Argonne National Laboratory
9700 South Cass Avenue
Argonne, Illinois 60439

ANL/MCS-TM-156

Proofs for Group and Abelian Group Single Axioms*

by

William W. McCune

Mathematics and Computer Science Division

Technical Memorandum No. 156

October 1991

*This work was supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy, under Contract W-31-109-Eng-38.

Proofs for Group and Abelian Group Single Axioms

William W. McCune

Abstract

This memorandum serves as a companion to the paper "Single axioms for groups and Abelian groups with various operations". That paper presents single axioms for groups and Abelian groups in terms of {product, inverse}, {division}, {double division, identity}, {double division, inverse}, {division, identity}, and {division, inverse}. Proofs that were omitted from that paper are presented here.

1 Introduction

In another paper [2], we summarized the results of an investigation into single axioms for groups and Abelian groups in terms of the following six sets of operations: {product, inverse}, {division}, {double division, identity}, {double division, inverse}, {division, identity}, and {division, inverse}. One or more new single axioms are presented for each of those twelve theories. However, [2] contains just two of proofs; we present the remaining proofs here.

Throughout both papers, we use α/β for (right) division, $\alpha \cdot \beta^{-1}$; and we use $\alpha \parallel \beta$ for double division, $\alpha^{-1} \cdot \beta^{-1}$. The following single axioms are presented in [2]. (The numbering of the equalities is the same as in [2].)

Ordinary Groups:

$$(x \cdot (y \cdot (((z \cdot z^{-1}) \cdot (u \cdot y)^{-1}) \cdot x))^{-1}) = u \text{ [2, Theorem 1]} \tag{3.1.1}$$
$$(((((x \cdot y) \cdot z)^{-1} \cdot x) \cdot y) \cdot (u \cdot u^{-1}))^{-1} = z \tag{3.1.2}$$
$$(x/((((y/y)/y)/z)/(((y/y)/x)/z))) = y \tag{3.1.3}$$
$$(((x/x)/(x/(y/(((x/x)/x)/z))))/z) = y \tag{3.1.4}$$
$$((x \parallel (((x \parallel y) \parallel z) \parallel (y \parallel e))) \parallel (e \parallel e)) = z \tag{3.1.5}$$
$$(x^{-1} \parallel ((x \parallel (y \parallel z))^{-1} \parallel (u \parallel (y \parallel u)))^{-1}) = z \tag{3.1.6}$$
$$((e/(x/(y/(((x/x)/x)/z))))/z) = y \tag{3.1.7}$$
$$((x/x)/(y/((z/(u/y))/u^{-1}))) = z \tag{3.1.8}$$

Abelian Groups:

$$(((x \cdot y) \cdot z) \cdot (x \cdot z)^{-1}) = y \text{ [2, Theorem 2]} \tag{3.2.1}$$
$$(x \cdot (((x \cdot y)^{-1} \cdot z) \cdot y)) = z \tag{3.2.2}$$
$$(x \cdot ((y \cdot z) \cdot (x \cdot z)^{-1})) = y \tag{3.2.3}$$
$$(x/((x/y)/(z/y))) = z \tag{3.2.4}$$
$$((x/((x/y)/z))/y) = z \tag{3.2.5}$$
$$((x/y)/((x/z)/y)) = z \tag{3.2.6}$$
$$((x \parallel ((z \parallel (x \parallel y)) \parallel (e \parallel y))) \parallel (e \parallel e)) = z \tag{3.2.7}$$
$$(x \parallel (((x \parallel y) \parallel z^{-1})^{-1} \parallel y)^{-1}) = z \tag{3.2.8}$$
$$((e/(((x/y)/z)/x))/z) = y \tag{3.2.9}$$
$$((x/(y/(x/z))^{-1})/z) = y \tag{3.2.10}$$

The theorem-proving program OTTER [1, 3] was used to construct sets of candidate axioms and to search for proofs that candidates are single axioms. The methodology is presented in [2].

OTTER frequently finds proofs that are much longer and more complex than need be, especially when back demodulation has a role. Most of the proofs presented here are not the first that were found; we used several tricks to find shorter proofs, including fine-tuning the weight limits, using the ratio strategy [3] to combine shortest-first and breadth-first searches, and disabling back demodulation.

Each of the theorems presented here states that a particular formula is a single axiom for a theory. In some of the cases, we derive a known single axiom for the theory. In other cases, we derive sufficient properties of product, inverse, and identity. If identity is not part of the theory, it can be defined, if it can be shown to exist, as $e = x/x$ or as $e = x \parallel x^{-1}$. If inverse is not part of the theory, it can be defined as $x^{-1} = e/x$ or as $x^{-1} = e \parallel x$. Product can be defined as $x \cdot y = x/(e/y)$ or as $x \cdot y = x^{-1} \parallel y^{-1}$.

Theorems 1 and 2 are in [2]; this memorandum starts with Theorem 3. The numbers within the OTTER derivations are not consecutive, because they reflect the sequence of retained equalities. Where an equality has two numbers, the first indicates its use as a simplifier, and the second its use for paramodulation. The justification $m \to n$ indicates substitution of an instance of the left side of m for an instance of a term in the left side of n, and $: m, n, \cdots$ indicates simplification with $m, n, \cdots$. The terms a, b, c, and d, which are present when a refutation is derived, are Skolem constants.

2 Ordinary Groups

THEOREM 3. *The theory of groups can be defined by the single axiom*

$$(((((x \cdot y) \cdot z)^{-1} \cdot x) \cdot y) \cdot (u \cdot u^{-1}))^{-1} = z. \tag{3.1.2}$$

Proof. Eq. (3.1.2) holds in groups. Consider the following derivation starting with (3.1.2).

8	$(((((x\cdot y)\cdot z)^{-1}\cdot x)\cdot y)\cdot(u\cdot u^{-1}))^{-1}=z$	[(3.1.2)]
10	$(((x\cdot(((y\cdot z)\cdot x)^{-1}\cdot y))\cdot z)\cdot(u\cdot u^{-1}))^{-1}=(v\cdot v^{-1})$	[8→8]
11	$(x\cdot x^{-1})=(y\cdot y^{-1})$	[10→10]
15,14	$((((x\cdot x^{-1})^{-1}\cdot y)\cdot z)\cdot(u\cdot u^{-1}))^{-1}=(y\cdot z)^{-1}$	[11→8]
16	$(((((x\cdot x^{-1})\cdot y)^{-1}\cdot z)\cdot z^{-1})\cdot(u\cdot u^{-1}))^{-1}=y$	[11→8]
19	$(((x\cdot x^{-1})\cdot y)\cdot(z\cdot z^{-1}))^{-1}=((u\cdot u^{-1})^{-1-1}\cdot y)^{-1}$	[11→14]
23	$((((x\cdot y)\cdot(z\cdot z^{-1})^{-1})^{-1}\cdot x)\cdot y)^{-1}=(u\cdot u^{-1})$	[14→10]
24	$(y\cdot y^{-1})^{-1}=(u\cdot u^{-1})^{-1}$	[11→16:15]
26	$((x\cdot x^{-1})\cdot(y\cdot y^{-1})^{-1})^{-1}=(z\cdot z^{-1})^{-1}$	[24→24]
29	$((x\cdot x^{-1})\cdot(y\cdot y^{-1})^{-1})=(z\cdot z^{-1})$	[24→11]
32	$(((x\cdot x^{-1})\cdot(y\cdot y^{-1})^{-1})\cdot(z\cdot z^{-1})^{-1})=(u\cdot u^{-1})$	[24→29]
33	$((x\cdot x^{-1})\cdot((y\cdot y^{-1})\cdot(z\cdot z^{-1})^{-1})^{-1})=(u\cdot u^{-1})$	[24→29]
35	$(((y\cdot y^{-1})^{-1}\cdot z)\cdot z^{-1})^{-1}=(v\cdot v^{-1})$	[29→10:15]
39	$((((x\cdot x^{-1})\cdot(y\cdot y^{-1})^{-1})\cdot z)\cdot(u\cdot u^{-1}))^{-1}=z^{-1}$	[35→8]
44	$(((x\cdot x^{-1})\cdot(y\cdot y^{-1})^{-1})\cdot(z\cdot z^{-1}))^{-1}=(u\cdot u^{-1})$	[35→23]
45	$(((((x\cdot y)\cdot(z\cdot z^{-1})^{-1})^{-1}\cdot x)\cdot y)\cdot(u\cdot u^{-1}))=(v\cdot v^{-1})$	[23→11]
49,48	$(((x\cdot x^{-1})\cdot y)\cdot(z\cdot z^{-1}))^{-1}=y^{-1}$	[33→39]
51	$((x\cdot x^{-1})\cdot(y\cdot y^{-1}))^{-1}=(z\cdot z^{-1})^{-1-1}$	[32→39]
52	$(y\cdot y^{-1})^{-1-1}=(u\cdot u^{-1})$	[44:49]
53	$((u\cdot u^{-1})^{-1-1}\cdot y)^{-1}=y^{-1}$	[19:49]
83,82	$((x\cdot x^{-1})\cdot y)^{-1}=y^{-1}$	[52→53]
86	$(y\cdot y^{-1})^{-1}=(z\cdot z^{-1})^{-1-1}$	[51:83]
90	$(y\cdot y^{-1})^{-1-1}=(z\cdot z^{-1})^{-1}$	[26:83]
94,93	$(((y^{-1}\cdot z)\cdot z^{-1})\cdot(u\cdot u^{-1}))^{-1}=y$	[16:83]
126,125	$((((x\cdot x^{-1})^{-1-1}\cdot y)\cdot z)\cdot(u\cdot u^{-1}))^{-1}=(y\cdot z)^{-1}$	[82→14]
141	$(x\cdot x^{-1})=(u\cdot u^{-1})^{-1-1}$	[35→86:94]
157	$((y\cdot y^{-1})^{-1}\cdot z)^{-1}=z^{-1}$	[141→39:126]
197	$(x\cdot x^{-1})^{-1}=(y\cdot y^{-1})$	[90→52]
308	$(((x\cdot x^{-1})\cdot y)\cdot((z\cdot z^{-1})^{-1}\cdot(u\cdot u^{-1})))^{-1}=y^{-1}$	[52→48]
352,351	$((u\cdot u^{-1})^{-1}\cdot x)=x$	[157→93:94]
354,353	$((u\cdot u^{-1})\cdot x)=x$	[82→93:94]
358,357	$(x\cdot(v\cdot v^{-1}))=x$	[48→93:94,354]
364	$((x\cdot y)\cdot y^{-1})^{-1}=((((u\cdot v)\cdot x)^{-1}\cdot u)\cdot v)$	[8→93:358,358]
366,365	$x^{-1-1}=x$	[197→93:358,358,358]
367	$((x^{-1}\cdot z)\cdot z^{-1})^{-1}=x$	[157→93:352,358]
390,389	$(y\cdot(z\cdot z^{-1})^{-1})^{-1}=y^{-1}$	[308:354,358]
405	$(((x\cdot y)^{-1}\cdot x)\cdot y)=(v\cdot v^{-1})$	[45:390,358]
444,443	$((x\cdot y)\cdot y^{-1})^{-1}=x^{-1}$	[365→367]
449	$((((z\cdot u)\cdot x)^{-1}\cdot z)\cdot u)=x^{-1}$	[364:444]
474	$((x\cdot y)\cdot y^{-1})=x$	[443→365:366]
482	$((z\cdot y)^{-1}\cdot z)=y^{-1}$	[405→474:354]
493,492	$(y\cdot x)^{-1}=(x^{-1}\cdot y^{-1})$	[482→474]

498	$(((z^{-1}\cdot(y^{-1}\cdot x^{-1}))\cdot x)\cdot y)=z^{-1}$	[449:493,493]
505	$((x\cdot(y\cdot z))\cdot(z^{-1}\cdot y^{-1}))=x$	[492→474]
531	$((x\cdot(y^{-1}\cdot z))\cdot(z^{-1}\cdot y))=x$	[365→505]
651	$(((x\cdot(y^{-1}\cdot z^{-1}))\cdot z)\cdot y)=x$	[365→498:366]
671	$((x\cdot y)\cdot z)=(x\cdot(y\cdot z))$	[531→651:366]

Eq. 11 asserts the existence of a unique element e such that for all x, $(x\cdot x^{-1})=e$, from 358 it follows that e is a right identity, and 671 asserts the associativity of product. ∎

THEOREM 4. *The theory of groups can be defined by the single axiom (in terms of division)*

$$(x/((((y/y)/y)/z)/(((y/y)/x)/z)))=y. \tag{3.1.3}$$

Proof. Eq. (3.1.3) holds in groups. Consider the following derivation starting with (3.1.3).

3	$(x/((((y/y)/y)/z)/(((y/y)/x)/z)))=y$	[(3.1.3)]
7	$(z/((((u/u)/u)/((((x/x)/x)/y)/(((x/x)/((u/u)/z))/y)))/x))=u$	[3→3]
9	$(u/(z/(((y/y)/u)/((((z/z)/z)/x)/(((z/z)/((y/y)/y))/x)))))=y$	[3→3]
44,43	$(y/(x/x))=y$	[3→7]
99	$(x/(((y/y)/y)/((y/y)/x)))=y$	[43→9]
123	$(((x/x)/x)/(((y/y)/y)/x))=y$	[99→99:44,44]
126,125	$((y/y)/((x/x)/x))=x$	[43→99:44]
127	$((u/((x/x)/((((u/u)/u)/z)/(x/z))))/(((y/y)/y)/x))=y$	[9→99:126,44]
133	$((z/(((z/z)/y)/(x/y)))/x)=z$	[9→99:44,126,44,44,126]
239	$(y/y)=(x/x)$	[43→125:44,44]
274,273	$((z/z)/((y/y)/x))=x$	[239→125]
465	$(((x/x)/x)/(y/x))=((z/z)/y)$	[273→123]
510	$((y/((z/z)/x))/x)=y$	[239→133:44]
513,512	$((z/x)/((y/y)/x))=z$	[273→510]
543	$(u/((z/z)/((((u/u)/u)/y)/(((z/z)/x)/y))))=x$	[512→127:513,44]
571,570	$((u/u)/(x/y))=(y/x)$	[512→465:513,274]
587	$(u/((((z/z)/y)/x)/(((u/u)/u)/x)))=y$	[543:571]

Eq. 587 is a generalization of single axiom (1.1) in [2]. ∎

THEOREM 5. *The theory of groups can be defined by the single axiom (in terms of division)*

$$(((x/x)/(x/(y/(((x/x)/x)/z))))/z)=y. \tag{3.1.4}$$

Proof. Eq. (3.1.4) holds in groups. The following refutation starts with (3.1.4) and the denial of single axiom (1.1) in [2].

2	$(b/((((b/b)/c)/a)/(((b/b)/b)/a))) \neq c$	[denial of (1.1)]
5,4	$(((x/x)/(x/(y/(((x/x)/x)/z))))/z) = y$	[(3.1.4)]
6	$((u/u)/(u/(y/(((u/u)/u)/(((z/z)/z)/x))))) = (((z/z)/(z/y))/x)$	[4→4]
12	$((((y/y)/(y/z))/x)/(((y/y)/y)/x)) = z$	[6→4]
14	$(((((x/x)/(x/x))/((x/x)/y))/(((x/x)/x)/(x/x)))/x) = y$	[12→12]
18,17	$(y/((((y/y)/(y/y))/(y/y))/(((y/y)/y)/((y/y)/x)))) = x$	[12→12]
25	$(((y/y)/x)/((y/y)/x)) = (((y/y)/(y/y))/(y/y))$	[17→4]
26	$(x/x) = (((y/y)/(y/y))/(y/y))$	[17→25:18]
27	$(((x/x)/(x/x))/(x/x)) = (((y/y)/(y/y))/(y/y))$	[26→26]
29,28	$(((((z/z)/(z/z))/(z/z))/x)/(((y/y)/y)/x)) = y$	[26→12]
34	$(y/y) = (u/u)$	[27→28:29]
37	$(((x/x)/(x/x))/(x/x)) = (y/y)$	[26→28]
48	$((((z/z)/((x/x)/y))/(((x/x)/x)/(x/x)))/x) = y$	[34→14]
54	$(((z/z)/x)/(((y/y)/y)/x)) = y$	[34→12]
62	$((y/y)/(y/(x/x))) = ((y/y)/y)$	[34→6:5]
70	$((((y/y)/(y/u))/x)/(((z/z)/y)/x)) = u$	[34→12]
83	$(b/((((b/b)/c)/a)/(((x/x)/b)/a))) \neq c$	[34→2]
90	$((y/y)/(x/x)) = (z/z)$	[34→37]
96,95	$(((u/u)/x)/(((z/z)/y)/x)) = y$	[34→54]
99	$(((z/z)/x)/((y/y)/x)) = (u/u)$	[90→54]
113	$((z/z)/(y/(x/x))) = ((y/y)/y)$	[34→62]
120	$(((u/u)/(((z/z)/x)/y))/x) = y$	[95→95]
126	$(y/(((v/v)/u)/(((z/z)/y)/(x/x)))) = u$	[95→95]
137	$((v/v)/(((u/u)/z)/((y/y)/(x/x)))) = z$	[99→95]
146,145	$(y/(u/u)) = y$	[113→95:96]
149,148	$((v/v)/((u/u)/z)) = z$	[137:146,146]
150	$(y/(((v/v)/u)/((z/z)/y))) = u$	[126:146]
155	$((y/((x/x)/x))/x) = y$	[48:146,149]
164	$((u/x)/((y/y)/x)) = u$	[99→155:149]
166	$((z/((y/y)/x))/x) = z$	[34→155]
174,173	$((v/v)/(x/y)) = (y/x)$	[120→166:149]
179	$(((u/y)/x)/(((z/z)/y)/x)) = u$	[70:174]
187	$((z/(x/y))/(y/x)) = z$	[173→166]
203,202	$(x/((u/z)/((y/y)/x))) = (z/u)$	[187→150]
251,250	$((v/x)/(y/x)) = (v/y)$	[164→179:174,146]
255	$c \neq c$	[83:251,203,146]

Therefore, (1.1) can be derived from (3.1.4). ∎

THEOREM 6. *The theory of groups can be defined by the single axiom (in terms of double division and identity)*

$$((x \parallel (((x \parallel y) \parallel z) \parallel (y \parallel e))) \parallel (e \parallel e)) = z. \tag{3.1.5}$$

Proof. Eq. (3.1.5) holds in groups. Eq. (1) is the mirror image of (2.2) in [2], and it is therefore a single axiom:

$$(y \parallel (((e \parallel x) \parallel (((e \parallel u) \parallel u) \parallel z)) \parallel ((e \parallel y) \parallel x))) = z. \tag{1}$$

The following refutation starts with (3.1.5) and the denial of (1).

3	$(a \parallel (((e \parallel b) \parallel (((e \parallel c) \parallel c) \parallel d)) \parallel ((e \parallel a) \parallel b))) \neq d$	[denial of (1)]
5,4	$((x \parallel (((x \parallel y) \parallel z) \parallel (y \parallel e))) \parallel (e \parallel e)) = z$	[(3.1.5)]
9,8	$((((x \parallel e) \parallel z) \parallel y) \parallel (z \parallel e)) = ((x \parallel y) \parallel (e \parallel e))$	[4→4]
11	$(x \parallel (((y \parallel x) \parallel (e \parallel e)) \parallel e)) = ((y \parallel (e \parallel e)) \parallel (e \parallel e))$	[4→8:9]
13	$(((x \parallel e) \parallel ((x \parallel y) \parallel (e \parallel e))) \parallel (e \parallel e)) = y$	[8→4]
15	$(((x \parallel e) \parallel y) \parallel (e \parallel e)) = (((x \parallel z) \parallel y) \parallel (z \parallel e))$	[4→13]
18	$(((x \parallel y) \parallel z) \parallel (y \parallel e)) = (((x \parallel u) \parallel z) \parallel (u \parallel e))$	[15→15]
20,19	$(((y \parallel z) \parallel ((y \parallel x) \parallel (e \parallel e))) \parallel (z \parallel e)) = x$	[13→15]
25	$((e \parallel e) \parallel ((x \parallel (e \parallel e)) \parallel e)) = (x \parallel (e \parallel e))$	[4→11:5]
28,27	$((e \parallel e) \parallel (x \parallel e)) = x$	[19→25:20]
31	$(((e \parallel x) \parallel e) \parallel (x \parallel e)) = e$	[27→19]
34,33	$((e \parallel (e \parallel e)) \parallel (e \parallel e)) = (e \parallel (e \parallel e))$	[27→11]
35	$((e \parallel (y \parallel (e \parallel e))) \parallel (e \parallel e)) = (y \parallel e)$	[27→19:9]
42	$(((e \parallel e) \parallel ((x \parallel y) \parallel ((x \parallel e) \parallel e))) \parallel (e \parallel e)) = y$	[27→4]
53	$((((e \parallel (e \parallel e)) \parallel x) \parallel (e \parallel (e \parallel e))) \parallel (x \parallel e)) = (e \parallel e)$	[33→19:34]
61	$(((e \parallel e) \parallel e) \parallel e) = e$	[31→35:28]
64,63	$((e \parallel e) \parallel e) = e$	[27→35:28]
65	$(((y \parallel e) \parallel ((y \parallel x) \parallel (e \parallel e))) \parallel e) = ((e \parallel x) \parallel (e \parallel e))$	[19→35]
73	$(((e \parallel e) \parallel x) \parallel e) = x$	[4→35]
76,75	$(e \parallel e) = e$	[61:64]
78,77	$((e \parallel x) \parallel e) = x$	[73:76]
82	$(((x \parallel e) \parallel ((x \parallel y) \parallel e)) \parallel e) = y$	[65:76,76,78]
84	$(x \parallel (x \parallel e)) = e$	[53:76,76,76,76,78,76]
91	$((x \parallel y) \parallel ((x \parallel e) \parallel e)) = y$	[42:76,76,78]
101	$(x \parallel (((y \parallel x) \parallel e) \parallel e)) = ((y \parallel e) \parallel e)$	[11:76,76,76]
109	$(((e \parallel y) \parallel x) \parallel (y \parallel e)) = x$	[75→18:76,78]
113,112	$((((e \parallel x) \parallel z) \parallel y) \parallel (z \parallel e)) = ((x \parallel y) \parallel e)$	[77→18:76]
115,114	$((e \parallel x) \parallel x) = e$	[77→84]
116	$(a \parallel (((e \parallel b) \parallel (e \parallel d)) \parallel ((e \parallel a) \parallel b))) \neq d$	[3:115]
117	$((e \parallel x) \parallel ((y \parallel e) \parallel e)) = (((y \parallel z) \parallel x) \parallel (z \parallel e))$	[84→18]
119	$((((x \parallel y) \parallel z) \parallel (y \parallel e)) \parallel ((((x \parallel u) \parallel z) \parallel e) \parallel e)) = (u \parallel e)$	[18→91]
122	$(((e \parallel (e \parallel x)) \parallel y) \parallel x) = y$	[77→109]
131,130	$((x \parallel e) \parallel e) = x$	[122→82:113,78]
138	$((((x \parallel y) \parallel z) \parallel (y \parallel e)) \parallel ((x \parallel u) \parallel z)) = (u \parallel e)$	[119:131]
141,140	$(((y \parallel z) \parallel x) \parallel (z \parallel e)) = ((e \parallel x) \parallel y)$	[117:131]
143,142	$(x \parallel (y \parallel x)) = y$	[101:131,131]
149	$((e \parallel z) \parallel (e \parallel x)) = ((x \parallel z) \parallel e)$	[112:141]
150	$(((e \parallel z) \parallel x) \parallel ((x \parallel u) \parallel z)) = (u \parallel e)$	[138:141]
184	$((e \parallel x) \parallel ((x \parallel y) \parallel e)) = (e \parallel y)$	[149→142]
829,828	$(((e \parallel x) \parallel (e \parallel y)) \parallel ((e \parallel z) \parallel x)) = (y \parallel z)$	[184→150:131]
840	$d \neq d$	[116:829,143]

Therefore, (1) can be derived from (3.1.5). ∎

Theorem 7. *The theory of groups can be defined by the single axiom (in terms of double division and inverse)*

$$(x^{-1} \Vert ((x \Vert (y \Vert z))^{-1} \Vert (u \Vert (y \Vert u)))^{-1}) = z. \tag{3.1.6}$$

Proof. Eq. (3.1.6) holds in groups. The proof is constructed from two Otter derivations.

5	$(x^{-1} \Vert ((x \Vert (y \Vert z))^{-1} \Vert (u \Vert (y \Vert u)))^{-1}) = z$	[(3.1.6)]
7	$(x^{-1} \Vert ((x \Vert y)^{-1} \Vert (z \Vert (u^{-1} \Vert z)))^{-1}) = ((u \Vert (v \Vert y))^{-1} \Vert (w \Vert (v \Vert w)))^{-1}$	[5→5]
12,11	$((y \Vert (z \Vert (y^{-1} \Vert x)))^{-1} \Vert (u \Vert (z \Vert u)))^{-1} = x$	[5→7]
17	$((x \Vert (y \Vert (x^{-1} \Vert z)))^{-1-1} \Vert (z \Vert (u \Vert (v \Vert u)))^{-1}) = (y \Vert v)$	[11→5]
22,21	$((x \Vert (y \Vert (x^{-1} \Vert (z \Vert (u \Vert (z^{-1} \Vert v)))^{-1})))^{-1-1} \Vert v) = (y \Vert u)$	[11→17]
23	$(((x \Vert (y \Vert (x^{-1} \Vert z)))^{-1} \Vert (u \Vert (y \Vert z^{-1})))^{-1-1} \Vert v) = (u \Vert v)$	[17→21]
25	$((x \Vert (y \Vert (((x \Vert z)^{-1} \Vert (u \Vert z))^{-1} \Vert (v \Vert (u \Vert v)))^{-1}))^{-1-1} \Vert w) = (y \Vert w)$	[7→21]
27	$(y \Vert ((w \Vert (v6 \Vert (w^{-1} \Vert v7)))^{-1} \Vert (u \Vert (v6 \Vert v7^{-1})))^{-1-1}) = (y \Vert u)$	[23→21:22]
30,29	$((v \Vert (w \Vert (v^{-1} \Vert v6)))^{-1} \Vert (z \Vert (w \Vert v6^{-1})))^{-1-1} = z$	[27→11:12]
35	$((x \Vert (y \Vert (z \Vert (u \Vert ((x^{-1} \Vert z) \Vert u)))^{-1}))^{-1-1} \Vert v) = (y \Vert v)$	[11→25]
37	$((x \Vert (y \Vert z))^{-1-1} \Vert u) = ((v \Vert (y \Vert (v^{-1} \Vert ((x \Vert w)^{-1} \Vert (z \Vert w))^{-1})))^{-1-1} \Vert u)$	[17→25]
41	$(v \Vert (u \Vert (w \Vert (v6 \Vert ((v^{-1} \Vert w) \Vert v6)))^{-1}))^{-1-1} = u$	[35→29:30]
44,43	$(x \Vert (y \Vert (((x \Vert (z \Vert u))^{-1} \Vert (v \Vert (z \Vert v)))^{-1} \Vert (w \Vert (u \Vert w)))^{-1}))^{-1-1} = y$	[5→41]
45	$(x \Vert (y \Vert (x^{-1} \Vert z)))^{-1-1} = (u \Vert (y \Vert (u^{-1} \Vert z)))^{-1-1}$	[17→41]
58	$(y \Vert (z \Vert (y^{-1} \Vert ((u \Vert v)^{-1} \Vert (w \Vert v))^{-1})))^{-1-1} = (u \Vert (z \Vert w))^{-1-1}$	[37→43:44]
64	$((x \Vert (y \Vert z))^{-1-1} \Vert (((x \Vert u)^{-1} \Vert (z \Vert u))^{-1} \Vert (v \Vert (w \Vert v)))^{-1}) = (y \Vert w)$	[58→17]
66	$((x \Vert (y \Vert (z \Vert (u \Vert z))))^{-1} \Vert (v \Vert (y \Vert v)))^{-1} = (u \Vert x^{-1})$	[7→64]
84	$(x^{-1} \Vert (y \Vert x^{-1})) = (v \Vert (y \Vert v))$	[66→7:12]
86	$(x \Vert (y \Vert x)) = (v7 \Vert (y \Vert v7))$	[43→84:44]
88	$((x \Vert y) \Vert (z \Vert (x \Vert z))) = (u \Vert (y \Vert u))$	[86→86]
91,90	$((x \Vert (y \Vert (z \Vert y)))^{-1} \Vert (u \Vert ((v \Vert z) \Vert u)))^{-1} = (v \Vert x^{-1})$	[86→66]
114,113	$(x^{-1} \Vert ((y \Vert (z \Vert y))^{-1} \Vert (u \Vert (z \Vert u)))^{-1}) = x$	[86→5]
155,154	$(x^{-1} \Vert (z \Vert (y \Vert (z \Vert y))^{-1})) = x$	[88→113:91]
161,160	$(z \Vert (x \Vert (z^{-1} \Vert (u \Vert (y \Vert u))^{-1})))^{-1} = (x \Vert y)$	[21→113:114]
201	$(x \Vert y)^{-1-1} = (y^{-1} \Vert x^{-1})^{-1}$	[154→45:161]
274,273	$(y \Vert x)^{-1} = (x^{-1} \Vert y^{-1})$	[201→154:155]
285	$(x^{-1} \Vert (y \Vert ((z^{-1} \Vert y^{-1}) \Vert z^{-1}))) = x$	[154:274,274]
300	$(x^{-1} \Vert (y \Vert ((z^{-1} \Vert z^{-1-1}) \Vert y))) = x$	[86→285]
304	$((x^{-1} \Vert y^{-1}) \Vert (z \Vert ((u^{-1} \Vert u^{-1-1}) \Vert z))) = (y \Vert x)$	[273→300]
310	$(x^{-1} \Vert ((y \Vert (z^{-1} \Vert z^{-1-1})) \Vert (u \Vert (y \Vert u)))) = x$	[86→300]
312	$(((y^{-1} \Vert (z^{-1-1-1} \Vert z^{-1-1})) \Vert y^{-1}) \Vert x^{-1-1}) = x^{-1}$	[300→273:274,274,274]
316	$(x \Vert (y \Vert (x^{-1} \Vert y))) = (z \Vert ((u \Vert ((v^{-1} \Vert v^{-1-1}) \Vert u)) \Vert z))$	[300→88]
390	$(x^{-1} \Vert ((y \Vert ((z^{-1} \Vert z^{-1-1}) \Vert y)) \Vert (u^{-1} \Vert u))) = x$	[304→300]
410,409	$(x^{-1} \Vert (y \Vert (((z^{-1} \Vert (u^{-1-1-1} \Vert u^{-1-1})) \Vert z^{-1}) \Vert y))) = x$	[86→390:274,274,274]
431,430	$(v \Vert (x^{-1-1} \Vert v)) = x$	[312→88:410]
487	$(x^{-1} \Vert (y \Vert y^{-1})) = x$	[430→310:431]
534,533	$x^{-1-1} = x$	[430→487]
537,536	$(x \Vert (y \Vert x)) = y$	[86→487]

569	$(x \parallel x^{-1}) = (v^{-1} \parallel v)$	[316:537,534,537,537]

By 569, there exists a unique element e such that $(x \parallel x^{-1}) = (x^{-1} \parallel x) = e$. Define $(x \cdot y) = (x^{-1} \parallel y^{-1})$. By 534, we have

$$(x \parallel y) = (x^{-1} \cdot y^{-1}). \tag{2}$$

The second derivation starts with (3.1.6) and some additional properties.

5	$(x^{-1} \parallel ((x \parallel (y \parallel z))^{-1} \parallel (u \parallel (y \parallel u)))^{-1}) = z$	[(3.1.6)]
8,7	$x^{-1-1} = x$	[534 (prev. derivation)]
9	$(x^{-1} \parallel x) = e$	[569 (prev. derivation)]
11	$(x \parallel x^{-1}) = e$	[569 (prev. derivation)]
14,13	$(x \parallel y) = (x^{-1} \cdot y^{-1})$	[(2)]
15	$(x^{-1} \cdot x) = e$	[11:14,8]
17	$(x \cdot x^{-1}) = e$	[9:14,8]
19	$(x \cdot ((x^{-1} \cdot (y^{-1} \cdot z^{-1})^{-1}) \cdot (u^{-1} \cdot (y^{-1} \cdot u^{-1})^{-1})^{-1})) = z$	[5:14,14,14,14,14,8,14,8,8]
35	$(x \cdot ((x^{-1} \cdot (y^{-1} \cdot z^{-1})^{-1}) \cdot (u \cdot (y^{-1} \cdot u)^{-1})^{-1})) = z$	[7→19:8]
39	$(x \cdot ((x^{-1} \cdot (y \cdot z^{-1})^{-1}) \cdot (y^{-1} \cdot e^{-1})^{-1})) = z$	[15→19:8]
42,41	$(x \cdot e) = x$	[17→19]
44,43	$e^{-1} = e$	[15→41]
45	$(x \cdot ((x^{-1} \cdot (y \cdot z^{-1})^{-1}) \cdot y)) = z$	[39:44,42,8]
52	$(e \cdot ((e \cdot (x \cdot y^{-1})^{-1}) \cdot x)) = y$	[43→45]
61,60	$(x \cdot (x^{-1} \cdot y)) = y$	[17→45:44,42]
64	$(e \cdot y^{-1})^{-1} = y$	[41→45:61]
69,68	$(e \cdot x)^{-1} = (e \cdot x^{-1})$	[64→64]
71,70	$(e \cdot x^{-1}) = x^{-1}$	[7→64:69]
74	$(e \cdot ((x \cdot y^{-1})^{-1} \cdot x)) = y$	[52:71]
77,76	$(e \cdot x) = x$	[7→70:8]
79	$((x \cdot y^{-1})^{-1} \cdot x) = y$	[74:77]
81	$(x^{-1} \cdot (x \cdot y)) = y$	[7→60]
83	$((x \cdot y)^{-1} \cdot x) = y^{-1}$	[7→79]
90,89	$(y \cdot x)^{-1} = (x^{-1} \cdot y^{-1})$	[81→83]
91	$(x \cdot ((x^{-1} \cdot (z \cdot y)) \cdot y^{-1})) = z$	[35:90,8,8,90,8,61]
93	$((y^{-1} \cdot x^{-1}) \cdot x) = y^{-1}$	[83:90]
111	$((x \cdot y^{-1}) \cdot y) = x$	[7→93:8]
117	$((x \cdot y) \cdot y^{-1}) = x$	[7→111]
145	$(x \cdot ((x^{-1} \cdot y) \cdot z)) = (y \cdot z)$	[117→91:8]
153	$((x \cdot y) \cdot z) = (x \cdot (y \cdot z))$	[81→145]

Eqs. 15, 77, and 153, all from the second derivation, establish the result. ∎

THEOREM 8. *The theory of groups can be defined by the single axiom (in terms of division and identity)*

$$((e/(x/(y/(((x/x)/x)/z))))/z) = y. \tag{3.1.7}$$

Proof. Eq. (3.1.7) holds in groups and is obtained from (3.1.4) by replacing the first occurrence of (y/y) with e. Therefore, it is sufficient to derive $(x/x) = e$. The following sequence does so.

7,6	$((e/(x/(y/(((x/x)/x)/z))))/z) = y$	[(3.1.7)]
8	$(e/(u/(y/(((u/u)/u)/(((z/z)/z)/x))))) = ((e/(z/y))/x)$	[6→6]
10	$(e/((e/(y/((e/e)/e)))/x)) = ((e/(z/z))/(((y/y)/y)/x))$	[8→8]
12,11	$(((e/(y/z))/x)/(((y/y)/y)/x)) = z$	[8→6]
13	$((e/(z/z))/(((y/y)/y)/x)) = ((e/(u/u))/(((y/y)/y)/x))$	[10→10]
15	$(e/(z/z)) = (e/(u/u))$	[13→6:7]
19	$(y/y) = (z/z)$	[15→11:12]
28	$(e/(y/(x/x))) = ((y/y)/y)$	[19→8:7]
34	$((e/(z/(y/y)))/x) = (((z/z)/z)/x)$	[19→6]
39,38	$((e/(x/(z/(y/y))))/((x/x)/x)) = z$	[19→6]
42,41	$(((e/(y/u))/x)/(((z/z)/y)/x)) = u$	[19→11]
88	$((y/y)/(y/y)) = e$	[28→38:39]
99	$(x/x) = e$	[34→88:42] ∎

THEOREM 9. *The theory of groups can be defined by the single axiom (in terms of division and inverse)*

$$((x/x)/(y/((z/(u/y))/u^{-1}))) = z. \tag{3.1.8}$$

Proof. Eq. (3.1.8) holds in groups. The following refutation starts with (3.1.8) and the denial of (3.1.4), which is a single axiom in terms of division alone.

6	$(((b/b)/(b/(c/(((b/b)/b)/a))))/a) \neq c$	[denial of (3.1.4)]
7	$((x/x)/(y/((z/(u/y))/u^{-1}))) = z$	[(3.1.8)]
9	$((u/u)/(((y/(z/x))/z^{-1})/(y/x^{-1}))) = (v/v)$	[7→7]
10	$(x/x) = (y/y)$	[9→9]
11	$((u/u)/(((z/z)/y^{-1})/((y/x)/x^{-1}))) = (v/v)$	[10→9]
15,14	$((u/u)/(z/((y/y)/x^{-1}))) = (x/z)$	[10→7]
16	$((u/u)/(x/((z/(y/y))/x^{-1}))) = z$	[10→7]
22	$(((b/b)/(b/(c/(((x/x)/b)/a))))/a) \neq c$	[10→6]
27	$((u/((v/v)/u^{-1}))/(z/((y/y)/x^{-1}))) = (x/z)$	[14→14]
31	$(x/((y/y)/x^{-1})) = (z/z)$	[14→10]
36	$((y/y)/(x/x)) = (z/z)$	[10→11]
40	$((v/v)/((u/u)/((z/(y/y))/(x/x)^{-1}))) = z$	[36→7]
47	$((u/u)/(((z/z)/(y/y)^{-1})/(x/x^{-1-1}))) = (v/v)$	[31→9]
49,48	$((x/x)/z^{-1}) = z$	[31→7:15]
50	$((u/u)/((y/y)/(x/x^{-1-1}))) = (v/v)$	[47:49]
54,53	$((u/u)/(z/x)) = (x/z)$	[27:49,49]

55	$(x/x^{-1-1}) = (u/u)$	[50:54,54]
56	$((z/(y/y))/(x/x)^{-1}) = z$	[40:54,54]
59	$(((c/(((x/x)/b)/a))/b)/a) \neq c$	[22:54]
60	$(((z/(y/y))/x^{-1})/x) = z$	[16:54]
65	$(((z/(x/y))/x^{-1})/y) = z$	[7:54]
68	$(x/x)^{-1} = (y/y)$	[48→55]
102,101	$((x/y)/(u/u)) = (x/y)$	[53→53:54]
104	$((y/y)/x) = (x^{-1}/(z/z))$	[48→53]
163,162	$(z/(y/y)) = z$	[68→56:102]
181	$((y/y)/x) = x^{-1}$	[104:163]
183	$((z/x^{-1})/x) = z$	[60:163]
201	$x^{-1-1} = x$	[48→181]
219	$(((c/(b^{-1}/a))/b)/a) \neq c$	[183→59]
314,313	$(((z/(y^{-1}/x))/y)/x) = z$	[201→65]
332	$c \neq c$	[219:314]

Therefore, (3.1.4) can be derived from (3.1.8). From Eq. 181, which is derived from (3.1.2) alone, it follows that $^{-1}$ is inverse. ∎

3 Abelian Groups

THEOREM 10. *The theory of Abelian groups can be defined by the single axiom*

$$(x \cdot (((x \cdot y)^{-1} \cdot z) \cdot y)) = z. \tag{3.2.2}$$

Proof. Eq. (3.2.2) holds in Abelian groups. Consider the following derivation starting with (3.2.2).

8	$(x \cdot (((x \cdot y)^{-1} \cdot z) \cdot y)) = z$	[(3.2.2)]
10	$(x \cdot ((y^{-1} \cdot z) \cdot (((x \cdot u)^{-1} \cdot y) \cdot u))) = z$	[8→8]
12	$((((x \cdot z)^{-1} \cdot u)^{-1} \cdot y) \cdot u) = (x \cdot (y \cdot z))$	[8→8]
14	$((x^{-1} \cdot y) \cdot x) = y$	[8→10]
18,17	$(((y^{-1} \cdot z)^{-1} \cdot x) \cdot z) = (x \cdot y)$	[8→14]
19	$(u \cdot (x \cdot y)) = (x \cdot (u \cdot y))$	[12:18]
23	$((z^{-1} \cdot y) \cdot (x \cdot z)) = (x \cdot y)$	[14→19]
27	$((x \cdot (y^{-1} \cdot z)) \cdot y) = (x \cdot z)$	[19→14]
48	$((x \cdot y) \cdot z) = (x \cdot (y \cdot z))$	[8→27:18]
65,64	$(z^{-1} \cdot (y \cdot (x \cdot z))) = (x \cdot y)$	[23→48]
67	$(y^{-1} \cdot (x \cdot y)) = x$	[14→48]
82	$(z \cdot y) = (y \cdot z)$	[48→67:65]
88	$(x \cdot (y^{-1} \cdot y)) = x$	[19→67]
106	$(y \cdot (y^{-1} \cdot x)) = x$	[14→82]
129	$(x \cdot (y \cdot y^{-1})) = x$	[82→88]
189	$(x \cdot x^{-1}) = (y \cdot y^{-1})$	[129→106]

By 189, there exists a unique element e such that for all x, $(x \cdot x^{-1}) = e$; it follows from 129 that e is a right identity; and 48 and 82, respectively, assert the associativity and commutativity of product. ∎

THEOREM 11. *The theory of Abelian groups can be defined by the single axiom*

$$(x \cdot ((y \cdot z) \cdot (x \cdot z)^{-1})) = y. \tag{3.2.3}$$

Proof. Eq. (3.2.3) holds in Abelian groups. Consider the following derivation starting with (3.2.3).

8	$(x \cdot ((y \cdot z) \cdot (x \cdot z)^{-1})) = y$	[(3.2.3)]
10	$(x \cdot (y \cdot (x \cdot ((y \cdot z) \cdot (u \cdot z)^{-1}))^{-1})) = u$	[8→8]
14	$(x \cdot (y \cdot y^{-1})) = x$	[8→10]
17	$(x \cdot (y \cdot x^{-1})) = y$	[14→10]
21	$((x \cdot y) \cdot (z \cdot (z \cdot y)^{-1})) = x$	[17→10]
38,37	$((x \cdot y) \cdot (z \cdot x^{-1})) = (z \cdot y)$	[21→10]
39	$((z \cdot (u \cdot (u \cdot y)^{-1})) \cdot y) = z$	[21→8:38]
61	$((x \cdot y) \cdot z) = ((x \cdot z) \cdot y)$	[37→39]
66	$(y \cdot (z \cdot (z \cdot x^{-1})^{-1})) = (x \cdot y)$	[39→17]
76	$((y \cdot (x \cdot (y \cdot z)^{-1})) \cdot z) = x$	[17→61]
92	$(x \cdot ((y \cdot x^{-1}) \cdot z)) = (y \cdot z)$	[61→17]
127,126	$(z \cdot (y \cdot (z \cdot x^{-1})^{-1})) = (x \cdot y)$	[76→17]
131	$(x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)$	[76→92:127]
133,132	$(x \cdot (((y \cdot x^{-1}) \cdot z) \cdot u)) = ((y \cdot z) \cdot u)$	[61→92]
139,138	$((x \cdot (y \cdot z)^{-1}) \cdot z) = (x \cdot y^{-1})$	[37→92]
141,140	$(y \cdot (y \cdot x^{-1})^{-1}) = x$	[21→92:139]
146	$(x \cdot z) = (z \cdot x)$	[66:141]
172	$((y \cdot y^{-1}) \cdot x) = x$	[14→146]
262	$(x \cdot x^{-1}) = (y \cdot y^{-1})$	[14→172]
388	$((y \cdot z) \cdot u) = (y \cdot (z \cdot u))$	[131→92:133]

By 262, there exists a unique element e such that for all x, $(x \cdot x^{-1}) = e$; it follows from 14 that e is a right identity; and 146 and 388, respectively, assert the commutativity and associativity of product. ∎

THEOREM 12. *The theory of Abelian groups can be defined by the single axiom (in terms of division)*

$$(x/((x/y)/(z/y))) = z. \tag{3.2.4}$$

Proof. Eq. (3.2.4) holds in Abelian groups. The following sequence derives Tarski's single axiom [2, (2.3)].

3	$(x/((x/y)/(z/y))) = z$	[(3.2.4)]

5	$(u/((u/((z/y)/(x/y)))/x)) = z$	[3→3]
7	$(z/(y/(u/((z/x)/(y/x))))) = u$	[3→3]
10,9	$((y/(x/z))/z) = (y/x)$	[3→3]
15,14	$((z/(y/(u/x)))/u) = ((z/y)/x)$	[9→9]
17	$(u/(u/z)) = z$	[5:15,10]
29,28	$((z/x)/(y/x)) = (z/y)$	[17→9]
30	$(z/(y/(u/(z/y)))) = u$	[7:29] ∎

Theorem 13. *The theory of Abelian groups can be defined by the single axiom (in terms of division)*

$$((x/((x/y)/z))/y) = z. \tag{3.2.5}$$

Proof. Eq. (3.2.5) holds in Abelian groups. The following refutation starts with (3.2.5) and Tarski's single axiom [2, (2.3)].

2	$(b/(a/(c/(b/a)))) \neq c$	[denial of (2.3)]
3	$((x/((x/y)/z))/y) = z$	[(3.2.5)]
5	$(((u/((u/x)/z))/(z/y))/x) = y$	[3→3]
7	$((z/x)/((z/y)/x)) = y$	[3→3]
31,30	$(y/(y/x)) = x$	[3→5]
45	$((z/x)/(y/x)) = (z/y)$	[30→7]
77,76	$(x/(z/(y/x))) = (y/z)$	[30→45]
81	$c \neq c$	[2:77,31] ∎

Theorem 14. *The theory of Abelian groups can be defined by the single axiom (in terms of division)*

$$((x/y)/((x/z)/y)) = z. \tag{3.2.6}$$

Proof. Eq. (3.2.6) holds in Abelian groups. The single axiom (3.2.5) can be derived from (3.2.6) in one step by paramodulation into term $((x/z)/y)$. ∎

Theorem 15. *The theory of Abelian groups can be defined by the single axiom (in terms of double division and identity)*

$$((x \parallel ((z \parallel (x \parallel y)) \parallel (e \parallel y))) \parallel (e \parallel e)) = z. \tag{3.2.7}$$

Proof. Eq. (3.2.7) holds in Abelian groups and is obtained from (3.1.5), which is a single axiom for ordinary groups, by commuting two terms. It is sufficient to derive commutativity of $\parallel$, because then (3.1.5) and commutativity of product follow easily.

4,3	$((x \parallel ((z \parallel (x \parallel y)) \parallel (e \parallel y))) \parallel (e \parallel e)) = z$	[(3.2.7)]
7	$((e \parallel (x \parallel (e \parallel e))) \parallel (e \parallel e)) = (y \parallel ((x \parallel (y \parallel z)) \parallel (e \parallel z)))$	[3→3]

8	$(y \parallel (((x \parallel (e \parallel e)) \parallel (y \parallel z)) \parallel (e \parallel z))) = x$	[3→7]
11,10	$((e \parallel y) \parallel (e \parallel e)) = (e \parallel (y \parallel (e \parallel e)))$	[7→3:4]
13,12	$(e \parallel (((x \parallel (e \parallel y)) \parallel (e \parallel y)) \parallel (e \parallel e))) = x$	[3→10]
14	$((x \parallel (e \parallel ((x \parallel e) \parallel (e \parallel e)))) \parallel (e \parallel e)) = e$	[10→3]
16	$(e \parallel (e \parallel (e \parallel (e \parallel (e \parallel (e \parallel (e \parallel e))))))) = e$	[10→8:11,11,11,11,11]
19,18	$(e \parallel (((x \parallel y) \parallel y) \parallel (e \parallel e))) = x$	[12→12:13]
20	$(e \parallel (((x \parallel (e \parallel e)) \parallel y) \parallel y)) = x$	[18→8:19]
24	$(((x \parallel e) \parallel x) \parallel (e \parallel e)) = e$	[18→14]
27,26	$(e \parallel (e \parallel e)) = e$	[24→24]
29,28	$(e \parallel e) = e$	[16:27,27,27]
30	$(((x \parallel e) \parallel x) \parallel e) = e$	[24:29]
34	$(e \parallel (((x \parallel e) \parallel y) \parallel y)) = x$	[20:29]
36	$(e \parallel (((x \parallel y) \parallel y) \parallel e)) = x$	[18:29]
42	$(x \parallel (((y \parallel e) \parallel (x \parallel z)) \parallel (e \parallel z))) = y$	[8:29]
46	$(e \parallel ((e \parallel x) \parallel x)) = ((y \parallel e) \parallel y)$	[30→34]
48,47	$(e \parallel ((e \parallel x) \parallel x)) = e$	[28→34]
49	$((y \parallel e) \parallel y) = e$	[46:48]
52,51	$(e \parallel (e \parallel x)) = x$	[49→34]
57	$((e \parallel x) \parallel x) = e$	[47→51:29]
59	$(x \parallel (e \parallel x)) = e$	[51→57]
62,61	$(e \parallel (x \parallel e)) = x$	[59→36:52]
64	$(x \parallel (((x \parallel y) \parallel y) \parallel e)) = e$	[36→36:62]
71,70	$((x \parallel y) \parallel y) = x$	[36:62]
72	$(x \parallel (x \parallel e)) = e$	[64:71]
81	$(x \parallel (y \parallel x)) = y$	[72→42:71,62]
103	$(x \parallel y) = (y \parallel x)$	[70→81] ∎

THEOREM 16. *The theory of Abelian groups can be defined by the single axiom (in terms of double division and inverse)*

$$(x \parallel (((x \parallel y) \parallel z^{-1})^{-1} \parallel y)^{-1}) = z. \tag{3.2.8}$$

Proof. Eq. (3.2.8) holds in Abelian groups. Consider the following sequence starting with (3.2.8) and the denial of (3.1.6), which is a single axiom for ordinary groups.

3	$(a^{-1} \parallel ((a \parallel (b \parallel c))^{-1} \parallel (d \parallel (b \parallel d)))^{-1}) \neq c$	[denial of (3.1.6)]
4	$(x \parallel (((x \parallel y) \parallel z^{-1})^{-1} \parallel y)^{-1}) = z$	[(3.2.8)]
6	$(x \parallel ((y \parallel z^{-1})^{-1} \parallel (((x \parallel u) \parallel y^{-1})^{-1} \parallel u)^{-1})^{-1}) = z$	[4→4]
8	$(x \parallel (y^{-1} \parallel z)^{-1}) = ((((x \parallel z) \parallel u) \parallel y^{-1})^{-1} \parallel u)$	[4→4]
10,9	$((((y \parallel z) \parallel u) \parallel ((y \parallel z) \parallel x^{-1})^{-1})^{-1} \parallel u) = x$	[4→8]
11	$(((x \parallel y) \parallel (x \parallel w^{-1})^{-1})^{-1} \parallel y) = w$	[9→9:10]
24	$((x \parallel y^{-1})^{-1} \parallel x^{-1}) = y$	[4→6]
26	$(((((z \parallel x^{-1})^{-1} \parallel y) \parallel u) \parallel z^{-1})^{-1} \parallel u) = (x^{-1} \parallel y)$	[6→11]

28	$(x^{-1} \parallel (y \parallel x^{-1})^{-1-1}) = y$	[24→24]
32	$(((((x \parallel y) \parallel z) \parallel u^{-1})^{-1} \parallel z)^{-1} \parallel x^{-1}) = (u^{-1} \parallel y)$	[8→24]
35,34	$(((y \parallel z) \parallel x^{-1})^{-1} \parallel z) = (x^{-1} \parallel y^{-1})$	[4→24]
38	$((u^{-1} \parallel (x \parallel y)^{-1})^{-1} \parallel x^{-1}) = (u^{-1} \parallel y)$	[32:35]
40	$(x^{-1} \parallel ((x \parallel y^{-1})^{-1} \parallel z)^{-1}) = (y^{-1} \parallel z)$	[26:35]
49	$(x \parallel (z^{-1} \parallel x^{-1})^{-1}) = z$	[4:35]
51	$((x^{-1} \parallel y^{-1-1}) \parallel x^{-1}) = y$	[49→49]
56,55	$(x \parallel y^{-1})^{-1} = (x^{-1} \parallel y^{-1-1})$	[24→28]
57	$(x \parallel (y^{-1-1} \parallel x^{-1-1})) = y$	[49:56]
65	$(x^{-1} \parallel ((x^{-1} \parallel y^{-1-1}) \parallel z)^{-1}) = (y^{-1} \parallel z)$	[40:56]
67	$((x^{-1-1} \parallel (y \parallel z)^{-1-1}) \parallel y^{-1}) = (x^{-1} \parallel z)$	[38:56]
73	$(x^{-1} \parallel y^{-1}) = (y^{-1} \parallel x^{-1})$	[51→65]
74	$(x \parallel (x^{-1-1} \parallel y^{-1-1})) = y$	[73→57]
76	$((x^{-1-1} \parallel y^{-1}) \parallel y^{-1}) = x$	[73→51]
78	$(x^{-1-1} \parallel x^{-1}) = (y^{-1} \parallel y^{-1-1})$	[76→65]
81,80	$((x^{-1} \parallel x^{-1-1}) \parallel y^{-1}) = y$	[78→76]
86,85	$y^{-1-1} = y$	[78→51:81]
101	$(x \parallel (x \parallel y)) = y$	[74:86,86]
102,101	$(x \parallel (x \parallel y)) = y$	[74:86,86]
106	$((x \parallel (y \parallel z)) \parallel y^{-1}) = (x^{-1} \parallel z)$	[67:86,86]
116,115	$(x \parallel (y \parallel x)) = y$	[57:86,86]
117	$(x \parallel y^{-1})^{-1} = (x^{-1} \parallel y)$	[55:86]
122	$(a^{-1} \parallel ((a \parallel (b \parallel c))^{-1} \parallel b)^{-1}) \neq c$	[3:116]
130	$(x \parallel y) = (y \parallel x)$	[115→101]
244,243	$(x \parallel y)^{-1} = (x^{-1} \parallel y^{-1})$	[85→117]
262	$(a^{-1} \parallel ((a \parallel (b \parallel c)) \parallel b^{-1})) \neq c$	[122:244,244,244,244,86,244,86,86]
418	$c \neq c$	[106→262:102]

Therefore, (3.1.6) can be derived from (3.2.8). Commutativity of product follows from 130, which is derived from (3.2.8) alone. ∎

THEOREM 17. *The theory of Abelian groups can be defined by the single axiom (in terms of division and identity)*

$$((e/(((x/y)/z)/x))/z) = y. \tag{3.2.9}$$

Proof. Eq. (3.2.9) holds in Abelian groups. Consider the following derivation starting with (3.2.9).

9,8	$((e/(((x/y)/z)/x))/z) = y$	[(3.2.9)]
12	$((e/(y/e))/x) = (((z/y)/x)/z)$	[8→8]
17	$(((y/((e/x)/z))/z)/y) = x$	[8→12]
33	$((((x/(e/y))/e)/x)/e) = y$	[12→17]
51	$(((((x/e)/e)/x)/e)/e) = e$	[12→33]
72,71	$(e/e) = e$	[51→51:9]

85	$((((x/e)/e)/x)/e) = e$	[71→33]
88,87	$(((y/e)/x)/y) = (e/x)$	[71→12:72]
140,139	$((e/y)/e) = (e/y)$	[85→87:88,72]
160,159	$(e/(e/x)) = x$	[87→33:72,140,140]
174,173	$(x/x) = e$	[87→8:160]
183,182	$(x/e) = x$	[173→33:174,160]
190	$(y/(y/x)) = x$	[173→8:160]
205,204	$(e/(x/y)) = (y/x)$	[190→87:183]
206	$((z/x)/((z/y)/x)) = y$	[190→8:205]

Eq. 206 is (3.2.6), which is a single axiom for Abelian groups in terms of division alone. It follows from Eq. 174 that e is the identity element. ∎

Theorem 18. *The theory of Abelian groups can be defined by the single axiom (in terms of division and inverse)*

$$((x/(y/(x/z))^{-1})/z) = y. \tag{3.2.10}$$

Proof. Eq. (3.2.10) holds in Abelian groups. Consider the following derivation starting with (3.2.10).

9	$((x/(y/(x/z))^{-1})/z) = y$	[(3.2.10)]
10	$(((u/(y/(u/x))^{-1})/(z/y)^{-1})/x) = z$	[9→9]
18	$(x/(y/x)^{-1}) = y$	[9→10]
20	$((x/y)^{-1}/x^{-1}) = y$	[18→18]
27	$(y^{-1}/x^{-1}) = (y/x)^{-1}$	[18→20]
33	$((y/x)/y)^{-1} = x$	[20→27]
34	$(y/(x/y^{-1}))^{-1} = x$	[18→27]
51	$((z/y)/x) = ((z/x)/y)$	[33→9]
58	$((x/y^{-1})/x) = y$	[34→18]
62	$((y/y)/x^{-1}) = x$	[34→9]
87	$((z/((y/y)/x))/x) = z$	[62→9]
562	$((y/x)/x^{-1}) = y$	[62→87]
581	$((y/y)/x) = x^{-1}$	[87→33]
623	$(y/(y/x)) = x$	[562→58]
654	$((z/x)/((z/y)/x)) = y$	[51→623]

Eq. 654 is (3.2.6), which is a single axiom for Abelian groups in terms of division alone. From 581, it follows that $^{-1}$ is inverse. ∎

References

[1] W. McCune. Otter 2.0 Users Guide. Tech. Report ANL-90/9, Argonne National Laboratory, Argonne, IL, March 1990.

[2] W. McCune. Single axioms for groups and Abelian groups with various operations. Preprint MCS-P270-1091, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL, 1991.

[3] W. McCune. What's New in OTTER 2.2. Tech. Memo. ANL/MCS-TM-153, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL, July 1991.