THE GINZBURG–LANDAU EQUATIONS OF SUPERCONDUCTIVITY IN THE LIMIT OF WEAK COUPLING NEAR THE UPPER CRITICAL FIELD

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Abstract. This article is concerned with the Ginzburg–Landau (GL) equations of superconductivity. The equations provide a mathematical model for the study of magnetic flux vortices in superconductors. The focus is on the asymptotic case when the charge of the superconducting charge carriers (Cooper pairs) is vanishingly small and the applied magnetic field approaches the upper critical field. It is shown that the GL model reduces in the limit to the frozen-field model, where the superconducting phenomena are affected by the electromagnetic phenomena, but not vice versa. The convergence is second order in the small parameter. The analytical results are confirmed in some numerical examples.

1 Introduction

Superconducting materials hold great promise for technological applications. Especially since the discovery of the so-called high-temperature superconductors in the 1980s, much research has been devoted to understanding the behavior of these new materials. While conventional superconductors require liquid helium (3–4 degrees Kelvin) to remain in the superconducting state, high-temperature superconductors can be cooled with liquid nitrogen (76 degrees Kelvin)—a clear economic advantage. Unfortunately, high-temperature superconductors are ceramic materials, which are difficult to manufacture into films and wires, but progress is being made all the time.

High-temperature superconductors belong to the class of type-II superconductors. Unlike type-I superconductors, type-II superconductors can sustain magnetic flux in their interior, but the magnetic flux is restricted to quantized amounts—filaments that are encircled by a current. The current shields the magnetic flux from the bulk, which is perfectly superconducting. The configuration resembles that of a vortex in a fluid, and the superconductor is said to be in the *vortex state*. The

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vortices, and especially their dynamics, determine the current-carrying capabilities of the superconductor.

Vortices can be studied at various levels of detail. The most detailed description is given by the Ginzburg–Landau (GL) model which, although phenomenological and not based on any microscopic quantum-mechanical theory, can be used to study both the dynamics and the structure of vortex systems in realistic superconductor configurations [1, 2]. Numerical simulations based on the GL model are, however, extremely time consuming, and it is desirable to use simpler models whenever possible.

In this article, we consider the GL model in the limit when the charge of the superconducting charge carriers (Cooper pairs) goes to zero and the applied magnetic field approaches the upper critical field. The upper critical field marks the transition from the superconducting to the normal state: When the applied field is below the upper critical field, a type-II superconductor admits vortex solutions; above the upper critical field, it loses all superconducting properties and becomes a normal metal. The assumption that the charge of the Cooper pairs is small leads to a weak coupling between the order parameter and the electromagnetic field.

We prove that the GL model reduces in the limit to the "frozen-field model," where the superconducting phenomena are determined by the magnetic field, but not vice versa. The equation for the order parameter is the same as in the GL model (except that the field is now prescribed) and admits vortex solutions as before. We also prove that the convergence rate is second order in the small parameter (the charge of a Cooper pair or, in dimensionless units, the inverse of the Ginzburg–Landau parameter) in a suitable Sobolev space norm. This convergence rate is confirmed in some numerical examples.

The frozen-field model has been used successfully for numerical simulations of vortex systems [3]. Our analysis shows how the model is obtained as an asymptotic limit of the GL model, circumscribes the domain of its validity, and suggests a systematic procedure for the derivation of higher-order approximations.

For more background on the physics of superconductivity we refer the reader to the monograph by Tinkham [4]. The original source for the Ginzburg–Landau equations of superconductivity is [5]. A good introduction to the mathematics of the Ginzburg–Landau equations was given by Du, Gunzburger, and Peterson [6]. The dynamics of the Ginzburg–Landau equations have been studied by several authors; we mention the articles by Du [7], Tang and Wang [8], and Fleckinger-Pellé et al. [9], where further references can be found. The article by Du and Gray [10] is closely related to the present investigation.

Section 2 contains an introduction to the Ginzburg–Landau equations, Section 3 the analysis (with a summary in Section 3.5), and Section 4 the numerics.

2 Ginzburg-Landau Equations

In the Ginzburg–Landau theory of superconductivity, the state of a superconducting medium is described by a complex scalar-valued *order parameter* ψ and a real vector-valued *vector potential* $\boldsymbol{A}$. The thermodynamic properties of the superconductor are determined by the Gibbs free energy G, which is defined by the relation

$$\mathrm{e}^{-(1/kT)G(T,\boldsymbol{H})} = \int \mathrm{e}^{-(1/kT)\int(\mathcal{F}[\psi,\psi^*,\boldsymbol{A}]-(1/4\pi)(\nabla\times\boldsymbol{A})\cdot\boldsymbol{H})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}\,\mathcal{D}\psi\mathcal{D}\psi^*\mathcal{D}\boldsymbol{A}. \tag{2.1}$$

Here, k is Boltzmann's constant, T the temperature, and $\boldsymbol{H}$ the applied magnetic field; $\mathcal{F}$ is the free-energy density, and the integral is a functional integral over all admissible states $(\psi, \boldsymbol{A})$. The free energy is the sum of the kinetic energy, the condensation energy, and the field energy; its density is given by the expression

$$\mathcal{F}[\psi,\psi^*,\boldsymbol{A}] = \frac{1}{2m_s}\left|\left(\frac{\hbar}{i}\nabla - \frac{q_s}{c}\boldsymbol{A}\right)\psi\right|^2 + \left(\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4\right) + \frac{1}{8\pi}|\nabla\times\boldsymbol{A}|^2. \tag{2.2}$$

The constants m_s and q_s are the mass and charge, respectively, of a Cooper pair (the superconducting charge carriers, also referred to as superelectrons); c is the speed of light; and $\hbar$ is Planck's constant divided by 2π. A Cooper pair is made up of two electrons, each with charge $-e$ (e is the elementary charge); hence, q_s is negative, $q_s = -|q_s|$. The parameters α and β are material parameters; α changes sign at the critical temperature T_c, $\alpha(T) < 0$ for $T < T_c$ (superconducting state) and $\alpha(T) > 0$ for $T > T_c$ (normal state); β is only weakly temperature dependent and positive for all T. These parameters are defined phenomenologically, but they can be expressed in terms of measurable quantities, such as the superconducting *coherence length* ξ and the London *penetration depth* λ,

$$\xi = \left(\frac{\hbar^2}{2m_s|\alpha|}\right)^{1/2}, \quad \lambda = \left(\frac{m_s c^2\beta}{4\pi q_s^2|\alpha|}\right)^{1/2}. \tag{2.3}$$

The coherence length and the London penetration depth define the respective characteristic length scales for the order parameter and the magnetic induction. Both depend on the temperature T and diverge as T approaches the critical temperature T_c, because of the factor $|\alpha|^{-1/2}$. However, their ratio is, to a good approximation, independent of temperature. This ratio is the *Ginzburg-Landau parameter*,

$$\kappa = \lambda/\xi. \tag{2.4}$$

In high-T_c superconductors, κ is of the order of 50–100.

As long as thermal fluctuations can be ignored, the equilibrium state at a point $(T, \boldsymbol{H})$ in the phase plane is found by minimizing the expression

$$\mathcal{F}[\psi, \psi^*, \boldsymbol{A}] - (1/4\pi)(\nabla \times \boldsymbol{A}) \cdot \boldsymbol{H} \tag{2.5}$$

with respect to ψ (or its complex conjugate, ψ^*) and $\boldsymbol{A}$. Thus we obtain the Ginzburg–Landau (GL) equations,

$$\frac{1}{2m_s}\left(\frac{\hbar}{i}\nabla - \frac{q_s}{c}\boldsymbol{A}\right)^2 \psi + \alpha\psi + \beta|\psi|^2\psi = 0, \tag{2.6}$$

$$-\frac{c}{4\pi}\nabla \times \nabla \times \boldsymbol{A} + \boldsymbol{J}_s + \frac{c}{4\pi}\nabla \times \boldsymbol{H} = 0, \tag{2.7}$$

where the supercurrent density, $\boldsymbol{J}_s$, is given by

$$\boldsymbol{J}_s = \frac{q_s\hbar}{2im_s}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{q_s^2}{m_s c}|\psi|^2\boldsymbol{A} = \frac{q_s}{m_s}\Re\left[\psi^*\left(\frac{\hbar}{i}\nabla - \frac{q_s}{c}\boldsymbol{A}\right)\psi\right]. \tag{2.8}$$

Notice that Eq. (2.7) is Ampère's law, $\boldsymbol{J} = (c/4\pi)\nabla \times \boldsymbol{B}$, where $\boldsymbol{B}$ is the magnetic induction, $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, and the current $\boldsymbol{J}$ is the sum of the supercurrent $\boldsymbol{J}_s$ and the transport current $\boldsymbol{J}_t = \nabla \times \boldsymbol{H}$. The natural boundary conditions are

$$\boldsymbol{n} \cdot \boldsymbol{J}_s = 0, \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A}) = \boldsymbol{n} \times \boldsymbol{H}. \tag{2.9}$$

They express the fact that superelectrons cannot leave the superconductor, while the tangential components of the magnetic field must be continuous across the boundary (if no surface currents are present there).

The GL equations are invariant under a *gauge transformation*

$$G_\chi : (\psi, \boldsymbol{A}) \mapsto \left(\psi e^{i(q_s/\hbar c)\chi}, \boldsymbol{A} + \nabla\chi\right). \tag{2.10}$$

The *gauge* χ can be any sufficiently smooth function of position. This gauge invariance does not affect the physically measurable quantities (the magnetic induction $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ or the magnetization $\boldsymbol{M} = \boldsymbol{B} - \boldsymbol{H}$ and the current density $\boldsymbol{J}$), but implies that the solution of the GL equations is not unique. Uniqueness requires an additional constraint, which is imposed through a gauge choice. The choice is, in principle, arbitrary; a common choice is the London gauge, where $\boldsymbol{A}$ is divergence free everywhere in the superconductor and tangential at the boundary.

2.1 Time-Dependent GL Equations

The time-dependent Ginzburg–Landau (TDGL) equations describe how a superconductor relaxes to the ground state. Because gauge invariance needs to be maintained, the TDGL equations are nontrivial generalizations of the GL equations. That is, the TDGL equations cannot be obtained from the GL equations simply by replacing 0 in the right-hand side of Eqs. (2.6) and (2.7) by the time derivatives of ψ and $\boldsymbol{A}$, respectively.

When the state of a superconductor varies with time, we must deal with the full electromagnetic field, not just the magnetic field. This necessitates the introduction of a third state variable, in addition to the order parameter and the vector potential, namely, the *electric potential* ϕ. The electromagnetic variables—the magnetic induction $\boldsymbol{B}$, the current density $\boldsymbol{J}$, and the electric field $\boldsymbol{E}$—are given in terms of $\boldsymbol{A}$ and ϕ by the expressions

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{J} = \frac{c}{4\pi}\nabla \times \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi. \tag{2.11}$$

The electric field represents a measurable quantity that, like the magnetic induction and the current, must be gauge invariant in time. The proper generalization of the gauge transformation (2.10) to the time-dependent domain is therefore

$$G_\chi : (\psi, \boldsymbol{A}, \phi) \mapsto \left(\psi e^{i(q_s/\hbar c)\chi}, \boldsymbol{A} + \nabla\chi, \phi - \frac{1}{c}\frac{\partial \chi}{\partial t}\right), \tag{2.12}$$

and the TDGL equations are

$$\gamma\hbar\left(\frac{\partial}{\partial t} + i\frac{q_s}{\hbar}\phi\right)\psi + \frac{1}{2m_s}\left(\frac{\hbar}{i}\nabla - \frac{q_s}{c}\boldsymbol{A}\right)^2\psi + \alpha\psi + \beta|\psi|^2\psi = 0, \tag{2.13}$$

$$\sigma\left(-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi\right) - \frac{c}{4\pi}\nabla \times \nabla \times \boldsymbol{A} + \boldsymbol{J}_s + \frac{c}{4\pi}\nabla \times \boldsymbol{H} = 0, \tag{2.14}$$

where $\boldsymbol{J}_s$ is again given by Eq. (2.8). We assume that the applied field $\boldsymbol{H}$ is stationary. The TDGL equations require the specification of two additional material parameters, the normal state conductivity σ and the mobility coefficient γ. The latter is dimensionless and related to the diffusion coefficient D, $\gamma = \hbar/2m_sD$.

If we interpret Eq. (2.14) again as Ampère's law, we see that the current is now made up of three parts: the supercurrent $\boldsymbol{J}_s$, the transport current $\boldsymbol{J}_t$, and a normal current $\boldsymbol{J}_n$ given by $\boldsymbol{J}_n = \sigma\boldsymbol{E}$ (Ohm's law). Hence, we are using a quasistatic version of Maxwell's equations, where the time derivative of the electric field is ignored. The TDGL equations were first given by Schmid [11] in 1966 and subsequently derived from the microscopic theory of superconductivity by Gor'kov and Eliashberg [12]. Our notation is the same as in Gor'kov and Kopnin [13].

The choice of a proper gauge for the TDGL equations has been a subject of considerable debate. The choice is a matter of convenience and may depend on the problem under investigation. In this article we adopt a gauge in which, at any time, the electric potential and the divergence of the vector potential satisfy the identity

$$\sigma\phi + (c/4\pi)\nabla \cdot \boldsymbol{A} = 0 \tag{2.15}$$

everywhere in the domain, while $\boldsymbol{A}$ is tangential at the boundary. This choice is realized by identifying the gauge χ with a solution of the linear parabolic equation

$$\frac{\sigma}{c}\frac{\partial\chi}{\partial t} - \frac{c}{4\pi}\Delta\chi = \sigma\phi + \frac{c}{4\pi}\nabla \cdot \boldsymbol{A}, \tag{2.16}$$

subject to the condition $\boldsymbol{n}\cdot\nabla\chi = -\boldsymbol{n}\cdot\boldsymbol{A}$ on the boundary. In [9], it was shown that the TDGL equations, subject to the constraint (2.15), define a dynamical system under suitable regularity conditions on $\boldsymbol{H}$. (In the more general case, where $\boldsymbol{H}$ varies not only in space but also in time, the TDGL equations define a dynamical process.) This dynamical system has a global attractor, which consists of the stationary points of the dynamical system and the heteroclinic orbits connecting such stationary points. Furthermore, it was shown that every solution on the attractor satisfies the condition $\nabla \cdot \boldsymbol{A} = 0$ (and, therefore, also $\phi = 0$). Thus, in the limit as $t \to \infty$, every solution of the TDGL equations satisfies the GL equations in the London gauge.

2.2 Nondimensional TDGL Equations

In this section, we render the TDGL equations dimensionless by choosing units for the independent and dependent variables. Since we are interested in the collective behavior of vortices in the bulk of a superconductor in the limit of weak coupling ($q_s \to 0$), we take care to choose the units in such a way that they remain of order one as $q_s \to 0$. (We recall that q_s is negative, $q_s = -2e$.)

As $q_s \to 0$, the coherence length ξ remains of order one, while the penetration depth λ increases like $|q_s|^{-1}$; see Eq. (2.3). This suggests taking the coherence length ξ as the unit of length.

To maintain the diffusion coefficient $D = \hbar/2\gamma m_s = \xi^2(\gamma\hbar/|\alpha|)^{-1}$ at order one, we measure time in units of $\gamma\hbar/|\alpha|$.

The real and imaginary parts of the order parameter are conveniently measured in units of $\psi_0 = (|\alpha|/\beta)^{1/2}$, which is the value of ψ that minimizes the free energy in the absence of a field.

Next, consider the magnetic field. A fundamental quantity in the theory of type-II superconductors is the flux quantum Φ_0,

$$\Phi_0 = \frac{hc}{|q_s|} = 2\pi \frac{\hbar c}{|q_s|}. \tag{2.17}$$

The flux quantum is the unit of magnetic flux carried by a vortex. Together with the coherence length and penetration depth, it defines three characteristic field strengths: the *lower critical field* H_{c1}, the *thermodynamical critical field* H_c, and the *upper critical field* H_{c2},

$$H_{c1} = \frac{\Phi_0}{4\pi\lambda^2 \ln \kappa}, \quad H_c = \frac{\Phi_0}{2\pi\xi\lambda\sqrt{2}}, \quad H_{c2} = \frac{\Phi_0}{2\pi\xi^2}. \tag{2.18}$$

Below H_{c1}, a superconductor is in the ideal superconducting (Meissner) state, where it does not support magnetic flux in the bulk; above H_{c2}, it is in the normal state, where the magnetic flux is distributed uniformly in the bulk; between H_{c1} and H_{c2}, it is in the vortex state, where magnetic flux is present, but in quantized units (fluxoids) that are shielded from the superconducting bulk by an encircling supercurrent. The thermodynamical critical field H_c is intermediate between H_{c1} and H_{c2} and is defined by the identity $H_c^2/8\pi = \frac{1}{2}\psi_0^2|\alpha|$; the quantity in the left member is the energy per unit volume associated with H_c, the quantity in the right member is the minimum condensation energy, which is attained when $\psi = \psi_0$, and H_c is defined so these two quantities are in balance.

As $q_s \to 0$, H_{c1} goes to 0 like $|q_s|$, H_c remains of order one, and H_{c2} grows like $|q_s|^{-1}$. This suggests that we define field strengths in terms of H_c. In fact, it is convenient to absorb a factor $\sqrt{2}$, so we adopt $H_c\sqrt{2}$ or, equivalently, $\hbar c/\xi\lambda|q_s|$ as the unit for the magnetic field strength.

With the coherence length as the unit of length and $H_c\sqrt{2}$ as the unit of field strength, it follows that the vector potential is measured in units of $\xi H_c\sqrt{2}$. Furthermore, energy densities are measured in units of $H_c^2/4\pi$, which is the same as $|\alpha|\psi_0^2$.

Finally, we define the scalar potential ϕ in units of $(1/\gamma\psi_0^2\kappa|q_s|)(H_c^2/4\pi)$. Notice that this unit remains of order one as $q_s \to 0$, because $\kappa|q_s|$ is of order one. On the other hand, the product $q_s\phi$, which represents an energy density, tends to zero as $q_s \to 0$. (It remains finite on the scale of the penetration depth.)

Table 1 summarizes the relations between the original variables and their nondimensional (primed) counterparts. We adopt the latter as the new variables and work until further notice on the nondimensional problem. We omit all primes.

Table 1: Nondimensionalization.

Independent variables	$\boldsymbol{x} = \xi \boldsymbol{x}'$ $t = (\gamma\hbar/\lvert\alpha\rvert)t'$
Dependent variables	$\psi = \psi_0 \psi'$ $\boldsymbol{A} = (\xi H_c \sqrt{2})\boldsymbol{A}'$ $\phi = (1/\gamma\psi_0^2\kappa\lvert q_s\rvert)(H_c^2/4\pi)\phi'$
Electromagnetic variables	$\boldsymbol{B} = (H_c\sqrt{2})\boldsymbol{B}'$ $\boldsymbol{J} = (cH_c\sqrt{2}/4\pi\xi)\boldsymbol{J}'$ $\boldsymbol{E} = (1/\gamma\psi_0^2\kappa\lvert q_s\rvert)(H_c^2/4\pi\xi)\boldsymbol{E}'$
Applied Field Normal conductivity	$\boldsymbol{H} = (H_c\sqrt{2})\boldsymbol{H}'$ $\sigma = (\gamma m_s c^2/2\pi\hbar)\sigma'$

The nondimensional TDGL equations are

$$\left(\frac{\partial}{\partial t} - \frac{i}{\kappa}\phi\right)\psi - \left(\nabla + \frac{i}{\kappa}\boldsymbol{A}\right)^2\psi - (1-|\psi|^2)\psi = 0, \tag{2.19}$$

$$\sigma\left(-\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi\right) + \frac{1}{\kappa}\boldsymbol{J}_s - \nabla\times\nabla\times\boldsymbol{A} + \nabla\times\boldsymbol{H} = \boldsymbol{0}, \tag{2.20}$$

where

$$\boldsymbol{J}_s = -\frac{1}{2i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{1}{\kappa}|\psi|^2\boldsymbol{A} = -\Im\left[\psi^*\left(\nabla + \frac{i}{\kappa}\boldsymbol{A}\right)\psi\right]. \tag{2.21}$$

The (nondimensional) gauge condition is

$$\sigma\phi + \nabla\cdot\boldsymbol{A} = 0, \tag{2.22}$$

and the electromagnetic variables are given by the expressions

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \quad \boldsymbol{J} = \nabla\times\nabla\times\boldsymbol{A}, \quad \boldsymbol{E} = -\partial_t\boldsymbol{A} - \nabla\phi. \tag{2.23}$$

The values of the lower and upper critical fields are

$$H_{c1} = (2\kappa\ln\kappa)^{-1}, \quad H_{c2} = \kappa. \tag{2.24}$$

Equations (2.19) and (2.20), together with the gauge condition (2.22), must be satisfied everywhere in a domain Ω (the superconducting domain measured in units of ξ). At the boundary $\partial\Omega$ of Ω, we have the conditions

$$\boldsymbol{n}\cdot\boldsymbol{J}_s = 0, \quad \boldsymbol{n}\times(\nabla\times\boldsymbol{A}) = \boldsymbol{n}\times\boldsymbol{H}, \quad \boldsymbol{n}\cdot\boldsymbol{A} = 0. \tag{2.25}$$

Here, $\boldsymbol{n}$ is the local unit normal vector.

2.3 Link Variables

The combination $\nabla + (i/\kappa)\boldsymbol{A}$ plays a fundamental role in the theory; we refer to it as the $\boldsymbol{A}$-gradient and denote it by a special symbol,

$$\nabla_{\mathbf{A}} = \nabla + \frac{i}{\kappa}\boldsymbol{A}. \tag{2.26}$$

Similarly, we define the $\boldsymbol{A}$-Laplacian,

$$\Delta_{\mathbf{A}} = \nabla_{\mathbf{A}} \cdot \nabla_{\mathbf{A}} = \left(\nabla + \frac{i}{\kappa}\boldsymbol{A}\right)^2. \tag{2.27}$$

The relation between the $\boldsymbol{A}$-Laplacian and the ordinary Laplacian is most easily illustrated by means of the *link variables*,

$$\begin{aligned} U_x(x,y,z) &= \exp\left(\frac{i}{\kappa}\int^x A_x(\xi,y,z)\right)\,\mathrm{d}\xi, \\ U_y(x,y,z) &= \exp\left(\frac{i}{\kappa}\int^y A_y(x,\eta,z)\right)\,\mathrm{d}\eta, \\ U_z(x,y,z) &= \exp\left(\frac{i}{\kappa}\int^z A_z(x,y,\zeta)\right)\,\mathrm{d}\zeta. \end{aligned} \tag{2.28}$$

(We omit the argument t.) The integrals are evaluated with respect to an arbitrary reference point. Each U_μ ($\mu = x, y, z$) is complex valued and unimodular, $U_\mu^* = U_\mu^{-1}$. The vectors $\boldsymbol{A}$ and $\boldsymbol{U}$ may be used interchangeably. With a slight abuse of notation, we have

$$\boldsymbol{U} = \mathrm{e}^{(i/\kappa)\int \boldsymbol{A}}, \quad \nabla_{\mathbf{A}} = U^*\nabla U, \quad \Delta_{\mathbf{A}} = U^*\Delta U. \tag{2.29}$$

Sometimes, we refer to the $\boldsymbol{A}$-Laplacian as a "twisted Laplacian."

2.4 Auxiliary Equations

The quantity $|\psi|^2$ corresponds to the density of the superconducting charge carriers. Its evolution is governed by the equation

$$\frac{\partial|\psi|^2}{\partial t} - 2\,\mathrm{Re}\,[\psi^*\Delta_{\mathbf{A}}\psi] = 2(1-|\psi|^2)|\psi|^2 \tag{2.30}$$

or, equivalently,

$$\frac{\partial|\psi|^2}{\partial t} - \Delta|\psi|^2 = 2(1-|\psi|^2)|\psi|^2 - 2\,|\nabla_{\mathbf{A}}\psi|^2\,. \tag{2.31}$$

The last equation shows that, if $|\psi| \leq 1$ initially, then $|\psi| \leq 1$ at all times.

Equations for the divergence and the curl of $\boldsymbol{A}$ follow from Eq. (2.20),

$$\sigma \left(-\frac{\partial (\nabla \cdot \boldsymbol{A})}{\partial t} - \Delta \phi \right) + \frac{1}{\kappa} \nabla \cdot \boldsymbol{J}_s = 0, \tag{2.32}$$

$$-\sigma \frac{\partial (\nabla \times \boldsymbol{A})}{\partial t} + \Delta (\nabla \times \boldsymbol{A}) + \frac{1}{\kappa} \nabla \times \boldsymbol{J}_s = 0. \tag{2.33}$$

Here we have used the facts that the divergence of a curl is zero, the curl of a gradient is zero, and $-\nabla \times \nabla \times \boldsymbol{a} = \Delta \boldsymbol{a} - \nabla(\nabla \cdot \boldsymbol{a})$ for any vector $\boldsymbol{a}$ and assumed, without loss of generality, that the applied field $\boldsymbol{H}$ is divergence free and harmonic, $\nabla \cdot \boldsymbol{H} = 0$ and $\Delta \boldsymbol{H} = 0$. In the gauge (2.22), Eq. (2.32) becomes an evolution equation for ϕ,

$$\sigma^2 \frac{\partial \phi}{\partial t} - \sigma \Delta \phi + \frac{1}{\kappa} \nabla \cdot \boldsymbol{J}_s = 0. \tag{2.34}$$

Equation (2.33) is an evolution equation for the magnetic induction,

$$-\sigma \frac{\partial \boldsymbol{B}}{\partial t} + \Delta \boldsymbol{B} + \frac{1}{\kappa} \nabla \times \boldsymbol{J}_s = 0. \tag{2.35}$$

Expressions for $\nabla \cdot \boldsymbol{J}_s$ and $\nabla \times \boldsymbol{J}_s$ are readily obtained from Eq. (2.21),

$$\nabla \cdot \boldsymbol{J}_s = -\Im \left[\psi^* \Delta_{\mathbf{A}} \psi \right] = -\Im \left[\psi^* \left(\frac{\partial}{\partial t} - \frac{i}{\kappa} \phi \right) \psi \right]. \tag{2.36}$$

$$\nabla \times \boldsymbol{J}_s = -\Im \left[(\nabla_{\mathbf{A}} \psi)^* \times (\nabla_{\mathbf{A}} \psi) \right] - \frac{1}{\kappa} |\psi|^2 \nabla \times \boldsymbol{A}. \tag{2.37}$$

2.5 Energy Inequalities

Associated with Eqs. (2.19) and (2.20) is an energy inequality. Let E be defined by the integral

$$E(t) = \int_\Omega \left[|\nabla_{\mathbf{A}} \psi|^2 + \frac{1}{2}(1 - |\psi|^2)^2 + |\nabla \times \boldsymbol{A} - \boldsymbol{H}|^2 \right] \mathrm{d}\boldsymbol{x}, \quad t \geq 0. \tag{2.38}$$

Then

$$E'(t) = -2 \int_\Omega \left[\left| \left(\frac{\partial}{\partial t} - \frac{i}{\kappa} \phi \right) \psi \right|^2 + \sigma \left| \frac{\partial \boldsymbol{A}}{\partial t} + \nabla \phi \right|^2 \right] \mathrm{d}\boldsymbol{x}, \quad t > 0. \tag{2.39}$$

Hence, $E'(t) \leq 0$, and therefore $E(t) \leq E(0)$ for all $t \geq 0$. If, in addition, $\boldsymbol{A}$ and ϕ are related by the gauge condition (2.22), we have a similar result for the extended functional

$$F(t) = \int_\Omega \left[|\nabla_{\mathbf{A}}\psi|^2 + \frac{1}{2}(1-|\psi|^2)^2 + |\nabla \times \boldsymbol{A} - \boldsymbol{H}|^2 + 2(\nabla \cdot \boldsymbol{A})^2 \right] \, \mathrm{d}\boldsymbol{x}, \quad t \geq 0, \tag{2.40}$$

namely,

$$F'(t) = -2 \int_\Omega \left[\left| \left(\frac{\partial}{\partial t} + \frac{i}{\kappa\sigma}(\nabla \cdot \boldsymbol{A}) \right) \psi \right|^2 + \sigma \left| \frac{\partial \boldsymbol{A}}{\partial t} \right|^2 + \frac{1}{\sigma} \left| \nabla(\nabla \cdot \boldsymbol{A}) \right|^2 \right] \, \mathrm{d}\boldsymbol{x}, \quad t > 0, \tag{2.41}$$

so $F(t) \leq F(0)$ for all $t \geq 0$.

3 Asymptotic Analysis

We now consider the TDGL equations in the limit as $q_s \to 0$ (*weak coupling*), when the applied field is near the upper critical field.

3.1 Scaling the Problem

We begin by establishing a scaling for the various variables. The scaling is done by means of the dimensionless GL parameter κ, which grows like $|q_s|^{-1}$.

Since the applied field is near $H_{c2} = \kappa$, we begin by scaling $\boldsymbol{H}$ by a factor κ, $\boldsymbol{H} = \kappa \boldsymbol{H}'$. By scaling the vector potential by the same factor κ, we achieve that the electromagnetic variables are all of the same order.

The scalar potential is proportional to the charge density of the Cooper pairs, which is $O(|q_s|)$ as $q_s \to 0$. Hence, $\kappa\phi$ remains of order one. This suggests scaling ϕ by a factor κ^{-1}.

Table 2 summarizes the relation between the current (nondimensional) variables and their scaled (primed) counterparts. We adopt the latter as the new variables and work until further notice on the scaled problem. We omit all primes.

After scaling, the relevant parameter is κ^2, rather than κ, so we introduce ε,

$$\varepsilon = \kappa^{-2}. \tag{3.1}$$

Table 2: Scaling.

Applied field	$\boldsymbol{H} = \kappa \boldsymbol{H}'$
Dependent variables	$\psi = \psi'$ $\boldsymbol{A} = \kappa \boldsymbol{A}'$ $\phi = \kappa^{-1}\phi'$
Electromagnetic variables	$\boldsymbol{B} = \kappa \boldsymbol{B}'$ $\boldsymbol{J} = \kappa \boldsymbol{J}'$ $\boldsymbol{E} = \kappa \boldsymbol{E}'$

The scaled TDGL equations are

$$(\partial_t - i\varepsilon\phi)\,\psi - \Delta_{\mathbf{A}}\psi - (1 - |\psi|^2)\psi = 0, \tag{3.2}$$

$$\sigma\,(-\partial_t \boldsymbol{A} - \varepsilon\nabla\phi) + \varepsilon \boldsymbol{J}_s - \nabla\times\nabla\times\boldsymbol{A} + \nabla\times\boldsymbol{H} = \mathbf{0}, \tag{3.3}$$

where

$$\boldsymbol{J}_s = -\frac{1}{2i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2\boldsymbol{A} = -\Im\left[\psi^*\nabla_{\mathbf{A}}\psi\right]. \tag{3.4}$$

The gauge condition is

$$\varepsilon\sigma\phi + \nabla\cdot\boldsymbol{A} = 0, \tag{3.5}$$

and the electromagnetic variables are given by the expressions

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}, \quad \boldsymbol{J} = \nabla\times\nabla\times\boldsymbol{A}, \quad \boldsymbol{E} = -\partial_t\boldsymbol{A} - \varepsilon\nabla\phi. \tag{3.6}$$

The boundary conditions associated with Eqs. (3.2) and (3.3) are

$$\boldsymbol{n}\cdot\nabla_{\mathbf{A}}\psi = 0, \quad \boldsymbol{n}\times(\nabla\times\boldsymbol{A}) = \boldsymbol{n}\times\boldsymbol{H}, \quad \boldsymbol{n}\cdot\boldsymbol{A} = 0. \tag{3.7}$$

Furthermore,

$$\boldsymbol{U} = \mathrm{e}^{i\int\boldsymbol{A}}, \quad \nabla_{\mathbf{A}} = \nabla + i\boldsymbol{A} = U^*\nabla U, \quad \Delta_{\mathbf{A}} = U^*\Delta U. \tag{3.8}$$

3.2 Reducing the Problem

Next, we separate the contribution to the vector potential $\boldsymbol{A}$ from the (stationary) applied field $\boldsymbol{H}$.

For any $\boldsymbol{H} \in [L^2(\Omega)]^n$, let $\boldsymbol{A}_{\mathbf{H}}$ be the minimizer of the convex quadratic form $Q_{\mathbf{H}} \equiv Q_{\mathbf{H}}[\boldsymbol{A}]$,

$$Q_{\mathbf{H}}[\boldsymbol{A}] = \int_\Omega \left[(\nabla\cdot\boldsymbol{A})^2 + |\nabla\times\boldsymbol{A} - \boldsymbol{H}|^2\right]\,\mathrm{d}\boldsymbol{x}, \tag{3.9}$$

on $\mathrm{dom}(Q_{\mathbf{H}}) = \{\boldsymbol{A} \in [W^{1,2}(\Omega)]^n : \boldsymbol{n} \cdot \boldsymbol{A} = 0 \text{ on } \partial\Omega\}$. The minimizer exists, is unique, and satifies the boundary-value problem

$$-\nabla \times \nabla \times \boldsymbol{A}_{\mathbf{H}} + \nabla \times \boldsymbol{H} = \mathbf{0}, \quad \nabla \cdot \boldsymbol{A}_{\mathbf{H}} = 0 \quad \text{in } \Omega, \tag{3.10}$$

$$\boldsymbol{n} \times (\nabla \times \boldsymbol{A}_{\mathbf{H}}) = \boldsymbol{n} \times \boldsymbol{H}, \quad \boldsymbol{n} \cdot \boldsymbol{A}_{\mathbf{H}} = 0 \quad \text{on } \partial\Omega, \tag{3.11}$$

in the dual of $\mathrm{dom}(Q_{\mathbf{H}})$ with respect to the inner product in $[L^2(\Omega)]^n$. The vector $\boldsymbol{A}_{\mathbf{H}}$ yields the magnetic field $\boldsymbol{B}_{\mathbf{H}}$ that would have been present in the absence of a superconductor,

$$\boldsymbol{B}_{\mathbf{H}} = \nabla \times \boldsymbol{A}_{\mathbf{H}}. \tag{3.12}$$

We separate its contribution to the field and current by making the substitution

$$\boldsymbol{A} = \boldsymbol{A}_{\mathbf{H}} + \boldsymbol{A}'. \tag{3.13}$$

Table 3 summarizes the relation between the current (scaled, nondimensional) variables and their reduced (primed) counterparts. We adopt the latter as the new variables and work until further notice on the reduced problem. We omit all primes.

Table 3: Reduction.

Dependent variables	$\psi = \psi'$ $\boldsymbol{A} = \boldsymbol{A}_{\mathbf{H}} + \boldsymbol{A}'$ $\phi = \phi'$
Electromagnetic variables	$\boldsymbol{B} = \boldsymbol{B}_{\mathbf{H}} + \boldsymbol{B}'$ $\boldsymbol{J} = \nabla \times \boldsymbol{B}_{\mathbf{H}} + \boldsymbol{J}'$ $\boldsymbol{E} = \boldsymbol{E}'$

The reduced TDGL equations are

$$(\partial_t - i\varepsilon\phi)\,\psi - \Delta_{\mathbf{A_H+A}}\psi - (1 - |\psi|^2)\psi = 0, \tag{3.14}$$

$$\sigma\,(-\partial_t \boldsymbol{A} - \varepsilon\nabla\phi) + \varepsilon \boldsymbol{J}_s - \nabla \times \nabla \times \boldsymbol{A} = \mathbf{0}, \tag{3.15}$$

where

$$\boldsymbol{J}_s = -\frac{1}{2i}(\psi^*\nabla\psi - \psi\nabla\psi^*) - |\psi|^2(\boldsymbol{A}_{\mathbf{H}} + \boldsymbol{A}). \tag{3.16}$$

The gauge condition is

$$\varepsilon\sigma\phi + \nabla \cdot \boldsymbol{A} = 0, \tag{3.17}$$

and the electromagnetic variables are given by the expressions

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \quad \boldsymbol{J} = \nabla \times \nabla \times \boldsymbol{A}, \quad \boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla\phi. \tag{3.18}$$

The boundary conditions associated with Eqs. (3.14) and (3.15) are

$$\boldsymbol{n} \cdot \nabla_{\mathbf{A}}\psi = 0, \quad \boldsymbol{n} \times (\nabla \times \boldsymbol{A}) = \mathbf{0}, \quad \boldsymbol{n} \cdot \boldsymbol{A} = 0. \tag{3.19}$$

3.3 Existence, Uniqueness, and Regularity

The existence and uniqueness of a solution of Eqs. (3.14)–(3.17), subject to the boundary conditions (3.19), can be shown with the same techniques as in [9]. First, we eliminate the scalar potential from the problem by incorporating the gauge condition (3.17) in the differential equations (3.14) and (3.15),

$$\partial_t \psi + i\sigma^{-1}(\nabla \cdot \boldsymbol{A})\psi - \Delta_{\mathbf{A_H+A}}\psi - (1-|\psi|^2)\psi = 0, \tag{3.20}$$

$$-\sigma\partial_t \boldsymbol{A} + \Delta \boldsymbol{A} + \varepsilon \boldsymbol{J}_s = \boldsymbol{0}. \tag{3.21}$$

Next, we reformulate this problem as an abstract functional equation for the vector

$$u = (\psi, \boldsymbol{A}), \tag{3.22}$$

considered as a mapping from the time domain $[0, \infty)$ to the Hilbert space

$$\mathcal{L}^2 = [L^2(\Omega)]^2 \times [L^2(\Omega)]^n. \tag{3.23}$$

The vector u satisfies an ordinary differential equation,

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = f(u),\ t > 0, \tag{3.24}$$

where A is the linear operator associated with the quadratic form $Q \equiv Q[u]$,

$$Q[u] = \int_\Omega \left[|\nabla\psi|^2 + \sigma^{-1}\left((\nabla\cdot\boldsymbol{A})^2 + |\nabla\times\boldsymbol{A}|^2\right)\right]\ \mathrm{d}\boldsymbol{x} \tag{3.25}$$

on $\mathrm{dom}(Q) = \{u = (\psi, \boldsymbol{A}) \in \mathcal{L}^2 : \boldsymbol{n}\cdot\boldsymbol{A} = 0 \text{ on } \partial\Omega\}$ and f is a nonlinear function of ψ and $\boldsymbol{A}$. Given any $f = (\varphi, \sigma^{-1}\boldsymbol{F}) \in \mathcal{L}^2$, the equation $Au = f$ is equivalent with the system of uncoupled boundary-value problems

$$-\Delta\psi = \varphi \text{ in } \Omega, \qquad \boldsymbol{n}\cdot\nabla\psi = 0 \text{ on } \partial\Omega, \tag{3.26}$$

$$-\Delta\boldsymbol{A} = \boldsymbol{F} \text{ in } \Omega, \qquad \boldsymbol{n}\times\boldsymbol{A} = \boldsymbol{0},\ \boldsymbol{n}\cdot\boldsymbol{A} = 0 \text{ on } \partial\Omega, \tag{3.27}$$

in the dual of $\mathrm{dom}(Q)$ with respect to the inner product in $\mathcal{L}^2$. The operator A is selfadjoint and positive definite in $\mathcal{L}^2$; hence, its fractional powers $A^{\theta/2}$ are well defined, they are unbounded if $\theta \geq 0$, and $\mathrm{dom}(A^{\theta/2})$ is a closed linear subspace of $\mathcal{W}^{\theta,2} = [W^{\theta,2}(\Omega)]^2 \times [W^{\theta,2}(\Omega)]^n$; see [14, Section 1.4].

The basic regularity assumption in [9], which we also adopt in the present investigation, is $\boldsymbol{H} \in [W^{\alpha,2}(\Omega)]^n$ for some $\alpha \in (\frac{1}{2}, 1)$. The appropriate framework for the analysis of the existence, uniqueness, and regularity of the solution of Eq. (3.24) is then the Hilbert space

$$\mathcal{W}^{1+\alpha,2} = [W^{1+\alpha,2}(\Omega)]^2 \times [W^{1+\alpha,2}(\Omega)]^n. \tag{3.28}$$

If Ω is bounded in $\mathbf{R}^n$ ($n = 2, 3$), then $\mathcal{W}^{1+\alpha,2}$ is continuously imbedded in $\mathcal{W}^{1,2} \cap \mathcal{L}^\infty$.

The vector $\boldsymbol{A}_\mathbf{H}$ introduced in Eqs. (3.10), (3.11) defines a (constant) vector $u_\mathbf{H}$,

$$u_\mathbf{H} = (0, \boldsymbol{A}_\mathbf{H}). \tag{3.29}$$

Theorem 3.1 *If $\boldsymbol{H} \in [W^{\alpha,2}(\Omega)]^n$, $\alpha \in (\frac{1}{2}, 1)$, then $u_\mathbf{H} \in \mathcal{W}^{1+\alpha,2}$.*

Proof. The mapping $\boldsymbol{H} \mapsto \boldsymbol{A}_\mathbf{H}$ is linear, time independent, and continuous from $[W^{\alpha,2}(\Omega)]^n$ to $[W^{1+\alpha,2}(\Omega)]^n$; see [15]. ∎

Theorem 3.2 *For any $u_0 \in dom(A^{(1+\alpha)/2})$ and $T > 0$, there exists a unique $u \in C([0,T]; \mathcal{W}^{1+\alpha,2})$ satisfying Eq. (3.24) for all $t \in [0,T]$ and the initial condition $u(0) = u_0$. If $u = (\psi, \boldsymbol{A})$, then ψ satisfies the "maximum principle,"*

$$|\psi(x,t)| \leq \max\{1, \|\psi(0)\|_\infty\}, \quad (x,t) \in \overline{\Omega} \times [0,T]. \tag{3.30}$$

Proof. See [9, Theorem 1]. ∎

Besides the maximum principle (3.30), there are some important inequalities involving the derivatives of ψ and $\boldsymbol{A}$, which are derived from the extended energy functional. This functional, which was defined in Eq. (2.40) for the TDGL equations (2.19)–(2.22), is given in terms of the current variables by the expression

$$F(t) = \int_\Omega \left[|\nabla_{\mathbf{A_H+A}}\psi|^2 + \frac{1}{2}(1 - |\psi|^2)^2 + |\nabla \times \boldsymbol{A}|^2 + 2(\nabla \cdot \boldsymbol{A})^2 \right] \mathrm{d}\boldsymbol{x}, \quad t \geq 0. \tag{3.31}$$

If $\boldsymbol{A}$ and ψ satisfy Eqs. (3.14)–(3.19), then

$$F'(t) = -2 \int_\Omega \left[\left| \left(\frac{\partial}{\partial t} + \frac{i}{\sigma}(\nabla \cdot \boldsymbol{A}) \right) \psi \right|^2 + \frac{1}{\varepsilon} \left[\sigma \left| \frac{\partial \boldsymbol{A}}{\partial t} \right|^2 + \frac{1}{\sigma} |\nabla(\nabla \cdot \boldsymbol{A})|^2 \right] \right] \mathrm{d}\boldsymbol{x}, \quad t > 0. \tag{3.32}$$

The integrand is nonnegative, so $F'(t) \le 0$ and, consequently,

$$\int_0^T \int_\Omega \left[|\nabla_{\mathbf{A_H+A}} \psi|^2 + \frac{1}{2}(1-|\psi|^2)^2 + |\nabla \times \boldsymbol{A}|^2 + 2(\nabla \cdot \boldsymbol{A})^2 \right] (\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x}\mathrm{d}t \le F(0)T, \tag{3.33}$$

for any $T > 0$. Furthermore, as long as $F(t)$ exists, we have the identity

$$2\int_0^t \int_\Omega \left[\left| \left(\frac{\partial}{\partial t} + \frac{i}{\sigma}(\nabla \cdot \boldsymbol{A}) \right) \psi \right|^2 + \frac{1}{\varepsilon} \left[\sigma \left| \frac{\partial \boldsymbol{A}}{\partial t} \right|^2 + \frac{1}{\sigma} |\nabla(\nabla \cdot \boldsymbol{A})|^2 \right] \right] (\boldsymbol{x}, s) \, \mathrm{d}\boldsymbol{x}\mathrm{d}s$$

$$= F(0) - F(t). \tag{3.34}$$

Hence,

$$\int_0^T \int_\Omega \left[\left| \frac{\partial \psi}{\partial t} \right|^2 + \left| \frac{\partial \boldsymbol{A}}{\partial t} \right|^2 + |\nabla(\nabla \cdot \boldsymbol{A})|^2 \right] (\boldsymbol{x}, t) \, \mathrm{d}\boldsymbol{x}\mathrm{d}t \le \left(1 + \varepsilon \frac{\sigma + \sigma^{-1}}{2} + \frac{M^2 T}{\sigma^2} \right) F(0), \tag{3.35}$$

for any $T > 0$ such that $M \equiv M(T) = \sup\{|\psi(x,t)| : (x,t) \in \Omega \times [0,T]\}$ is finite.

3.4 Convergence as $\varepsilon \downarrow 0$

The solution $u = (\psi, \boldsymbol{A})$ of Eq. (3.24) defined in Theorem 3.2 depends on ε. In this section we investigate its limit as $\varepsilon \downarrow 0$. We assume that the applied field $\boldsymbol{H}$ is independent of ε.

We rescale once more to make the ε-dependence more explicit, putting

$$\boldsymbol{A} = \varepsilon \boldsymbol{A}'. \tag{3.36}$$

Table 4 summarizes the relation between the current (reduced, scaled, nondimensional) variables and their rescaled (primed) counterparts. We adopt the latter as the new variables and work until further notice on the rescaled problem. We omit all primes.

The rescaled TDGL equations, including the gauge, are

$$\partial_t \psi + i\sigma^{-1}(\nabla \cdot (\varepsilon \boldsymbol{A}))\psi - \Delta_{\mathbf{A_H}+\varepsilon\mathbf{A}} \psi - (1-|\psi|^2)\psi = 0, \tag{3.37}$$

$$-\sigma \partial_t \boldsymbol{A} + \Delta \boldsymbol{A} + \boldsymbol{J}_s = \boldsymbol{0}, \tag{3.38}$$

Table 4: Rescaling.

Dependent variables	$\psi = \psi'$ $\boldsymbol{A} = \varepsilon \boldsymbol{A}'$ $\phi = \varepsilon \phi'$
Electromagnetic variables	$\boldsymbol{B} = \varepsilon \boldsymbol{B}'$ $\boldsymbol{J} = \boldsymbol{J}'$ $\boldsymbol{E} = \varepsilon \boldsymbol{E}'$

where

$$\boldsymbol{J}_s = -\frac{1}{2i}(\psi^* \nabla \psi - \psi \nabla \psi^*) - |\psi|^2 (\boldsymbol{A}_{\mathbf{H}} + \varepsilon \boldsymbol{A}). \tag{3.39}$$

We reformulate the equations as a differential equation for the vector

$$u_\varepsilon = (\psi, \boldsymbol{A}) : [0, \infty) \to \mathcal{L}^2. \tag{3.40}$$

The equation has the same form as Eq. (3.24), but the ε-dependence of the nonlinear function in the right member is more explicit,

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = f_0(u) + \varepsilon f_1(u), \tag{3.41}$$

where

$$f_i(u) = (\varphi_i(\psi, \boldsymbol{A}), \sigma^{-1} \boldsymbol{F}_i(\psi, \boldsymbol{A})), \quad i = 0, 1, \tag{3.42}$$

with

$$\begin{aligned}
\varphi_0(\psi, \boldsymbol{A}) &= 2i \boldsymbol{A}_{\mathbf{H}} \cdot (\nabla \psi) - |\boldsymbol{A}_{\mathbf{H}}|^2 \psi + (1 - |\psi|^2)\psi, &(3.43)\\
\varphi_1(\psi, \boldsymbol{A}) &= i(1 - \sigma^{-1})(\nabla \cdot \boldsymbol{A})\psi + 2i \boldsymbol{A} \cdot (\nabla \psi) - (\boldsymbol{A}_{\mathbf{H}} \cdot \boldsymbol{A})\psi - |\boldsymbol{A}|^2 \psi, &(3.44)\\
\boldsymbol{F}_0(\psi, \boldsymbol{A}) &= 0, &(3.45)\\
\boldsymbol{F}_1(\psi, \boldsymbol{A}) &= -\frac{1}{2i}(\psi^* \nabla \psi - \psi \nabla \psi^*) - |\psi|^2 (\boldsymbol{A}_{\mathbf{H}} + \varepsilon \boldsymbol{A}). &(3.46)
\end{aligned}$$

We compare the solution u_ε of Eq. (3.41) with the solution u_0 of the reduced equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Au = f_0(u). \tag{3.47}$$

The existence, uniqueness, and regularity properties of u_0 are the same as for u_ε.

Theorem 3.3 *There exists a positive constant C such that*

$$\|u_\varepsilon(t) - u_0(t)\|_{\mathcal{W}^{1+\alpha,2}} \leq C \left(\|u_\varepsilon(0) - u_0(0)\|_{\mathcal{W}^{1+\alpha,2}} + \varepsilon \right), \quad t \in [0, T]. \tag{3.48}$$

Proof. Let B_R be the ball of radius R centered at the origin in $\mathcal{W}^{1+\alpha,2}$. Let $u_\varepsilon \in B_R$ and $u_0 \in B_R$ satisfy Eqs. (3.41) and (3.47), respectively, with initial data $u_\varepsilon(0)$ and $u_0(0)$. The difference $v = u_\varepsilon - u_0$ satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} + Av = f_0(u_\varepsilon) - f_0(u_0) + \varepsilon f_1(u_\varepsilon) \tag{3.49}$$

or, equivalently, the integral equation

$$v(t) = \mathrm{e}^{-tA}v(0) + \int_0^t \mathrm{e}^{-(t-s)A}[f_0(u_\varepsilon) - f_0(u_0) + \varepsilon f_1(u_\varepsilon)](s)\ \mathrm{d}s. \tag{3.50}$$

From the integral equation we obtain the estimate

$$\begin{aligned} \|v(t)\|_{\mathcal{W}^{1+\alpha,2}} \ &\leq\ \|\mathrm{e}^{-tA}\|_{\mathcal{W}^{1+\alpha,2}}\|v(0)\|_{\mathcal{W}^{1+\alpha,2}} + \int_0^t \|A^{(1+\alpha)/2}\mathrm{e}^{-(t-s)A}\|_{\mathcal{W}^{1+\alpha,2}} \\ &\quad \times [\|f_0(u_\varepsilon) - f_0(u_0)\|_{L^2} + \varepsilon\|f_1(u_\varepsilon)\|_{L^2}]\,(s)\ \mathrm{d}s. \end{aligned} \tag{3.51}$$

The operator norms satisfy the inequalities

$$\|\mathrm{e}^{-tA}\|_{\mathcal{W}^{1+\alpha,2}} \leq 1, \quad \|A^{(1+\alpha)/2}\mathrm{e}^{-(t-s)A}\|_{\mathcal{W}^{1+\alpha,2}} \leq C(t-s)^{-(1+\alpha)/2}; \tag{3.52}$$

see [14, Theorem 1.4.3]. Furthermore, adding and subtracting terms, we have

$$\begin{aligned} f_0(u_\varepsilon) - f_0(u_0) \ &=\ \Big(2i\boldsymbol{A}_{\mathbf{H}}\cdot(\nabla(\psi_\varepsilon - \psi_0)) - |\boldsymbol{A}_{\mathbf{H}}|^2(\psi_\varepsilon - \psi_0) \\ &\quad +\ (1 - |\psi_\varepsilon|^2 - |\psi_0|^2)(\psi_\varepsilon - \psi_0) - \psi_\varepsilon\psi_0(\psi_\varepsilon^* - \psi_0^*),\ 0\Big), \end{aligned} \tag{3.53}$$

where

$$\begin{aligned} &\|2i\boldsymbol{A}_{\mathbf{H}}\cdot(\nabla(\psi_\varepsilon - \psi_0))\|_{L^2} \leq 2\|\boldsymbol{A}_{\mathbf{H}}\|_{L^\infty}\|\psi_\varepsilon - \psi_0\|_{W^{1,2}} \\ &\qquad \leq C\|\psi_\varepsilon - \psi_0\|_{W^{1+\alpha,2}} \leq C\|u_\varepsilon - u_0\|_{\mathcal{W}^{1+\alpha,2}}, \\ &\||\boldsymbol{A}_{\mathbf{H}}|^2(\psi_\varepsilon - \psi_0)\|_{L^2} \leq C\|\boldsymbol{A}_{\mathbf{H}}\|_{L^\infty}^2\|\psi_\varepsilon - \psi_0\|_{L^\infty} \\ &\qquad \leq C\|\psi_\varepsilon - \psi_0\|_{W^{1+\alpha,2}} \leq C\|u_\varepsilon - u_0\|_{\mathcal{W}^{1+\alpha,2}}, \end{aligned}$$

and the other terms are estimated similarly. Here, C is some (generic) positive constant, which may depend on $\boldsymbol{H}$ and Ω but not on u_ε or u_0. (In these inequalities we have used the continuity of the imbedding of $W^{1+\alpha,2}$ into $W^{1,2}\cap L^\infty$.) The result is an inequality of the type

$$\|f_0(u_\varepsilon) - f_0(u_0)\|_{L^2} \leq C\|u_\varepsilon - u_0\|_{\mathcal{W}^{1+\alpha,2}}, \tag{3.54}$$

showing that f_0 is Lipschitz from $\mathcal{W}^{1+\alpha,2}$ to $\mathcal{L}^2$.

Using the same types of estimates, we show that f_1 is bounded from $\mathcal{W}^{1+\alpha,2}$ to $\mathcal{L}^2$, so there exists a positive constant C such that

$$\|f_1(u_\varepsilon)\|_{L^2} \le C. \tag{3.55}$$

Combining the estimates (3.52), (3.54), and (3.55) with the inequality (3.51), we conclude that there exist positive constants C_1 and C_2 such that

$$\|v(t)\|_{\mathcal{W}^{1+\alpha,2}} \le \|v(0)\|_{\mathcal{W}^{1+\alpha,2}} + \varepsilon C_1 t^{(1-\alpha)/2} + C_2 \int_0^t (t-s)^{-(1+\alpha)/2} \|v(s)\|_{\mathcal{W}^{1+\alpha,2}} \,\mathrm{d}s. \tag{3.56}$$

Applying Gronwall's inequality, we obtain the estimate

$$\|v(t)\|_{\mathcal{W}^{1+\alpha,2}} \le C\left(\|v(0)\|_{\mathcal{W}^{1+\alpha,2}} + \varepsilon\right), \quad t \in [0,T], \tag{3.57}$$

for some positive constant C. ∎

It follows from Theorem 3.3 that, if the initial data are such that $\|u_\varepsilon(0) - u_0(0)\|_{\mathcal{W}^{1+\alpha,2}} = o(1)$ as $\varepsilon \downarrow 0$, then

$$\lim_{\varepsilon \to 0} u_\varepsilon = u_0 \tag{3.58}$$

in $C([0,T];\mathcal{W}^{1+\alpha,2})$ for any $T > 0$. In particular, if $\|u_\varepsilon(0) - u_0(0)\|_{\mathcal{W}^{1+\alpha,2}} = O(\varepsilon)$, then the convergence in Eq. (3.58) is $O(\varepsilon)$.

3.5 Summary of the Analytical Results

We relate the results of this section back to the dimensionless TDGL equations (2.19)–(2.22). First, the hypotheses about the data:

(H1) Ω is bounded in $\mathbf{R}^n$ $(n = 2, 3)$, with a sufficiently smooth boundary $\partial\Omega$; for example, $\partial\Omega$ of class $C^{1,1}$.

(H2) The parameters κ and σ are real and positive.

(H3) $\boldsymbol{H}$ is independent of time; as a function of position, it satisfies the regularity condition $\boldsymbol{H} \in [W^{\alpha,2}(\Omega)]^n$ for some $\alpha \in (\frac{1}{2}, 1)$.

The TDGL equations (2.19)–(2.22) define an initial-value problem for the pair $(\psi, \boldsymbol{A})$; at any time, the third variable ϕ is found from the gauge, $\phi = -\sigma^{-1}(\nabla \cdot \boldsymbol{A})$.

Suppose the initial data for ψ and $\boldsymbol{A}$ are $(\psi(0), \boldsymbol{A}(0))$. If these initial data are sufficiently smooth, the initial-value problem has a unique solution $(\psi(t), \boldsymbol{A}(t))$ for all $t > 0$. In particular, if $(\psi(0), \boldsymbol{A}(0)) \in \mathrm{dom}(A^{(1+\alpha)/2})$, where A is the linear operator associated with the quadratic form Q defined in Eq. (3.25), then $\psi \in C([0,T]; [W^{1+\alpha,2}(\Omega)]^2)$ and $\boldsymbol{A} \in C([0,T]; [W^{1+\alpha,2}(\Omega)]^n)$ for any $T > 0$. The solution is such that $\boldsymbol{A}$ is the sum of a constant vector $\boldsymbol{A}_{\mathbf{H}} \in [W^{1+\alpha,2}(\Omega)]^n$ and a time-varying vector. The former is the solution of the boundary-value problem

$$-\nabla \times \nabla \times \boldsymbol{A} + \nabla \times \boldsymbol{H} = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{A} = 0 \quad \text{in } \Omega, \tag{3.59}$$

$$\boldsymbol{n} \times (\nabla \times \boldsymbol{A}) = \boldsymbol{n} \times \boldsymbol{H}, \quad \boldsymbol{n} \cdot \boldsymbol{A} = 0 \quad \text{on } \partial\Omega. \tag{3.60}$$

It yields the magnetic field $\boldsymbol{B}_{\mathbf{H}} = \nabla \times \boldsymbol{A}_{\mathbf{H}}$ that would have been present in the absence of the superconductor and contributes a vector $\nabla \times \boldsymbol{B}_{\mathbf{H}}$ to the current density $\boldsymbol{J}$.

We consider the limiting case of weak coupling ($q_s \to 0$), when the applied field $\boldsymbol{H}$ is close to the upper critical field H_{c2}. This case is described more precisely by the following hypotheses.

(H4) The parameter κ satisfies the strong inequality $\kappa \gg 1$.

(H5) The applied field, vector potential, and scalar potential satisfy the asymptotic relations

$$\boldsymbol{H} = O(\kappa), \ \boldsymbol{A} = O(\kappa), \ \phi = O(\kappa^{-1}) \quad \text{as } \kappa \to \infty. \tag{3.61}$$

The order relation for $\boldsymbol{H}$ is consistent with the fact that $H_{c2} = \kappa$ in the system of units associated with Eqs. (2.19)–(2.22). The order relation for $\boldsymbol{A}$ is consistent with the fact that the induced magnetic field $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ must be of the same order as $\boldsymbol{H}$. The order relation for ϕ is consistent with the conservation law for the Cooper pairs. Notice that the ratio $\phi/(\nabla \cdot \boldsymbol{A}) = O(\kappa^{-2})$, so the scalar potential enters the asymptotic scene well after the vector potential.

We denote the solution of the TDGL equations by $(\psi_\kappa, \boldsymbol{A}_\kappa)$, to indicate its dependence on κ. The order parameter ψ_κ is $O(1)$. The vector potential $\boldsymbol{A}_\kappa$ is again the sum of the constant vector $\boldsymbol{A}_{\mathbf{H}}$ and a time-varying component. Because $\boldsymbol{H}$ and $\boldsymbol{A}_{\mathbf{H}}$ are of the same order, we have $\boldsymbol{A}_{\mathbf{H}} = O(\kappa)$.

The solution $(\psi_\kappa, \boldsymbol{A}_\kappa)$ can be compared asymptotically with the vector $(\psi_\infty, \boldsymbol{A}_\infty)$, where $\boldsymbol{A}_\infty = \boldsymbol{A}_{\mathbf{H}}$ and $\psi_\infty \in [W^{1+\alpha,2}(\Omega)]^2$ is the solution of the differential equation

$$\frac{\partial \psi}{\partial t} - \left(\nabla + \frac{i}{\kappa}\boldsymbol{A}_\infty\right)^2 \psi - (1 - |\psi|^2)\psi = 0, \tag{3.62}$$

subject to the boundary condition

$$\boldsymbol{n} \cdot \left(\nabla + \frac{i}{\kappa} \boldsymbol{A}_\infty \right) \psi = 0. \tag{3.63}$$

Since ψ_κ and $\boldsymbol{A}_\kappa$ are of different orders as $\kappa \to \infty$, the comparison is more conveniently made after a normalization. The natural way to normalize is to measure $\boldsymbol{B}_\kappa = \nabla \times \boldsymbol{A}_\kappa$ relative to the applied field $\boldsymbol{H}$: both are of order κ, so their ratio is of order one, the same as for ψ.

The vector $(\psi_\infty, \boldsymbol{B}_\infty)$ with $\boldsymbol{B}_\infty = \nabla \times \boldsymbol{A}_\infty$ is known as the "frozen-field approximation." The scalar potential associated with this approximation is zero. (Recall that $\nabla \cdot \boldsymbol{A}_{\mathbf{H}} = 0$.)

Theorem 3.4 *There exists a positive constant C such that*

$$\|\psi_\kappa(t) - \psi_\infty(t)\|_{W^{1+\alpha,2}} + \frac{\|\boldsymbol{B}_\kappa(t) - \boldsymbol{B}_\infty\|_{W^{\alpha,2}}}{\|\boldsymbol{H}\|_{W^{\alpha,2}}} \tag{3.64}$$

$$\leq C \left(\|\psi_\kappa(0) - \psi_\infty(0)\|_{W^{1+\alpha,2}} + \frac{\|\boldsymbol{B}_\kappa(0) - \boldsymbol{B}_\infty\|_{W^{\alpha,2}}}{\|\boldsymbol{H}\|_{W^{\alpha,2}}} + \frac{1}{\kappa^2} \right), \tag{3.65}$$

for all $t \in [0, T]$, $T > 0$.

It follows that $(\psi_\kappa, \boldsymbol{B}_\kappa)$ converges to $(\psi_\infty, \boldsymbol{B}_\infty)$ uniformly in t on $[0, T]$ in the topology of $[W^{1+\alpha,2}(\Omega)]^2 \times [W^{\alpha,2}(\Omega)]^n$ as soon as the initial data satisfy the asymptotic estimates $\|\psi_\kappa(0) - \psi_\infty(0)\|_{W^{1+\alpha,2}} = o(1)$ and $\|\boldsymbol{B}_\kappa(0) - \boldsymbol{B}_\infty\|_{W^{\alpha,2}} = o(\kappa)$ as $\kappa \to \infty$. Under slightly sharper conditions we obtain the order of convergence.

Corollary 3.1 *If*

$$\|\psi_\kappa(0) - \psi_\infty(0)\|_{W^{1+\alpha,2}} = O\left(\frac{1}{\kappa^2}\right) \quad \textit{and} \quad \frac{\|\boldsymbol{B}_\kappa(0) - \boldsymbol{B}_\infty\|_{W^{\alpha,2}}}{\|\boldsymbol{H}\|_{W^{\alpha,2}}} = O\left(\frac{1}{\kappa^2}\right)$$

as $\kappa \to \infty$, then

$$\|\psi_\kappa(t) - \psi_\infty(t)\|_{W^{1+\alpha,2}} + \frac{\|\boldsymbol{B}_\kappa(t) - \boldsymbol{B}_\infty\|_{W^{\alpha,2}}}{\|\boldsymbol{H}\|_{W^{\alpha,2}}} = O\left(\frac{1}{\kappa^2}\right), \tag{3.66}$$

uniformly on compact intervals.

The asymptotic approximation procedure can be continued to higher order, as can be seen from a formal expansion. The equations for the order parameter and the vector potential decouple, and at each order one finds first the vector potential, then the order parameter. The vector potential satisfies a linear heat equation; for example, the first correction beyond $\boldsymbol{A}_\infty$ is $\kappa^{-1}\boldsymbol{A}$, where $\boldsymbol{A}$ satisfies the equation

$$-\sigma\partial_t\boldsymbol{A} + \Delta\boldsymbol{A} = \Im\left[\psi_\infty^* \nabla_{\mathbf{A}_\infty}\psi_\infty\right]. \tag{3.67}$$

Remark. Before concluding this section, we note that our analysis differs at several points from the analysis of Ref. [10]. First, our scaling is slightly different and, we believe, more in tune with the physics; second, our regularity assumptions on the applied field are weaker; third, our proofs are more direct; and fourth, our results hold in a stronger topology.

4 Numerical Solution

A parallel code for solving Eqs. (3.2)–(3.4) has been developed as part of a project for large-scale simulations of vortex dynamics in superconductors. Details on these simulations and on the code will be published elsewhere; here, we give only a brief overview of the numerical methods and the results of some numerical simulations to illustrate the analytical results of the preceding section.

The algorithm uses finite differences on a staggered grid, making all approximations accurate to second order in the mesh widths, and an implicit method for the time integration, making the algorithm (essentially) unconditionally stable. The code, written in C++, has been designed for a multiprocessing environment; it uses MPI for message passing.

We restrict the discussion to rectangular two-dimensional configurations that are periodic in one direction and open in the other. The configurations are assumed to be infinite in the third, orthogonal direction, which is also the direction of the applied magnetic field, $\boldsymbol{H} = (0, 0, H_z)$.

4.1 Computational Grid

The computational grid is uniform, with equal mesh sizes in the x and y direction, $h_x = h_y = h$. A vertex on the grid is denoted by $\boldsymbol{x}_{i,j} = (x_i, y_j)$ and is the point of reference for the grid cell shown in Fig. 1. The indices run through the values

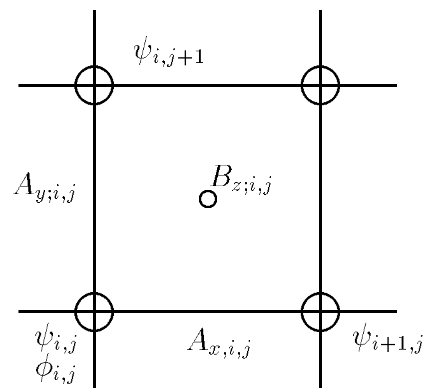

Figure 1: Computational grid cell and definition of the discrete variables.

$i = 1, \ldots, N_x$ and $j = 1, \ldots, N_y$. We assume periodicity in the x direction and take the grid so the vertices with $j = 1$ and $j = N_y$ are located on the open boundary of the superconductor. Thus, the size of the domain is $S = N_x(N_y - 1)h^2$.

4.2 Discrete Variables

The discrete variables are introduced so that all derivatives are given by second-order accurate central-difference approximations. The scalar variables ψ and ϕ are defined on the vertices of the grid,

$$\psi_{i,j} = \psi(\boldsymbol{x}_{i,j}), \quad \phi_{i,j} = \phi(\boldsymbol{x}_{i,j}). \tag{4.1}$$

(We use the same symbol for the original field and its discrete counterpart.) Vectors are defined at the midpoints of the links connecting adjacent vertices,

$$A_{x;i,j} = A_x(\boldsymbol{x}_{i,j} + \tfrac{1}{2}h_x\boldsymbol{e}_x), \quad A_{y;i,j} = A_y(\boldsymbol{x}_{i,j} + \tfrac{1}{2}h_y\boldsymbol{e}_y). \tag{4.2}$$

Here, $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ denote the unit vectors in the x and y direction, respectively. The definition of the discrete supercurrent $\boldsymbol{J}_s$ is completely analogous. The link variables, defined in Eq. (3.8), are obtained from the vector potential,

$$U_{x;i,j} = \mathrm{e}^{iA_{x;i,j}h_x}, \quad U_{y;i,j} = \mathrm{e}^{iA_{y;i,j}h_y}. \tag{4.3}$$

They are therefore also defined on the links. Finally, the magnetic induction $\boldsymbol{B}$, which is a vector perpendicular to the plane and given by the curl of the vector potential, is defined at the center of a grid cell,

$$B_{z;i,j} = B_z(\boldsymbol{x}_{i,j} + \tfrac{1}{2}h_x\boldsymbol{e}_x + \tfrac{1}{2}h_y\boldsymbol{e}_y). \tag{4.4}$$

The definition of the discrete variables is also illustrated in Fig. 1.

Note that, because of the location of the grid relative to the boundaries, all scalar variables, as well as the x components of all vectors (A_x, U_x, $J_{s,x}$, and so forth), are defined on a $N_x \times N_y$ grid, whereas the y components of all vectors and the magnetic induction B_z are defined on a $N_x \times (N_y - 1)$ grid.

4.3 Boundary Conditions

The boundary conditions are the discrete analogs of Eq. (3.7). We assume periodicity in the x direction, so we need to consider only the boundaries at $y = y_1$ and $y = y_{N_y}$.

The boundary condition for the order parameter, $\boldsymbol{n} \cdot \nabla_{\mathbf{A}}\psi = 0$, becomes

$$U_{y;i,1}\psi_{i,2} - \psi_{i,1} = 0, \quad \psi_{i,N_y} - U^*_{y;i,N_y-1}\psi_{i,N_y-1} = 0, \tag{4.5}$$

for $i = 1, \ldots, N_x$. For the vector potential, we require that $\partial_y A_x = H_z$ and A_y is constant ($A_y = 0$) on the boundary.

4.4 Operators

The gradient of a scalar is a vector and is therefore defined at the midpoint of a link connecting two adjacent vertices. Thus,

$$(\nabla\phi)_{x;i,j} = (\partial_x\phi)(\boldsymbol{x}_{i,j} + \tfrac{1}{2}h_x\boldsymbol{e}_x) = h_x^{-1}(\phi_{i+1,j} - \phi_{i,j}), \tag{4.6}$$

with an analogous definition for the y component. The gauge-invariant gradient $\nabla_{\mathbf{A}} = \nabla + i\boldsymbol{A}$ is defined in a similar way, with

$$(\nabla_{\mathbf{A}}\psi)_{x;i,j} = h_x^{-1}(\psi_{i+1,j}U_{x;i,j} - \psi_{i,j}). \tag{4.7}$$

Thus, the discrete version of the twisted Laplacian $\Delta_{\mathbf{A}}$ is

$$\begin{aligned}(\Delta_{\mathbf{A}}\psi)_{i,j} &= h_x^{-2}(\psi_{i+1,j}U_{x;i,j} - 2\psi_{i,j} + \psi_{i-1,j}U^*_{x;i-1,j}) \\ &\quad + h_y^{-2}(\psi_{i,j+1}U_{y;i,j} - 2\psi_{i,j} + \psi_{i,j-1}U^*_{x;i,j-1}).\end{aligned} \tag{4.8}$$

The discrete version of the (normal) Laplacian is defined in the usual way,

$$(\Delta\psi)_{i,j} = h_x^{-2}(\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}) + h_y^{-2}(\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}). \tag{4.9}$$

The magnetic induction, which is the curl of the vector potential, takes the form

$$B_{z;i,j} = h_x^{-1}(A_{y;i+1,j} - A_{y;i,j}) - h_y^{-1}(A_{x;i,j+1} - A_{x;i,j}). \tag{4.10}$$

We also need the divergence of the vector potential, which is given by

$$(\nabla \cdot \boldsymbol{A})_{i,j} = h_x^{-1}(A_{x;i,j} - A_{x;i-1,j}) + h_y^{-1}(A_{y;i,j} - A_{y;i,j-1}). \tag{4.11}$$

4.5 Algorithm

For numerical purposes, it is useful to think of the TDGL equations (3.2) and (3.3) as two separate equations, which are only coupled through certain fields and variables. The electromagnetic potentials ϕ and $\boldsymbol{A}$ are treated as static variables in the order parameter equation, which takes the form

$$(\partial_t - i\varepsilon\phi)\,\psi - \Delta_{\mathbf{A}}\psi - \left(1 - |\psi|^2\right)\psi = 0. \tag{4.12}$$

The local nonlinear part of this equation,

$$(\partial_t - i\varepsilon\Phi)\,\psi - \left(1 - |\psi|^2\right)\psi = 0, \tag{4.13}$$

is integrated in the simplest possible manner,

$$\psi_{i,j}(t + \Delta t) = \mathrm{e}^{-\varepsilon\phi_{i,j}\Delta t}\left\{\psi_{i,j}(t) + \Delta t\left(1 - |\psi_{i,j}|^2\right)\psi_{i,j}\right\}. \tag{4.14}$$

The nonlocal part,

$$\partial_t\psi - \Delta_{\mathbf{A}}\psi = 0, \tag{4.15}$$

is integrated using a backward Euler method, where the linear equation system is solved with the conjugate gradient method.

The equation for the vector potential,

$$\partial_t\boldsymbol{A} = \Delta\boldsymbol{A} + \varepsilon\boldsymbol{J}_s, \tag{4.16}$$

is linear and depends only indirectly on the order parameter through the supercurrent. If we treat the supercurrent as a static variable, we can integrate the equation easily, again using the backward Euler method. In the actual implementation, we use the fact that the equation is linear and the system is periodic to do a fast Fourier transform in the x direction. This leaves us with a tridiagonal system to solve in the y direction. This procedure is considerably faster than using an iterative method, such as the conjugate gradient method.

4.6 Numerical Results

The main result of the asymptotic analysis, which is summarized in Corollary 3.1, is that the solution for a system with a finite κ converges to the solution of the frozen-field approximation as $\kappa \to \infty$ with a convergence rate of the order of κ^{-2}. We illustrate this result numerically, using a rectangular sample that is periodic in the x direction and open in the y direction, with $N_x = N_y = 128$. We take $h_x = h_y = \frac{1}{2}\xi$, so the sample measures 64 coherence lengths in the periodic direction and 63.5 coherence lengths across. (The coherence length ξ is defined in Eq. (2.3).)

First, we considered this system with $\kappa = 200$ and an applied magnetic field that produced an almost perfect lattice, $H_z = 0.088\kappa$. With a relatively large value of κ, the surface barrier for vortex entry is low, and the system equilibrates relatively fast [16, 17]. The equilibration required 5×10^4 time steps with $\Delta t = 0.4$. A contour plot of the density of Cooper pairs $|\psi|^2$ at equilibrium is shown in Fig. 2. We then

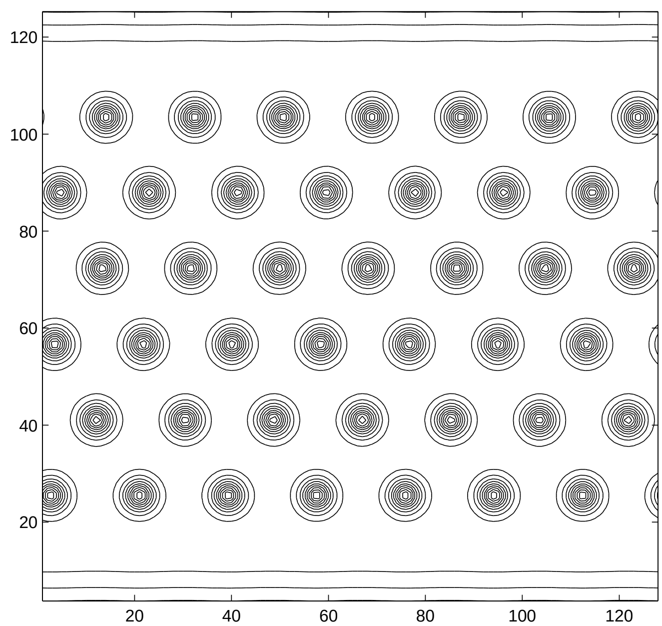

Figure 2: Contours of the density of Cooper pairs, $|\psi|^2$, for a system with $\kappa = 200$.

started from the configuration of Fig. 2 to find the equilibrium configuration for other values of κ, which required another 3×10^4 time steps. Because the purpose of the computations was to illustrate the results of the asymptotic analysis for $\kappa \to \infty$, we used only fairly large values of κ, varying κ from $\kappa_{\min} = 40$ to $\kappa_{\max} = 800$. In this range, the ground states are comparable and similar to the one shown in Fig. 2. Since the magnetization of a sample is proportional to $1/\kappa^2$, the vortex density decreases with κ; below $\kappa_{\min}$, the equilibrium state has fewer vortices, and a comparison becomes meaningless.

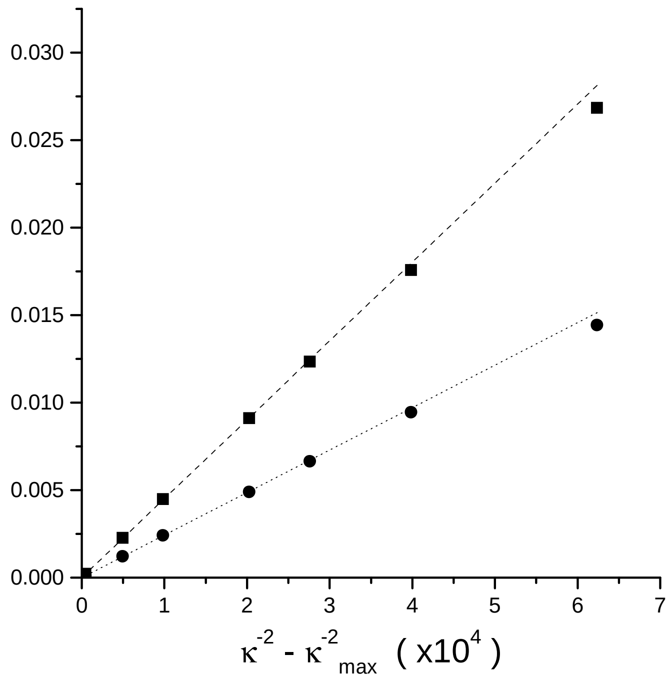

Figure 3: The quantities $\delta\psi$ (solid squares) and δB_z (solid discs) for $\kappa = 40, 50, 60, 70, 100, 140, 400, 800$. The straight lines correspond to the asymptotic $1/\kappa^2$ behavior.

Figure 3 gives the computed values of the quantities

$$\delta\psi = \|\psi_\kappa - \psi_{\kappa_{\max}}\|_{L^2}, \quad \delta B_z = \frac{\|B_{z,\kappa} - B_{z,\kappa_{\max}}\|_{L^2}}{\|H_z\|_{L^2}}, \tag{4.17}$$

for different values of κ. The agreement with the predicted $1/\kappa^2$ behavior is perfect down to $\kappa \approx 40$.

Figure 4 shows the average $< A_{x,\kappa} - A_{x,\infty} >$, taken over x, as a function of y in the bulk of the sample, for different values of κ.

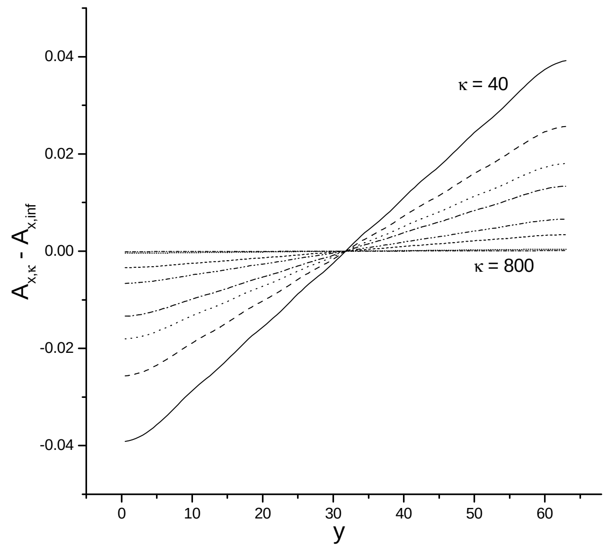

Figure 4: The average $< A_{x,\kappa} - A_{x,\infty} >$ vs. y for $\kappa = 40, 50, 60, 70, 100, 140, 400, 800$.

4.7 Discussion of the Numerical Results

The results of the numerical simulations confirm the quadratic convergence rates of the order parameter and the magnetic induction as $\kappa \to \infty$. The asymptotic behavior becomes manifest already around $\kappa = 40$. Given the fact that superconducting materials have a Ginzburg–Landau parameter of order 50–100, we may conclude that the frozen-field approximation is a practical alternative in many applications.

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