Let $p > 1$ and $f(u)$ be a C^1 function for $u > 0$ satisfying the condition*

$$\frac{f(u)}{u^p} \text{ is a non-increasing function of } u. \tag{3.1}$$

Suppose that (2.1) satisfies the uniqueness condition (U) and $u(t)$ is a positive solution of (2.1) in (a, b) such that

$$u'(a) \leq 0. \tag{3.2}$$

Then the solution of the initial value problem

$$v''(t) + q(t)f(v(t)) + r(t)v(t) = 0, \tag{3.3}$$

$$v(a) = u(a), \quad v'(a) = u'(a), \tag{3.4}$$

satisfies the inequality

$$v(t) \leq u(t), \tag{3.5}$$

before $v(t)$ changes sign.

Proof. The boundary condition (3.2) implies that $v(t)$ is decreasing in t in the interval (a, B). By (3.1), the function $f(v(t))/v^p(t)$ is therefore an increasing function of t in (a, B). Rewriting (3.3) as

$$v''(t) + \frac{f(v(t))}{v^p(t)} q(t)v^p(t) + r(t)u(t) = 0, \tag{3.6}$$

we bring it into the form of (2.25), with $k(t) = f(v(t))/v^p(t)$. Theorem 1 now applies. ∎

As pointed out in [2], all functions of the form $f(u) = u^p + \lambda u^q$, or more generally $f(u) = \sum \lambda_i u^{q_i}$, with $0 < q, q_i < p, \lambda, \lambda_i > 0$ satisfy (3.1).

Let Ω denote the unit ball in R^n. Brezis and Nirenberg in [5] studied the nonlinear eigenvalue problem

$$\Delta u + u^p + \lambda u^q = 0 \quad \text{in } \Omega, \tag{3.7}$$

with Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega, \tag{3.8}$$

where $1 < q < p = \dfrac{n+2}{n-2}$. They obtained, using variational techniques, necessary and sufficient conditions on the value of λ for the existence of a solution. What is most interesting is that the necessary and sufficient range depends on the value of q as well as on n. Two or three distinct cases can be distinguished according to whether $n \geq 4$ or $n = 3$. For all values of n, the cutoff value of q for the first case is $n/(n-2)$, whereas for $n = 3$, the value $q = 1$ is in a separate category by itself.

An alternative approach using ordinary differential equation methods was adopted in [2] to reconfirm and to refine the results of Brezis and Nirenberg. It is known that any solution of the eigenvalue problem must be radially symmetric. Thus we are really dealing with an ordinary differential equation. A scaling in the independent variable further changes the problem to the equivalent one of studying the location of the first zero of the solutions of the initial value problem

$$u''(t) + \frac{n-1}{t}u'(t) + u^p(t) + u^q(t) = 0, \tag{3.9}$$

$$u(0) = \alpha, \quad u'(0) = 0. \tag{3.10}$$

As the initial height α varies from 0 to ∞, the first zero is tracked; the range of possible locations is related to the range of possible eigenvalues of (3.7).

We remark that the method presented below works without change for the more general equation

$$\Delta u + u^p + \sum_{i=1}^{N} c_i u_i^q + \lambda u^q = 0 \quad \text{in } \Omega, \tag{3.11}$$

where $c_i > 0$, and $q < q_i < p$, and the equivalent scaled equation

$$u''(t) + \frac{n-1}{t}u'(t) + u^p(t) + \sum_{i=1}^{N} C_i u_i^q + u^q(t) = 0. \tag{3.12}$$

It is natural to ask what the corresponding result of Brezis and Nirenberg is in cases where the main exponent p is non-critical. The variational approach has been successfully applied to the subcritical case to answer this question. See, for example, the work of Ambrosetti and Rabinowitz [1]. The shooting method of Atkinson and Peletier provides an alternative. In fact, in the case of p being subcritical, the extension of their key lemma as given by Theorem 2 is all we need. As our next result shows, the corresponding Brezis and Nirenberg result is simpler, comprising always two cases, $q = 1$ and $q > 1$.

THEOREM 3 *Suppose* $1 \le q < p < \dfrac{n+2}{n-2}$. *If* $q > 1$, *the set of the first zero of all solutions of* (3.9) (*more generally,* (3.9′)) *and* (3.10) *is* $(0, \infty)$. *Equivalently, the eigenvalue problem* (3.7) (*more generally,* (3.7′)) *and* (3.8) *has a solution for all* $\lambda > 0$.

If $q = 1$, *the set of the first zero of all solutions of* (3.9) (*more generally,* (3.9′)) *and* (3.10) *is* $(0, T)$ *where* T *is the first zero of the solution of the linear initial value problem*

$$\phi''(t) + \frac{n-1}{t}\phi'(t) + \phi(t) = 0, \tag{3.13}$$

$$\phi(0) = 1, \quad \phi'(0) = 0. \tag{3.14}$$

Equivalently, the eigenvalue problem (3.7) (*or* (3.7′)) *and* (3.8) *has a solution for all* $\lambda \in (0, T^2)$.

Proof. To emphasize the fact that the solution to (3.9) and (3.10) depends on the initial shooting height α, we write it as $u(t;\alpha)$. Its first zero is therefore also a function of α; we denote it by $b(\alpha)$. Let us first show that in all cases

$$\lim_{\alpha\to\infty} b(\alpha) = 0. \tag{3.15}$$

Let $U(t;\alpha)$ denote the solution of the initial value problem

$$U''(t;\alpha) + \frac{n-1}{t}U'(t;\alpha) + U^p(t;\alpha) = 0, \tag{3.16}$$

$$U(0) = \alpha, \quad U'(0) = 0, \tag{3.17}$$

and $B(\alpha)$ the first zero of $U(t;\alpha)$. Using the well-known Emden transform, we can rewrite (3.9) and (3.16) in a form similar to (1.13), for which we can apply Theorem 2 to conclude that

$$u(t;\alpha) \le U(t,\alpha) \quad \text{for all } t \in (0, b(\alpha)). \tag{3.18}$$

As a consequence,

$$b(\alpha) \le B(\alpha). \tag{3.19}$$

Incidentally this establishes the fact that all solutions of (3.9) and (3.10) must have a finite zero; in other words $b(\alpha) < \infty$. Now (3.15) follows if we can show that

$$\lim_{\alpha \to \infty} B(\alpha) = 0. \tag{3.20}$$

This is a well-known fact, since $U(t;\alpha)$ can be obtained from the special case $U(t;1)$ by scaling, namely,

$$U(t;\alpha) = \alpha U(\alpha^{(p-1)/2} t; 1), \tag{3.21}$$

and

$$B(\alpha) = \frac{B(1)}{\alpha^{(p-1)/2}}. \tag{3.22}$$

Next let us show that for $q > 1$,

$$\lim_{\alpha \to 0} b(\alpha) = \infty. \tag{3.23}$$

We exploit the method of scaling again. Define

$$v(t;\alpha) = \frac{1}{\alpha} u \left(\frac{t}{\alpha^{(q-1)/2}}; \alpha \right). \tag{3.24}$$

Then $v(t;\alpha)$ satisfies the initial value problem

$$v''(t;\alpha) + \frac{n-1}{t} v'(t;\alpha) + \alpha^{p-q} v^p(t;\alpha) + v^q(t;\alpha) = 0, \tag{3.25}$$

$$v(0) = 1, \quad v'(0) = 0. \tag{3.26}$$

Note that as $\alpha \to 0$, the coefficient of the term v^p in (3.25) goes to zero. Thus by continuity, $v(t;\alpha)$ converges uniformly to $U(t;1)$ in any finite interval. The first zero of $v(t;\alpha)$ therefore approaches $B(1)$. It follows that $b(\alpha)$, being the first zero of $v(t;\alpha)$ divided by $\alpha^{(q-1)/2}$, approaches ∞ as $\alpha \to 0$.

By (3.15) and (3.23), the set of $b(\alpha)$ contains arbitrarily large and arbitrarily small positive values. By connectedness, the set must therefore be $(0, \infty)$.

Now let us look at the case $q = 1$. In view of (3.15), it remain to show that $b(\alpha)$ is always less than T, but can be arbitrarily close to T. Writing (3.9) in the form of a "linear" equation:

$$u''(t) + \frac{n-1}{t} u'(t) + \left[u^{p-1}(t) + 1 \right] u(t) = 0, \tag{3.27}$$

we see that it oscillates more than (3.16), since the coefficient of the last term in (3.27) is larger than the corresponding coefficient of the last term in (3.16). Hence the first zero of $u(t;\alpha)$ is strictly less than the first zero of $U(t)$; in other words, $b(\alpha) < T$. On the other hand, $u(t;\alpha) \leq \alpha$ in $(0, b(\alpha))$ so that if α is very small, the first term of the expression inside the square brackets in (3.27) is very small, say less than some $\epsilon > 0$. Hence it oscillates slower than the equation

$$w''(t) + \frac{n-1}{t} w'(t) + [\epsilon + 1]\, w(t) = 0, \tag{3.28}$$

implying that $b(\alpha)$ is larger than the first zero of $w(t)$ (which is assumed to satisfy the initial conditions $w(0) = 1$ and $w'(0) = 0$). Since the first zero of $w(t)$ tends to T as $\epsilon \to 0$, so does $b(\alpha)$. This completes the proof of the theorem. ∎

After establishing existence, the next natural question to ask is how many solutions there are for each given λ or $b(\alpha)$. Results of Ni [11] and Ni and Nussbaum [13] imply uniqueness when $1 \leq q < p < \dfrac{n}{n-2}$. The situation is more complicated for larger values of p. So far most of the knowledge is derived from numerical computation. The number of solutions can be read from the graph plotting α against λ or $b(\alpha)$, the so-called bifurcation diagram. Atkinson and Peletier [2] proved that when $2 < n < 4$, p is critical, and $1 < q < (6-n)/(n-2)$, there are at least two solutions for each large $b(\alpha)$. The bifurcation curve looks like one branch of a hyperbola with the horizontal axis as one of its asymptotes, and the other end of the curve runs off to infinity at the top right-hand corner. Theorem 3 shows that if p is decreased from the critical value, this end of the curve will approach the vertical axis $b = 0$ instead. Hence if p is sufficiently close to the critical value, the bifurcation curve will maintain the hyperbolic shape of the curve for the critical exponent case within a bounded region, but the upper portion of the curve will, for large α, be bent back towards the vertical axis. This creates (at least) a doublefold curve. It follows that there are values of $b(\alpha)$ or λ for which there are at least three distinct solutions.

A similar phenomenon was observed by Ni and Nussbaum [13] when p is supercritical and q is subcritical. As p is further reduced, numerical evidence indicates that the doublefold curve gradually unfolds; below a certain threshold, depending on q, the bifurcation curve becomes strictly monotone, and uniqueness for the boundary value problems for all $b(\alpha)$ is regained. It will be interesting to see whether these facts can be verified theoretically and the threshold value can be determined.

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