Four Challenge Problems in Equational Logic

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The following four problems arose in some work I am doing with R. Padmanabhan of the University of Manitoba. All were a challenge for our prover Otter (we still don't have a proof for the fourth), and I'd be very interested to see how other equational provers handle them. See `ftp://info.mcs.anl.gov/pub/Otter/AAR28-challenge` for a directory of Otter input files and proofs.

Problem A. On commutators in group theory.

$$\left\{\begin{array}{l} ex = x \\ x^{-1}x = e \\ (xy)z = x(yz) \\ [x,y] = x^{-1}y^{-1}xy \\ x^3 = e \end{array}\right\} \Rightarrow \{[[[y,z],u],v] = e\}.$$

Otter, with a specialized formulation and strategy, takes several hours to find a proof.

Problem B. On semigroups.

$$\{(xy)z = x(yz),\ xy^3 = y^3x\} \Rightarrow \{(xy)^9 = x^9y^9\}.$$

Otter, with a specialized formulation and strategy, takes several minutes to find a proof.

Problem C. Quasilattice theory (QLT) is defined by

$$\left\{\begin{array}{ll} x \wedge x = x & x \vee x = x \\ x \wedge y = y \wedge x & x \vee y = y \vee x \\ (x \wedge y) \wedge z = x \wedge (y \wedge z) & (x \vee y) \vee z = x \vee (y \vee z) \\ (x \wedge (y \vee z)) \vee (x \wedge y) = x \wedge (y \vee z) & (x \vee (y \wedge z)) \wedge (x \vee y) = x \vee (y \wedge z) \end{array}\right\}.$$

The challenge is to prove one form of modularity from another:

$$\left\{\begin{array}{l} \text{QLT} \\ (x \wedge y) \vee (z \wedge (x \vee y)) = (x \vee y) \wedge (z \vee (x \wedge y)) \end{array}\right\} \Rightarrow \{x \wedge (y \vee (x \wedge z)) = (x \wedge y) \vee (x \wedge z)\}.$$

With a straightforward strategy, Otter takes about 10 minutes to find a proof. I have not tried to prove this theorem with an associative-commutative program.

Problem D. The quasivariety HBCK is defined by

$$\left\{\begin{array}{ll} (M3)\ \ x \cdot 1 = 1 & (M8)\ \ x \cdot x = 1 \\ (M4)\ \ 1 \cdot x = x & (M9)\ \ x \cdot (y \cdot z) = y \cdot (x \cdot z) \\ (M5)\ \ (x \cdot y) \cdot ((z \cdot x) \cdot (z \cdot y)) = 1 & (H)\ \ (x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \\ (M7)\ \ x \cdot y = 1,\ y \cdot x = 1\ \rightarrow\ x = y & \end{array}\right\}.$$

Consider

$$(J)\quad (((x \cdot y) \cdot y) \cdot x) \cdot x = (((y \cdot x) \cdot x) \cdot y) \cdot y.$$

Theorem (Blok and Ferreirim). HBCK is an equational class of algebras defined by the identities {M3,M4,M5,M8,M9,H,J}.

The challenge is to prove {M3,M4,M5,M7,M8,M9,H} $\Rightarrow$ {J}. I have not been able to get Otter to prove this. The only known proof is model theoretic.