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ASYMPTOTIC ANALYSIS

Working Note #1
BASIC CONCEPTS AND DEFINITIONS

by

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Preface

For some time, we have been interested in the development and application of asymptotic methods for the numerical solution of boundary value problems with critical parameters—that is, parameters that determine the nature of the solution in some critical way. We are thinking, for example, of fluid flow (viscosity), combustion (Lewis number), and superconductivity (Ginzburg-Landau parameter) problems. Their solution may remain smooth over a wide range of parameter values, but as the parameters approach critical values, complicated patterns may emerge. Boundary layers may develop, or the region over which the solution extends may take on the appearance of a patchwork of subregions; on each subregion, the solution is smooth, but between subregions the solution undergoes dramatic changes over very short distances. Shock layers in fluid flow are a visible manifestation of this type of behavior.

Boundary value problems with critical parameters pose some of the most challenging problems in computational science, and much effort is being spent on developing new techniques for their numerical solution. Some of the most useful techniques, in particular on parallel computing architectures, are based on domain decomposition. In a domain decomposition method, one partitions the domain into subdomains, approximates the solution on each subdomain, and assembles these solutions to obtain an approximate solution on the entire domain. Many criteria, involving considerations from linear algebra to computer architecture, go into the design of a useful domain decomposition method. Our aim is to explore the use of asymptotic methods.

Asymptotic analysis, in particular singular perturbation theory, is the study of boundary value problems involving critical parameters. It provides a methodology to identify and characterize boundary layers, transition layers, and initial layers; hence, our idea to use asymptotic methods in the design of domain decomposition algorithms.

We have organized two workshops on the subject of asymptotic analysis and domain decomposition: a workshop at Argonne, jointly sponsored by the Department of Energy and the National Science Foundation (February 1990), and a NATO Advanced Research Workshop in Beaune, France (May 1992). Proceedings of these workshops have been published (*Asymptotic analysis and the numerical solution of partial differential equations*, edited by H. G. Kaper and M. Garbey, Lecture Notes in Pure and Applied Mathematics – Vol. 130, Marcel Dekker, Inc., New York, 1991; *Asymptotic and numerical methods for partial differential equations*, edited by H. G. Kaper and M. Garbey, NATO ASI Series C: Mathematical and Physical Sciences – Vol. 384, Kluwer Academic Publishers, Dordrecht, Neth., 1993).

We currently have plans to develop a full-length book on the subject. To formulate our thoughts before final publication, we intend to produce a series of *Working Notes* on various relevant topics. Some of the notes will contain new material; others may offer new presentations of existing material. We certainly expect the notes to evolve in time; the notes may or may not appear eventually as chapters of the book. The notes are intended for our own use, but we will be happy to supply copies to interested colleagues.

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Abstract

In this note we introduce the basic concepts of asymptotic analysis. After some comments of historical interest (Section 1), we begin by defining the order relations O, o, and $O^\sharp$, which enable us to compare the asymptotic behavior of functions of a small positive parameter ϵ as $\epsilon \downarrow 0$ (Section 2). Next, we introduce order functions (Section 3), asymptotic sequences of order functions (Section 4), and more general gauge sets of order functions (Section 5) and define the concepts of an asymptotic approximation and an asymptotic expansion with respect to a given gauge set (Section 6). This string of definitions culminates in the introduction of the concept of a regular asymptotic expansion, also known as a Poincaré expansion, of a function $f : (0, \epsilon_0) \to X$, where X is a normed vector space of functions defined on a domain $D \in \mathbf{R}^N$. We conclude the note with the asymptotic analysis of an initial value problem whose solution is obtained in the form of a regular asymptotic expansion (Section 7).

1 From Euler to Poincaré

Asymptotic analysis is the art of comparing functions whose graphs "do not meet" (Gr. $\dot{\alpha}$-$\sigma\upsilon\nu$-$\pi\acute{\iota}\pi\tau\epsilon\iota\nu$); the graphs may get close, even arbitrarily close, but they may not have any point in common. In particular, it is the art of expressing the asymptotic behavior of functions that are defined implicitly—for example, as solutions of boundary value problems or initial value problems—in terms of functions whose asymptotic behavior as parameters or variables approach critical values is known.

Asymptotic analysis as a scientific method for comparing functions and their graphs goes back to the French mathematician Henri Poincaré, who, in the first volume of his monograph *Les méthodes nouvelles de la mécanique céleste* (1892), established a theoretical framework for the asymptotic approximation by series that are divergent in the customary sense [1, Chapter 7]. Divergent series had received the attention of various mathematicians in the eighteenth and nineteenth century; in fact, it is still instructive to consider Euler's publication [2] of 1754 on the subject. Euler discussed the series

$$S(x) = \sum_{n=0}^{\infty} (-1)^n n!\ x^n,$$

which diverges for all $x \in \mathbf{R}$ except $x = 0$. Euler observed that, for small $|x|$, successive terms of the series decrease quite rapidly; he asked what function might possibly be represented by the sum of the first few terms. Because $n! = \int_0^\infty e^{-t} t^n\, dt$, we have

$$S(x) = \sum_{n=0}^{\infty} (-1)^n \int_0^\infty e^{-t} (xt)^n\, dt,$$

so *if* we could interchange the order of the summation and the integration, it would follow that $S(x) = f(x)$, where

$$f(x) = \int_0^\infty \frac{e^{-t}}{1 + xt}\, dt.$$

Of course, the order of the summation and integration cannot be interchanged, so we have no right to conclude that $S(x) = f(x)$. On the other hand, f is well defined, even analytic in the complex plane cut along the negative x axis. Hence, it is not unreasonable to ask whether the sum $S(x)$, taken to a finite number of terms, represents some approximation to $f(x)$. Indeed, as Euler showed,

$$f(x) = S_m(x) + R_m(x),$$

where S_m is the partial sum,

$$S_m(x) = \sum_{n=0}^{m-1} (-1)^n n!\ x^n,$$

and the remainder R_m satisfies the estimates

$$|R_m(x)| \le \begin{cases} m!\ |x|^m & \text{if } \mathrm{Re}(x) \ge 0, \\ m!\ |x|^m \ \mathrm{cosec}(\arg(x)) & \text{if } \mathrm{Re}(x) < 0. \end{cases}$$

In either case, R_m is of the order of the first "neglected" term in S_m, and $R_m(x)$ tends to 0 as x tends to 0. Thus, Euler identified precisely those properties that Poincaré used over a century later to formalize the concept of an "asymptotic approximation."

We also mention the work by Pierre Du Bois–Reymond, who published a series of articles in 1870–71 on the foundation of a calculus with infinitesimally large quantities. Du Bois–Reymond singled out the crucial notion of "asymptotic scales," which was eventually given its rigorous and definitive form by Hardy [3]. References to Du Bois–Reymond's articles are given in [3, Appendix 1].

Thus, although the development of asymptotic analysis cannot be traced back to the Babylonians, there existed at least a considerable body of knowledge related to the subject prior to Poincaré. But the honors certainly go to Poincaré, who introduced the concept of an *asymptotic expansion.* This concept, which is broader than that of an asymptotic series, made it possible to give a rigorous meaning to approximations like Euler's and to exploit asymptotic methods for practical purposes. Today, the concept covers, in particular, many topics related to differential equations and perturbations involving a small (or large) parameter.

2 Order Relations

Throughout most of this book we will be comparing functions that are parameterized by a "small positive parameter," ϵ. That is, ϵ is confined to an interval $(0, \epsilon_0)$, where ϵ_0 is an arbitrarily small fixed positive number. "Generic constants" do not depend on ϵ (but may depend on ϵ_0).

We begin by considering continuous positive-valued functions that depend only on ϵ.

Definition 1 *Let f and g be continuous positive-valued functions on the interval $(0, \epsilon_0)$ for some $\epsilon_0 > 0$.* (i) *$f = O(g)$ as $\epsilon \downarrow 0$ if there exist an $\epsilon_0 > 0$ and a positive constant C, which may depend on ϵ_0, such that $f(\epsilon) < Cg(\epsilon)$ for all $\epsilon \in (0, \epsilon_0)$.* (ii) *$f = o(g)$ as $\epsilon \downarrow 0$ if, for every positive constant c, there exists an $\epsilon_0 > 0$, which may depend on c, such that $f(\epsilon) < cg(\epsilon)$ for all $\epsilon \in (0, \epsilon_0)$.* (iii) *$f = O^\sharp(g)$ as $\epsilon \downarrow 0$ if $f = O(g)$ and $f \neq o(g)$ as $\epsilon \downarrow 0$.*

The symbols O, o, and $O^\sharp$ are pronounced "big-oh," "little-oh," and "big-oh-sharp," respectively. The notation O and o goes back to Pfeiffer [4, p. 1–21], Bachmann [5, p. 401], and Landau [6, p. 61]; O and o are known as the *Landau symbols.* Sometimes, we shall use a notation due to Hardy [3], writing $f \preceq g$ instead of $f = O(g)$ and $f \prec g$ instead of $f = o(g)$. The symbol $O^\sharp$ is denoted O_s in Eckhaus [7]. If $f = O^\sharp(g)$, then there exist positive constants C_1 and C_2 such that $C_1 g(\epsilon) < f(\epsilon) < C_2 g(\epsilon)$ for all $\epsilon \in (0, \epsilon_0)$. We often omit the quantifier "as $\epsilon \downarrow 0$."

Exercises

1. Prove that the symbol O provides a partial ordering (that is, O is reflexive and transitive) on the set of all continuous positive-valued functions on $(0, \epsilon_0)$. Is the same true for the symbols o and $O^\sharp$?
2. Show by giving a counterexample that the partial ordering introduced by the symbol O on the set of all continuous positive-valued functions on $(0, \epsilon_0)$ is not a total ordering. (That is, there are elements f and g in the set for which neither $f = O(g)$ nor $g = O(f)$.)
3. The symbol O defines an equivalence relation, $f \approx g$, if $f = O(g)$ and $g = O(f)$. Prove that (a) $f \approx g$ implies that $f \neq o(g)$ and $g \neq o(f)$; (b) $f \approx g$ is sufficient, but not necessary for $f = O^\sharp(g)$.
4. Show by giving a counterexample that the two relations $f \preceq g$ and $f \not\approx g$ do not imply that $f \prec g$.

5. Compare, if possible, the functions $f(\epsilon) = \epsilon^2$ and $g(\epsilon) = \epsilon \sin^2(1/\epsilon) + \epsilon^3$.

The symbols O and o obey certain algebraic rules. Here are a few examples: (1) If $f = O(h)$ and $g = O(h)$, then $f + g = O(h)$ and $f - g = O(h)$. (2) If $f_1 = O(g_1)$ and $f_2 = O(g_2)$, then $f_1 f_2 = O(g_1 g_2)$. (3) If $f = O(g)$ and λ is a positive constant, then $f^\lambda = O(g^\lambda)$; other rules are found in the exercises. Similar rules hold for the symbol o.

Exercises

1. Show that, if $f = o(g)$ and both f and g tend to infinity as $\epsilon \downarrow 0$, then $g^{-1} = o(f^{-1})$.
2. Show that, if $f = O(g)$, then $F = O(G)$, where $F(\epsilon) = \int_0^\epsilon f(\eta)\, d\eta$ and $G(\epsilon) = \int_0^\epsilon g(\eta)\, d\eta$. (That is, order relations can be integrated with respect to parameters.)
3. Show by giving a counterexample that, if f and g are differentiable and $f = O(g)$, then it is not necessarily true that $f' = O(g')$. (That is, order relations cannot be differentiated with respect to parameters.)

Next, we consider variable functions whose values depend not only on ϵ, but also on additional variables, which we denote collectively by $x = (x_1, x_2, \ldots, x_N)$. The variable x ranges over a domain $D \subset \mathbf{R}^N$.[1] We consider these functions as maps from the interval $(0, \epsilon_0)$ into normed vector spaces of functions defined on D. That is, we associate a variable function f on $(0, \epsilon_0) \times D$ with a vector $f(\epsilon) \in X$ by making the identification

$$f(\epsilon)(x) = f(\epsilon, x), \quad x \in D.$$

Here, X is a normed vector space with the norm $\|\cdot\|_X$. We always assume that the map $\epsilon \mapsto \|f(\epsilon)\|_X$ is continuous.

Definition 2 *Let $f : (0, \epsilon_0) \to X$ and $g : (0, \epsilon_0) \to Y$ be continuous maps from the interval $(0, \epsilon_0)$ into the normed vector spaces X and Y, respectively.* (i) *$f = O(g)$ as $\epsilon \downarrow 0$ if $\|f(\epsilon)\|_X = O(\|g(\epsilon)\|_Y)$ as $\epsilon \downarrow 0$.* (ii) *$f = o(g)$ as $\epsilon \downarrow 0$ if $\|f(\epsilon)\|_X = o(\|g(\epsilon)\|_Y)$ as $\epsilon \downarrow 0$.* (iii) *$f = O^\sharp(g)$ as $\epsilon \downarrow 0$ if $\|f(\epsilon)\|_X = O^\sharp(\|g(\epsilon)\|_Y)$ as $\epsilon \downarrow 0$.*

The order relations for variable functions depend intimately on the choice of the normed vector spaces. One usually takes $X = Y$ when comparing variable functions f and g defined on the same domain D, but this choice is certainly not necessary. (Recall that nonequivalent norms define different topologies and therefore different function spaces.) In many situations we will be comparing a variable function defined on $(0, \epsilon_0) \times D$ with a positive-valued function defined on $(0, \epsilon_0)$. In such cases, we can take X to be any normed space of functions on D and $Y = \mathbf{R}_+$ with the ususal topology. Thus, the distinction between "variable" and "nonvariable" functions becomes irrelevant. We will usually indicate the normed vector spaces X and Y explicitly, unless the choice is clear from the context.

[1] A domain is an open set, which may be bounded or unbounded. The set of boundary points of D is ∂D, and $\overline{D} = D \cup \partial D$. Sometimes, we shall refer to functions defined on a set D that contains some or all of its boundary points; in that case, we assume that the function is defined on an open set that contains D or that one-sided limits are considered at the included boundary points.

Exercises

1. Consider the function $f(x, \epsilon) = e^{-x/\epsilon}$ on $(0,1) \times (0, \epsilon_0)$ as a map from $(0, \epsilon_0)$ into X. Show that (a) $f = O^\sharp(1)$ if $X = L^\infty(0,1)$; (b) $f = O^\sharp(\epsilon^{-1})$ if $X = W^{1,\infty}(0,1)$; (c) $f = O^\sharp(\epsilon^{1/2})$ if $X = L^2(0,1)$; (d) $f = O^\sharp(\epsilon^{-1/2})$ if $X = W^{1,2}(0,1)$. Here, L^p and $W^{p,q}$ are the usual Lebesgue and Sobolev spaces.

3 Order Functions

The fundamental idea in asymptotic analysis is to compare the behavior of one function as $\epsilon \downarrow 0$ with that of another whose behavior as $\epsilon \downarrow 0$ is known or at least simpler than that of the original function. For this purpose it is useful to define a set of comparison functions that is in some sense dense in the set of continuous functions.

Definition 3 *An* order function *is a monotone continuous positive-valued function on* $(0, \epsilon_0)$*. The set of all order functions is denoted by* $\mathcal{O}$*.*

The set $\mathcal{O}$ is sufficient for the asymptotic study of all continuous positive-valued functions, as the following theorem shows. The theorem was first formulated by Eckhaus [7].

Theorem 1 *For every continuous positive-valued function f on $(0, \epsilon_0)$ there exists an element $\delta \in \mathcal{O}$ such that $f = O^\sharp(\delta)$.*

Proof. If $\lim_{\epsilon\downarrow 0} f(\epsilon)$ exists and is (finite and) positive, it suffices to take $\delta(\epsilon) = 1$. If $\lim_{\epsilon\downarrow 0} f(\epsilon)$ exists and is 0, we take $\delta(\epsilon) = \sup\{f(\eta) : \eta \in (0, \epsilon)\}$. If $\lim_{\epsilon\downarrow 0} f(\epsilon)$ does not exist (as a finite number), we take $\delta(\epsilon) = \sup\{f(\eta) : \eta \in (\epsilon, \epsilon_0)\}$. ∎

Examples of order functions are 1, ϵ, $e^{-1/\epsilon}$, and $\log(1/\epsilon)$.

Exercises

1. Which of the following are order functions: (a) $\epsilon^{1/5}$, (b) $e^{1/\epsilon}$, (c) $1 + \sin(1/\epsilon)$, (d) $e^{-1/\epsilon}(1 + \sin(1/\epsilon))$, (e) $\log(\sin(\epsilon))$, (f) $\log(\sin(1/\epsilon))$.
2. Show that, if $f = O(\delta)$, then $f/\delta = O(1)$. Does the same property hold for the symbols o and $O^\sharp$?

4 Asymptotic Sequences and Asymptotic Series

As can be seen from the examples, some order functions have a finite limit, others an infinite limit as $\epsilon \downarrow 0$. In fact, it is not difficult to show that every order function has a (finite or infinite) limit as $\epsilon \downarrow 0$. This observation motivates us to view $\mathcal{O}$ as the union of two sets, $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$, where the elements of $\mathcal{O}_1$ have a finite limit and those of $\mathcal{O}_2$ have an infinite limit as $\epsilon \downarrow 0$. Each of these subsets is closed under the operations of addition and multiplication. Thus, we can form sequences of order functions by taking successive powers, for example, $\{\epsilon^n : n = 0, 1, \ldots\}$, $\{e^{-n/\epsilon} : n = 0, 1, \ldots\}$, and $\{(\log(1/\epsilon))^n : n = 0, 1, \ldots\}$. Sequences of this type, which are linearly ordered, play an important role in asymptotic analysis.

Definition 4 (i) *A linearly ordered sequence* $\{\delta_n : n = 0, 1, \ldots\}$ *of order functions* $\delta_n \in \mathcal{O}$ *is an* asymptotic sequence *if* $\delta_{n+1} = o(\delta_n)$ *for all* n. (ii) *A linearly ordered sequence* $\{f_n : n = 0, 1, \ldots\}$ *of functions* f_n *on* $(0, \epsilon_0) \times D$ *is an* asymptotic sequence *if there exists an asymptotic sequence of order functions* $\{\delta_n : n = 0, 1, \ldots\}$ *such that* $f_n = O^\sharp(\delta_n)$ *for all* n. *Here, the sequences can be finite or infinite.*

Asymptotic sequences generate asymptotic series by the usual linear operations.

Definition 5 *A sum* $\sum_n a_n f_n$ *is an* asymptotic series *if* $\{f_n : n = 0, 1, \ldots\}$ *is an asymptotic sequence and the coefficients* a_n *are all* $O^\sharp(1)$. *An asymptotic series can have a finite or infinite number of terms.*

The concept of an infinite asymptotic series is purely formal; nothing is said about the convergence or divergence of the series. In fact, the question of convergence of an asymptotic series plays no role in asymptotic analysis.

A relation of the type $\delta_{n+1} = o(\delta_n)$ is usually interpreted to mean that "δ_{n+1} is asymptotically smaller than δ_n." One might ask whether, in the set $\mathcal{O}$, there exist sequences of order functions that become "arbitrarily small" in the asymptotic sense. The following theorem, essentially due to Du Bois–Reymond, shows that this is not the case [3].

Theorem 2 *For every asymptotic sequence* $\{\delta_n : n = 0, 1, \ldots\}$ *there exists an order function* $\delta \in \mathcal{O}$ *such that* $\delta = o(\delta_n)$ *for all* n.

Proof. If $\lim_{\epsilon \downarrow 0} \delta_n(\epsilon)$ is positive for all sufficiently large n, it suffices to take $\delta(\epsilon) = 1$. Suppose $\delta_n = o(1)$ for all sufficiently large n. Then there certainly exists a monotonically decreasing sequence $\{\epsilon_n : n = 1, 2, \ldots\}$ of positive numbers $\epsilon_n \in (0, \epsilon_0)$ converging to zero, such that, for each n, $\delta_{n+1}(\epsilon) < \delta_n(\epsilon)$ for all $\epsilon \in (0, \epsilon_n)$. Applying a diagonal procedure, we define δ as a monotone continuous function on $(0, \epsilon_0)$ such that $\delta(\epsilon_n) = \delta_{n+1}(\epsilon_n)$. Then $\delta(\epsilon)/\delta_n(\epsilon) < \delta_{n+1}(\epsilon)/\delta_n(\epsilon)$ for all $\epsilon \in (0, \epsilon_n)$, where the upper bound tends to 0 as $\epsilon \downarrow 0$. ∎

As an example, consider the asymptotic sequence $\{\delta_n : n = 0, 1, \ldots\}$ with $\delta_n(\epsilon) = \epsilon^n$. Here $\delta = o(\delta_n)$ for $n = 0, 1, \ldots$ for $\delta(\epsilon) = e^{-1/\epsilon}$. Functions of the order of magnitude of δ are often called "transcendentally small."

Exercises

1. Show that, for any nontrivial function f on $(0, \epsilon_0) \times D$, there exists an order function $\delta \in \mathcal{O}$ such that $f = O^\sharp(\delta)$.

5 Gauge Sets

In an asymptotic sequence, there is a linear ordering of the elements as indicated by the index n. Such an ordering does not always exist in the more general concept of a gauge set, which we now define. (A "gauge" is an instrument for or a means of measuring. Hint for nonnative English speakers: "gauge" rhymes with "cage.")

Definition 6 (i) *A* gauge set *is a subset of $\mathcal{O}$ that is totally ordered with respect to the relation "$\prec$ or $=$." A gauge set is denoted by $\mathcal{E}$.* (ii) *A function $f : (0, \epsilon_0) \to X$ is* measurable *with respect to $\mathcal{E}$ if there exists an element $\delta \in \mathcal{E}$ such that $f = O^\sharp(\delta)$ as $\epsilon \downarrow 0$.*

Observe that the element δ in part (ii) of the definition is uniquely determined once the gauge set $\mathcal{E}$ is given.

An example of a nontrivial gauge set is $\mathcal{E} = \{\epsilon^m(\log(1/\epsilon))^{-n} : m, n = 0, 1, \ldots\}$. Although $\mathcal{E}$ is countable, its elements cannot be ordered as in an asymptotic sequence; here, the natural ordering is lexicographical—that is, if $\delta_{m,n}(\epsilon) = \epsilon^m(\log(1/\epsilon))^{-n}$, then $\delta_{p,q} = o(\delta_{m,n})$ if $p > m$ or if $p = m$ and $q > n$.

If an operation like multiplication or inversion maps the elements of a given gauge set into elements that are measurable with respect to the same gauge set, we say that the gauge set is *stable* under the given operation.

Exercises

1. Verify that the gauge set $\{\epsilon^m(\log(1/\epsilon))^{-n} : m, n = 0, 1, \ldots\}$ is stable under multiplication, differentiation, and integration.
2. Show that the set $\{e^{-p/\epsilon}(\log(1/\epsilon))^q : p, q \in \mathbf{Q}\}$ is a gauge set. ($\mathbf{Q}$ is the set of all rational numbers.) Is this set stable under integration?

6 Asymptotic Approximations and Asymptotic Expansions

Given a gauge set $\mathcal{E}$, we can define the concept of an asymptotic approximation.

Definition 7 *The function* $g : (0, \epsilon) \rightarrow X$ *is an* asymptotic approximation of the order of δ *of the function* $f : (0, \epsilon_0) \rightarrow X$ *if there exists a gauge* $\delta \in \mathcal{E}$ *such that* $g = O^\sharp(\delta)$ *and* $f - g = o(\delta)$.

If the function f in Definition 7 is itself also $O^\sharp(\delta)$, and the order function δ is clear from the context, we may use the special notation $f \sim g$ to denote that g is *asymptoticlly similar* to f. Thus, $f \sim g$ implies both $f - g = o(f)$ and $f - g = o(g)$.

If the gauge set $\mathcal{E}$ is an asymptotic sequence with the natural ordering, we can repeatedly apply the definition to find asymptotic approximations of successively higher order,

$$g = f - (((f - g_0) - g_1) - \cdots),$$

where $g_0 = O^\sharp(\delta_0)$, $g_1 = O^\sharp(\delta_1)$, $\ldots$. The approximation obtained after m steps is $g = \sum_{n=0}^{m-1} g_n$; by rescaling, we obtain what is called an asymptotic expansion.

Definition 8 *An* asymptotic expansion *of the function* $f : (0, \epsilon_0) \rightarrow X$ *is an asymptotic approximation* $g : (0, \epsilon_0) \rightarrow X$ *of the form* $g(\epsilon, x) = \sum_n \delta_n(\epsilon) f_n(\epsilon, x)$*, where each coefficient* $f_n : (0, \epsilon_0) \rightarrow X$ *satisfies the order relation* $f_n = O^\sharp(1)$. *The asymptotic expansion is said to be* to m terms *if the sum contains* m *terms* $(n = 0, \ldots, m-1)$*; here,* m *can be finite or infinite.*

The case where the functions f_n in the asymptic expansion are independent of ϵ is special and merits discussion.

Suppose $f : (0, \epsilon_0) \rightarrow X$ has an asymptotic approximation $g : (0, \epsilon_0) \rightarrow X$ of the form $g(\epsilon, x) = \sum_n \delta_n(\epsilon) f_n(x)$, where each $f_n \in X$ is independent of ϵ. Then,

$$\lim_{\epsilon \downarrow 0} \| f^{(n)}(\epsilon)/\delta_n(\epsilon) - f_n \|_X = 0, \quad n = 0, 1, \ldots,$$

where $f^{(0)}(\epsilon) = f(\epsilon)$ and $f^{(n)}(\epsilon) = f(\epsilon) - \sum_{p=0}^{n-1} \delta_p(\epsilon) f_p$ for $n = 1, 2, \ldots$. Hence, the fact that f has an asymptotic expansion with ϵ-independent coefficients implies that there exist nontrivial functions $f_0, f_1, \ldots$ in X such that each f_n is the limit as $\epsilon \downarrow 0$ of an expression involving f and the previous coefficients $f_0, \ldots, f_{n-1}$. In other words, the coefficients are uniquely determined (with respect to the given gauge set) and can be calculated explicitly by taking limits in X. This property is very special and motivates the final definition of this chapter.

Definition 9 *The function* $f : (0, \epsilon) \rightarrow X$ *has a* regular asymptotic expansion *on* D *if there exist an asymptotic sequence* $\{\delta_n : n = 0, 1, \ldots\}$ *of order functions* δ_n *and a nontrivial sequence* $\{f_n : n = 0, 1, \ldots\}$ *of elements* $f_n \in X$*, which do not depend on* ϵ*, such that the function* $Ef : (0, \epsilon_0) \rightarrow X$ *defined by the expression* $Ef(\epsilon) = \sum_n \delta_n(\epsilon) f_n$ *is an asymptotic approximation of* f *on* D.

We emphasize that "regularity" is an attribute of the asymptotic expansion of a function, not of the function itself. In fact, it is easy to think of examples of functions that have a regular asymptotic expansion, yet are singular in the sense of the classical theory of functions.

The discussion preceding Definition 9 is summarized in the following theorem.

Theorem 3 *The coefficients $f_n \in X$ in a regular asymptotic expansion $Ef = \sum_n \delta_n f_n$ of $f : (0, \epsilon_0) \to X$ are uniquely determined and found by taking limits in X,*

$$f_n = \lim_{\epsilon \downarrow 0} \frac{f^{(n)}(\epsilon)}{\delta_n(\epsilon)}, \quad n = 0, 1, \ldots,$$

where $f^{(0)}(\epsilon) = f(\epsilon)$ and $f^{(n)}(\epsilon) = f(\epsilon) - \sum_{p=0}^{n-1} \delta_p(\epsilon) f_p$ for $n = 1, 2, \ldots$.

A regular asymptotic approximation is sometimes called a *Poincaré expansion* although the expansions considered by Poincaré were of the general type, with coefficients that could be ϵ-dependent.

An asymptotic expansion may be defined up to a specified number of terms or up to a specified order of accuracy, depending on the particular application. The number of terms in the expansion may be finite or infinte; if it is infinite, nothing is said about the convergence or divergence of the expansion.

Roughly speaking, an asymptotic expansion can fail to be regular in either of two ways. It can be regular "almost everywhere" on D, that is, regular except on a subset of D of (N-dimensional) measure 0; or it can be "strictly singular" on D, that is, nowhere regular or regular outside some subset of D of positive measure. The latter situation occurs, for example, when f is oscillatory and the limits that define the expansion coefficients do not exist on D; the former situation is characteristic for problems with "boundary-layer behavior." The asymptotic methods that have been developed for these two types of problem are quite different and do not lend themselves to a comprehensive treatment. In this book, the focus will be entirely on functions that show boundary-layer behavior. Readers who are interested in oscillatory functions are referred to the literature on the subject; we mention the monographs of Nayfeh [8] and Roseau [9, 10].

Exercises

1. Find the asymptotic expansion of f, defined by $f(\epsilon, x) = (1 - \epsilon x/(1+\epsilon))^{-1}$ for $x \in [0, 1]$, with respect to the asymptotic sequence (a) $\{(\epsilon/(1+\epsilon))^n : n = 0, 1, \ldots\}$, and (b) $\{\epsilon^n : n = 0, 1, \ldots\}$.

7 Regular Initial Value Problems

We conclude this chapter with an example to illustrate the concept of regular approximations and regular expansions. The example is concerned with a family of initial value problems, parameterized by a small parameter ϵ. The gauge set is $\mathcal{E} = \{\epsilon^n : n = 0, 1, \ldots\}$.

Let $\{f(\epsilon) : \epsilon \in (0, \epsilon_0)\}$ be a family of vector fields mapping a domain U in an $(N+1)$-dimensional Euclidean vector space with coordinates (t, x), where $x = (x_1, \ldots, x_N)$, (the "extended phase space") into an N-dimensional Euclidean vector space with coordinates $(f_1, \ldots, f_N)$. The vector fields define a family of initial value problems,

$$\dot{x} = f(\epsilon, t, x),\ t > 0, \quad x(0) = \xi, \tag{7.1}$$

where ξ is fixed. (The symbol $\dot{}$ denotes differentiation with respect to t.)

Using the identification $x(\epsilon)(t) = x(\epsilon, t)$, we interpret (7.1) as an abstract initial value problem for the vector-valued function $x \in C((0, \epsilon_0); X)$, where $X = (C([0,T]), \|\cdot\|_\infty)$ for some $T > 0$. The values of $x(\epsilon)$ are vectors that belong to some bounded set $V \subset \mathbf{R}^N$, and if T is sufficiently small, then the cylinder $V_T = [0,T] \times V$ is entirely contained in U. We prove the following theorem.

Theorem 4 *If $f(\epsilon)$ has a regular asymptotic expansion $\sum_{n=0}^{m-1} \epsilon^n f_n$ on U, where f_n is continuous with respect to t and $(m+1-n)$ times continuously Fréchet differentiable with respect to x for $n = 0, \ldots, m-1$, then the solution $x(\epsilon)$ of (7.1) has a regular asymptotic expansion $\sum_{n=0}^{m-1} \epsilon^n x_n$ as $\epsilon \downarrow 0$. The leading coefficient x_0 is found by solving the differential equation $\dot{x}_0 = f_0(t, x_0)$ for $t > 0$, subject to the initial condition $x_0(0) = \xi$; the higher-order coefficients x_n $(n = 1, \ldots, m-1)$ are found by solving a linear inhomogeneous differential equation of the form $\dot{x}_n = f_0'(t, x_0(t))x_n + b_n(t)$ for $t > 0$, subject to the initial condition $x_n(0) = 0$. Here, $f_0'(t, x_0(t))$ is the Fréchet derivative of f_0 with respect to x at $x_0(t)$.*

Proof. We prove the theorem successively for $m = 1, 2, \ldots$.

(i) $m = 1$. Since $x(\epsilon)$ is a solution of the initial value problem, it satisfies the integral equation

$$x(\epsilon, t) = \xi + \int_0^t f_0(s, x(\epsilon, s))\, ds + \int_0^t f^{(1)}(\epsilon, s, x(\epsilon, s))\, ds,$$

where $f^{(1)}(\epsilon) = f(\epsilon) - f_0$. Let x_0 be the solution of the initial value problem

$$\dot{x}_0 = f_0(t, x_0),\ t > 0, \quad x_0(0) = \xi,$$

so $x_0(t) = \xi + \int_0^t f_0(s, x_0(s))\, ds$.

Consider the function $x^{(1)}(\epsilon) = x(\epsilon) - x_0$. It is defined in such a way that the equation

$$x^{(1)}(\epsilon, t) = \int_0^t \left[f_0(s, x(\epsilon, s)) - f_0(s, x_0(s)) \right] \, ds + \int_0^t f^{(1)}(\epsilon, s, x(\epsilon, s)) \, ds$$

is satisfied for all t. We estimate each integral in the right member.

According to the Mean Value Theorem for multidimensional mappings, we have

$$f_0(s, x(\epsilon, s)) - f_0(s, x_0(s)) = f_0'(s, x_0(s) + \sigma x^{(1)}(\epsilon, s)) x^{(1)}(\epsilon, s),$$

where f_0' is the Fréchet derivative of f_0 with respect to x and $\sigma = (\sigma_1, \ldots, \sigma_N)$ is a vector with components between 0 and 1. More precisely,

$$f_0'(s, x_0(s) + \sigma x^{(1)}(\epsilon, s)) = \begin{pmatrix} f_{0,1}' \left(s, x_0(s) + \sigma_1 x^{(1)}(\epsilon, s) \right) \\ \vdots \\ f_{0,N}' \left(s, x_0(s) + \sigma_N x^{(1)}(\epsilon, s) \right) \end{pmatrix};$$

see, for example, [11, Section 3.2]. The Fréchet derivative, which is a linear operator from $\mathbf{R}^N$ into itself, is uniformly bounded on the convex compact set V_T, so

$$|f_0(s, x(\epsilon, s)) - f_0(s, x_0(s))| \le C |x^{(1)}(\epsilon, s)|,$$

for some positive constant C. Thus we obtain the following estimate for the first integral:

$$\left| \int_0^t \left[f_0(s, x(\epsilon, s)) - f_0(s, x_0(s)) \right] \, ds \right| \le C \int_0^t |x^{(1)}(\epsilon, s)| \, ds.$$

Next, we observe that $f^{(1)}(\epsilon) = O(\epsilon)$ as $\epsilon \downarrow 0$, so there exists a positive constant c such that $f^{(1)}(\epsilon) \le c\epsilon$ on V_T. Thus we obtain the following estimate for the second integral:

$$\left| \int_0^t f^{(1)}(\epsilon, s, x(\epsilon, s)) \, ds \right| \le \epsilon c t.$$

Combining these two estimates, we conclude that

$$|x^{(1)}(\epsilon, t)| \le C \int_0^t \left| x^{(1)}(\epsilon, s) \right| \, ds + \epsilon c t.$$

Using Gronwall's inequality, we obtain the estimate

$$\left| x^{(1)}(\epsilon, t) \right| \le \epsilon c' \left(e^{Ct} - 1 \right);$$

hence, taking the supremum over all $t \in [0, T]$, we obtain

$$\|x^{(1)}(\epsilon)\| \le \epsilon c' \left(e^{CT} - 1 \right) \le \epsilon C'.$$

In other words, $x^{(1)}(\epsilon) = O(\epsilon)$. This proves the claim for $m = 1$.

(ii) $m = 2$. The proof is similar, but more complicated. We take x_0 from step (i) and define x_1 as the solution of the initial value problem

$$\dot{x}_1 = f_0'(t, x_0(t))x_1 + b_1(t),\ t > 0, \quad x_1(0) = 0,$$

where $b_1(t) = f_1(t, x_0(t))$. Thus, $x_1(t) = \int_0^t f_0'(s, x_0(s))x_1(s)\ ds + \int_0^t f_1(s, x_0(s))\ ds$.

With the definition $x^{(2)}(\epsilon) = x(\epsilon) - x_0 - \epsilon x_1$ we have

$$x^{(2)}(\epsilon, t) = \int_0^t \left[f_0(s, x(\epsilon, s)) - f_0(s, x_0(s)) - \epsilon f_0'(s, x_0(s))x_1(s)\right]\ ds$$

$$+\epsilon \int_0^t \left[f_1(s, x(\epsilon, s)) - f_1(s, x_0(s))\right]\ ds + \int_0^t f^{(2)}(\epsilon, s, x(\epsilon, s))\ ds.$$

Using the Mean Value Theorem, we rewrite the first integrand adding and subtracting terms,

$$x^{(2)}(\epsilon, t) = \int_0^t f_0'(s, x_0(s) + \sigma x^{(1)}(\epsilon, s))x^{(2)}(\epsilon, s)\ ds$$

$$+\epsilon \int_0^t [f_0'(s, x_0(s) + \sigma x^{(1)}(\epsilon, s)) - f_0'(s, x_0(s))]x_1(s)\ ds$$

$$+\epsilon \int_0^t \left[f_1(s, x(\epsilon, s)) - f_1(s, x_0(s))\right]\ ds + \int_0^t f^{(2)}(\epsilon, s, x(\epsilon, s))\ ds,$$

and estimate each of the integrals in the right member of this expression.

Because the Fréchet derivative is uniformly bounded on V_T, it follows immediately that

$$\left| \int_0^t f_0'(s, x_0(s) + \sigma x^{(1)}(\epsilon, s))x^{(2)}(\epsilon, s)\ ds \right| \leq C \int_0^t |x^{(2)}(\epsilon, s)|\ ds.$$

According to the Mean Value Theorem, there exists a vector $\tau = (\tau_1, \ldots, \tau_N)$, with components between 0 and 1, such that

$$f_0'(s, x_0(s) + \sigma x^{(1)}(\epsilon, s)) - f_0'(s, x_0(s)) = \begin{pmatrix} f_{0,1}'' \left(s, x_0(s) + \tau_1 x^{(1)}(\epsilon, s)\right) \\ \vdots \\ f_{0,N}'' \left(s, x_0(s) + \tau_N x^{(1)}(\epsilon, s)\right) \end{pmatrix} x^{(1)}(\epsilon, s).$$

The operator represented by the matrix is uniformly bounded on the convex compact set V_T, so

$$\left| \int_0^t [f_0'(s, x_0(s) + \sigma x^{(1)}(\epsilon, s)) - f_0'(s, x_0(s))]x_1(s)\ ds \right| \leq ct.$$

The third and fourth integral are estimated as in step (i); the former is bounded by a constant multiple of t, the latter by a constant multiple of ϵt.

Taking all the estimates together, we obtain

$$|x^{(2)}(\epsilon,t)| \leq C\int_0^t |x^{(2)}(\epsilon,s)|\,ds + \epsilon ct.$$

Hence, proceeding as in step (i), we find that $\|x^{(2)}(\epsilon)\| \leq \epsilon^2 C'$, so $x^{(2)}(\epsilon) = O(\epsilon^2)$. This proves the claim for $m = 2$.

(iii) The process can be continued for successive values of m. After m steps, one takes the coefficients $x_0, \ldots, x_{m-1}$ from all previous steps and introduces a new coefficient x_m by solving a linear initial value problem of the form

$$\dot{x}_m = f_0'(t, x_0(t))x_m + b_m(t),\ t > 0, \quad x_m(0) = 0.$$

The inhomogeneous term b_m is defined in terms of x_0, …, x_{m-1}. Then one considers the equation that is satisfied by the function $x^{(m)}(\epsilon,t) = x(\epsilon,t) - \sum_{n=0}^{m-1}\epsilon^n x_n(t)$ and estimates the various terms. Using Gronwall's inequality, one shows that $x^{(m)}(\epsilon) = O(\epsilon^m)$ as $\epsilon \downarrow 0$. ∎

The crucial point in the construction of the asymptotic expansion of $x(\epsilon)$ is the definition of the inhomogeneous term b_n. It can be obtained formally by the following process: (1) Define $y(\epsilon,t) = \sum_{n=0}^{\infty}\epsilon^n x_n(t)$; (2) expand $f_n(t, y(\epsilon,t))$ in a Taylor series expansion near $(t, x_0(t))$, for $n = 0, 1, \ldots$; (3) substitute the expansions in the formal sum $\sum_{n=0}^{\infty}\epsilon^n f_n(t, y(\epsilon,t))$; (4) rearrange the terms, grouping them in like powers of ϵ. The coefficient of ϵ^n then corresponds to the expression in the right-hand side of the differential equation for $\dot{x}_n$, i.e., $f_0'(t, x_0(t))x_n + b_n(t)$.

The general expression for b_n becomes increasingly complicated as n increases; for example,

$$b_2(t) = f_2(t, x_0(t)) + f_1'(t, x_0(t))x_1(t) + \tfrac{1}{2}f_0''(t, x_0(t))x_1(t)x_1(t),$$

$$\begin{aligned} b_3(t) = {} & f_3(t, x_0(t)) + f_2'(t, x_0(t))x_1(t) + \tfrac{1}{2}f_1''(t, x_0(t))x_1(t)x_1(t) + f_1'(t, x_0(t))x_2(t) \\ & + \frac{1}{6}f_0'''(t, x_0(t))x_1(t)x_1(t)x_1(t) + f_0''(t, x_0(t))x_1(t)x_2(t) + f_0'(t, x_0(t))x_3(t). \end{aligned}$$

These coefficients are best calculated on a case-by-case basis.

Note that, if the vector field is given as an (infinite) regular asymptotic expansion, the result of this procedure is an (infinite) regular asymptotic expansion of the solution. Nothing is said, however, about the convergence or divergence of the expansion in the classical sense.

Another caveat is in order. In the proof of the theorem, we considered the solution of the initial value problem only on a *finite* interval $[0, T]$; in fact, the arguments depended critically on the fact that T was finite. Consequently, nothing is said about the asymptotic behavior of global solutions—solutions that exist for all $t \geq 0$. It is indeed extremely risky to extend our results to an infinite time interval, as the limits $T \to \infty$ and $\epsilon \downarrow 0$ are not interchangeable without further restrictions on the vector field. (As it is, the conditions are already rather restrictive!)

In conclusion, we observe that the leading term in the asymptotic expansion, x_0, satisfies a nonlinear differential equation (the "unperturbed" equation) but that all higher-order terms satisfy a linear differential equation (the "variational" equation). This observation is intimately linked to the fact that the solution of a regularly perturbed differential equation depends continuously on the perturbation parameter ϵ—see, for example, [12, Sections 8.5 and 9.5].

Exercises

1. Construct an asymptotic expansion of the solution of the equation for the damped harmonic oscillator, $\ddot{x} + 2\epsilon\dot{x} + x = 0$, $t > 0$, that starts at $x(0) = a$ with velocity $\dot{x}(0) = 0$. Compare the expansion with the exact solution.
2. Consider the system
$$\dot{x} = \epsilon f_1(x,t) + \epsilon^2 f_2(x,t),\ t > 0, \quad x(0) = \xi,$$
$$\dot{y} = \epsilon f_1(y,t)\ t > 0, \quad y(0) = \xi,$$
Construct an asymptotic expansion of the solution (x, y) as $\epsilon \downarrow 0$.

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