Automated Equational Deduction with Otter

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Chapter 1

Introduction

This is the introduction.

Chapter 2

Otter and MACE

The theorem prover is Otter [6], and the program that searches for models and counterexamples is MACE [5]. Some references on automated reasoning with Otter are [13] and [2].

2.1 Otter

2.1.1 Notes on Otter Proof Notation

Double identifiers for dynamic demoudlators.

An input clause occurs twice, once ordinary, once demouldator.

Sequent/box notation for nonunits.

2.2 MACE

Chapter 3

Lattices and Lattice-like Structures

The following eight identities characterize lattice theory (LT).

$$
\begin{array}{ll}
x \wedge y = y \wedge x & x \vee y = y \vee x \\
(x \wedge y) \wedge z = x \wedge (y \wedge z) & (x \vee y) \vee z = x \vee (y \vee z) \\
x \wedge x = x & x \vee x = x \\
x \wedge (x \vee y) = x & x \vee (x \wedge y) = x
\end{array}
$$

The following four absorption identities, due to R. McKenzie [7], also characterize lattice theory.

$$
\begin{array}{ll}
y \vee (x \wedge (y \wedge z)) = y & y \wedge (x \vee (y \vee z)) = y \\
((x \wedge y) \vee (y \wedge z)) \vee y = y & ((x \vee y) \wedge (y \vee z)) \wedge y = y
\end{array}
$$

Note that all of McKenzie's identities are absorption laws.

Problem 68A. Uniqueness of the meet operation in a lattice. If $\wedge$, $\vee$, and $\cdot$ are binary operations such that both $(\mathrm{L}; \vee, \wedge)$ and $(\mathrm{L}; \vee, \cdot)$ are lattices, then $x \wedge y = x \cdot y$.

Proof (found by Otter 3.0.3+ on gyro at 73.67 seconds).

1	$A \wedge B \neq A \cdot B$
5	$x \wedge y = y \wedge x$
6	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$
10	$x \cdot y = y \cdot x$
11	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
15	$x \vee y = y \vee x$
18	$x \wedge (x \vee y) = x$
20	$x \vee (x \wedge y) = x$
22	$x \cdot (x \vee y) = x$

24	$x \vee (x \cdot y) = x$	
33	$x \wedge (y \vee x) = x$	[15 → 18]
37	$(x \vee y) \wedge x = x$	[5 → 18]
41	$(x \wedge y) \vee (x \wedge (y \wedge z)) = x \wedge y$	[6 → 20]
43	$x \vee (y \wedge x) = x$	[5 → 20]
45	$(x \wedge y) \vee x = x$	[15 → 20]
47	$x \cdot (y \vee x) = x$	[15 → 22]
49	$(x \vee y) \cdot x = x$	[10 → 22]
64	$x \vee (y \cdot x) = x$	[10 → 24]
66	$(x \cdot y) \vee x = x$	[15 → 24]
74	$(x \vee y) \wedge y = y$	[5 → 33]
84	$(x \wedge y) \vee y = y$	[15 → 43]
123	$x \cdot (y \cdot (z \vee (x \cdot y))) = x \cdot y$	[11 → 47]
130,129	$x \cdot (x \wedge y) = x \wedge y$	[45 → 49]
164	$x \wedge (x \cdot y) = x \cdot y$	[66 → 37]
168	$x \wedge (y \cdot x) = y \cdot x$	[64 → 74]
183,182	$x \cdot (y \wedge x) = y \wedge x$	[84 → 49]
478	$(x \wedge y) \vee (x \wedge (z \cdot y)) = x \wedge y$	[168 → 41]
8072	$(x \wedge y) \vee (x \cdot y) = x \wedge y$	[164 → 478]
8199	$x \wedge y = x \cdot y$	[8072 → 123 :183,130]
8201	$\square$	[8199,1]

Problem 68Q. Uniqueness of the meet operation in a quasilattice. If $\wedge$, $\vee$, and $\cdot$ are binary operations such that both (QL; $\vee, \wedge$) and (QL; $\vee, \cdot$) are quasilattices, then $x \wedge y = x \cdot y$.

Counterexample. The clauses

$x \vee x = x$
$x \wedge x = x$
$x \cdot x = x$
$x \vee y = y \vee x$
$x \wedge y = y \wedge x$
$x \cdot y = y \cdot x$
$(x \vee y) \vee z = x \vee (y \vee z)$
$(x \wedge y) \wedge z = x \wedge (y \wedge z)$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$(x \wedge (y \vee z)) \vee (x \wedge y) = x \wedge (y \vee z)$
$(x \vee (y \wedge z)) \wedge (x \vee y) = x \vee (y \wedge z)$
$(x \cdot (y \vee z)) \vee (x \cdot y) = x \cdot (y \vee z)$
$(x \vee (y \cdot z)) \cdot (x \vee y) = x \vee (y \cdot z)$
$A \cdot B \neq A \wedge B$

have the following model (found by MACE 1.0.0 on gyro at 0.36 seconds).

v:

```
    | 0 1 2
  --+------
  0 | 0 0 0
  1 | 0 1 0
  2 | 0 0 2
^:
    | 0 1 2
  --+------
  0 | 0 0 0
  1 | 0 1 0
  2 | 0 0 2
*:
    | 0 1 2
  --+------
  0 | 0 1 0
  1 | 1 1 1
  2 | 0 1 2
A: 0
B: 1
```

Problem 68B. If $(\mathrm{QL}; \vee, \wedge)$ and $(\mathrm{QL}; \wedge, \cdot)$ are quasilattices, then $(\mathrm{QL}; \vee, \cdot)$ is also a quasilattice.

Counterexample. The clauses

$$
\begin{aligned}
&x \vee x = x\\
&x \wedge x = x\\
&x \cdot x = x\\
&x \vee y = y \vee x\\
&x \wedge y = y \wedge x\\
&x \cdot y = y \cdot x\\
&(x \vee y) \vee z = x \vee (y \vee z)\\
&(x \wedge y) \wedge z = x \wedge (y \wedge z)\\
&(x \cdot y) \cdot z = x \cdot (y \cdot z)\\
&(x \wedge (y \vee z)) \vee (x \wedge y) = x \wedge (y \vee z)\\
&(x \vee (y \wedge z)) \wedge (x \vee y) = x \vee (y \wedge z)\\
&(x \cdot (y \vee z)) \vee (x \cdot y) = x \cdot (y \vee z)\\
&(x \vee (y \cdot z)) \cdot (x \vee y) = x \vee (y \cdot z)\\
&(A \wedge (B \cdot C)) \cdot (A \wedge B) \neq A \wedge (B \cdot C)
\end{aligned}
$$

have the following model (found by MACE 1.0.0 on gyro at 0.37 seconds).

```
v:
    | 0 1 2
  --+------
  0 | 0 0 2
  1 | 0 1 2
```

```
    2 | 2 2 2
 ^:
      | 0 1 2
    --+------
    0 | 0 1 0
    1 | 1 1 1
    2 | 0 1 2
 *:
      | 0 1 2
    --+------
    0 | 0 1 2
    1 | 1 1 2
    2 | 2 2 2
 A: 0
 B: 1
 C: 2
```

Problem 69. In lattice theory, a distributive law implies its dual.

$$\{\mathrm{LT}, x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\} \Rightarrow \{x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\}$$

Proof (found by Otter 3.0.3+ on gyro at 3.91 seconds).

4	$x \wedge y = y \wedge x$	
6,5	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	
9	$x \vee y = y \vee x$	
10	$(x \vee y) \vee z = x \vee (y \vee z)$	
12	$x \wedge (x \vee y) = x$	
14	$x \vee (x \wedge y) = x$	
16	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	
17	$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$	[flip 16]
19	$A \vee (B \wedge C) \neq (A \vee B) \wedge (A \vee C)$	
20	$x \wedge (y \vee x) = x$	[9 $\rightarrow$ 12]
22	$(x \vee y) \wedge x = x$	[4 $\rightarrow$ 12]
35	$x \vee (y \wedge x) = x$	[4 $\rightarrow$ 14]
39	$x \wedge (y \wedge x) = x \wedge y$	[14 $\rightarrow$ 20 :6]
53	$(x \wedge y) \vee y = y$	[9 $\rightarrow$ 35]
55	$x \vee ((y \wedge x) \vee z) = x \vee z$	[35 $\rightarrow$ 10, flip]
70	$(x \vee (y \vee z)) \wedge (x \vee y) = x \vee y$	[10 $\rightarrow$ 22]
91,90	$x \vee ((x \vee y) \wedge z) = (x \vee y) \wedge (x \vee z)$	[22 $\rightarrow$ 17]
145	$A \vee (C \wedge B) \neq (A \vee B) \wedge (A \vee C)$	[4 $\rightarrow$ 19]
346,345	$x \vee (y \wedge (x \vee z)) = x \vee (y \wedge z)$	[17 $\rightarrow$ 55]
502,501	$(x \vee y) \wedge (x \vee (z \wedge y)) = x \vee (z \wedge y)$	[53 $\rightarrow$ 70]
816	$x \vee (y \wedge z) = (x \vee z) \wedge (x \vee y)$	[39 $\rightarrow$ 90 :91,346,502, flip]
818	$\square$	[816,145]

Problem 70. Distributivity in quasilattices.

$$\{\text{QLT}, x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)\} \Rightarrow \{x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)\}$$

Proof (found by Otter 3.0.3+ on gyro at 7.61 seconds).

2	$x \wedge x = x$	
4	$x \wedge y = y \wedge x$	
5	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	
8,7	$x \vee x = x$	
9	$x \vee y = y \vee x$	
11,10	$(x \vee y) \vee z = x \vee (y \vee z)$	
14	$(x \vee (y \wedge z)) \wedge (x \vee y) = x \vee (y \wedge z)$	
16	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	
18,17	$(x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z)$	[flip 16]
19	$A \vee (B \wedge C) \neq (A \vee B) \wedge (A \vee C)$	
23	$x \wedge (x \wedge y) = x \wedge y$	[2 → 5, flip]
30	$x \vee (x \vee y) = x \vee y$	[7 → 10, flip]
43,42	$x \vee (y \vee x) = x \vee y$	[9 → 30]
51,50	$x \vee (y \vee (x \vee z)) = x \vee (y \vee z)$	[42 → 10 :11,11, flip]
58	$(x \vee ((x \vee y) \wedge z)) \wedge (x \vee y) = x \vee ((x \vee y) \wedge z)$	[30 → 14]
108	$x \vee (x \wedge y) = x \wedge (x \vee y)$	[2 → 17]
116	$x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee z)$	[23 → 17 :18, flip]
122,121	$(x \wedge y) \vee x = x \wedge (y \vee x)$	[2 → 17]
123	$x \wedge (y \vee z) = x \wedge (z \vee y)$	[9 → 17 :18]
163	$A \vee (C \wedge B) \neq (A \vee B) \wedge (A \vee C)$	[4 → 19]
181,180	$(x \vee (y \wedge z)) \wedge (x \vee (y \wedge (z \vee y))) = x \vee (y \wedge z)$	[14 → 108 :8,11,51,122, flip]
183	$x \vee (y \wedge x) = x \wedge (x \vee y)$	[4 → 108]
226	$(x \vee y) \wedge z = z \wedge (y \vee x)$	[4 → 123]
230	$x \wedge (y \vee z) = (z \vee y) \wedge x$	[flip 226]
916,915	$(x \vee (y \wedge z)) \wedge y = y \wedge (x \vee z)$	[4 → 116]
918,917	$x \vee ((x \vee y) \wedge z) = (x \vee y) \wedge (x \vee z)$	[58 :916, flip]
919	$x \vee (y \wedge (z \vee x)) = x \vee (y \wedge z)$	[116 → 183 :11,18,43,11,122,181]
1440	$x \vee (y \wedge z) = (x \vee z) \wedge (x \vee y)$	[230 → 919 :918, flip]
1442	$\square$	[1440,163]

Problem 70B. Bowden's inequality, $x \vee (y \wedge z) \geq (x \vee y) \wedge z$, implies distributivity in quasilattices.

Proof (found by Otter 3.0.3+ on gyro at 7.89 seconds).

3,2	$x \wedge x = x$
4	$x \wedge y = y \wedge x$
6,5	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$
8,7	$x \vee x = x$
9	$x \vee y = y \vee x$

11,10	$(x \vee y) \vee z = x \vee (y \vee z)$	
12	$(x \wedge (y \vee z)) \vee (x \wedge y) = x \wedge (y \vee z)$	
14	$(x \vee (y \wedge z)) \wedge (x \vee y) = x \vee (y \wedge z)$	
16	$(x \vee (y \wedge z)) \vee ((x \vee y) \wedge z) = x \vee (y \wedge z)$	
17	$x \vee ((y \wedge z) \vee ((x \vee y) \wedge z)) = x \vee (y \wedge z)$	[copy,16 :11]
21	$A \vee (B \wedge C) \neq (A \vee B) \wedge (A \vee C)$	
23	$x \wedge (x \wedge y) = x \wedge y$	[2 $\rightarrow$ 5, flip]
25	$x \wedge (y \wedge z) = y \wedge (z \wedge x)$	[4 $\rightarrow$ 5]
28	$x \wedge (y \wedge z) = z \wedge (x \wedge y)$	[flip 25]
29	$x \vee (y \vee z) = y \vee (x \vee z)$	[9 $\rightarrow$ 10 :11]
30	$x \vee (x \vee y) = x \vee y$	[7 $\rightarrow$ 10, flip]
32	$x \vee (y \vee z) = y \vee (z \vee x)$	[9 $\rightarrow$ 10]
39,38	$x \wedge (y \wedge x) = x \wedge y$	[4 $\rightarrow$ 23]
43,42	$x \vee (y \vee x) = x \vee y$	[9 $\rightarrow$ 30]
46	$x \wedge (y \wedge (x \wedge z)) = x \wedge (y \wedge z)$	[38 $\rightarrow$ 5 :6,6, flip]
48	$x \vee (y \vee (z \vee x)) = x \vee (y \vee z)$	[10 $\rightarrow$ 42]
52	$(x \wedge (y \vee z)) \vee (x \wedge z) = x \wedge (z \vee y)$	[9 $\rightarrow$ 12]
54	$((x \vee y) \wedge z) \vee (z \wedge x) = z \wedge (x \vee y)$	[4 $\rightarrow$ 12]
56	$x \vee (y \vee ((x \vee y) \wedge x)) = x \vee y$	[2 $\rightarrow$ 12 :11,3]
60	$(x \wedge ((x \wedge y) \vee z)) \vee (x \wedge y) = x \wedge ((x \wedge y) \vee z)$	[23 $\rightarrow$ 12]
63,62	$(x \wedge (y \vee z)) \vee (y \wedge x) = x \wedge (y \vee z)$	[4 $\rightarrow$ 12]
68	$(x \wedge (y \vee z)) \vee ((x \wedge y) \vee u) = (x \wedge (y \vee z)) \vee u$	[12 $\rightarrow$ 10, flip]
125	$x \vee (y \vee z) = z \vee (y \vee x)$	[9 $\rightarrow$ 32]
160,159	$x \vee (y \vee ((x \vee y) \wedge y)) = x \vee y$	[2 $\rightarrow$ 17 :3]
161	$x \vee ((y \wedge z) \vee ((y \vee x) \wedge z)) = x \vee (y \wedge z)$	[9 $\rightarrow$ 17]
230	$x \vee (((y \vee x) \wedge y) \vee y) = y \vee x$	[32 $\rightarrow$ 56]
249	$(x \vee y) \wedge (y \vee x) = x \vee y$	[2 $\rightarrow$ 52 :11,160, flip]
255	$(x \wedge (y \vee z)) \vee (z \wedge x) = x \wedge (z \vee y)$	[4 $\rightarrow$ 52]
290,289	$(x \vee y) \wedge (z \wedge (y \vee x)) = (x \vee y) \wedge z$	[249 $\rightarrow$ 46 :39, flip]
293	$x \wedge (y \vee z) = (z \vee y) \wedge x$	[249 $\rightarrow$ 28 :290]
322,321	$(x \vee y) \wedge (y \vee (x \wedge z)) = y \vee (x \wedge z)$	[14 $\rightarrow$ 293, flip]
350	$(x \wedge y) \vee (y \wedge x) = y \wedge x$	[7 $\rightarrow$ 54 :8]
365,364	$((x \vee y) \wedge x) \vee x = x \wedge (x \vee y)$	[2 $\rightarrow$ 54]
368	$x \vee (y \wedge (y \vee x)) = y \vee x$	[230 :365]
405,404	$(x \wedge y) \vee (z \vee (y \wedge x)) = (x \wedge y) \vee z$	[350 $\rightarrow$ 48 :43, flip]
408	$x \vee (y \wedge z) = (y \wedge z) \vee ((z \wedge y) \vee x)$	[350 $\rightarrow$ 125]
411	$x \vee (y \wedge z) = (z \wedge y) \vee x$	[350 $\rightarrow$ 29 :405]
413,412	$(x \wedge y) \vee ((y \wedge x) \vee z) = (y \wedge x) \vee z$	[350 $\rightarrow$ 10, flip]
414	$(x \wedge y) \vee z = z \vee (y \wedge x)$	[flip 408 :413]
430	$x \vee (y \wedge (x \vee y)) = y \vee x$	[9 $\rightarrow$ 368]
521,520	$(x \wedge y) \vee ((x \vee z) \wedge y) = y \wedge (x \vee z)$	[54 $\rightarrow$ 411, flip]
536	$x \vee (y \wedge (z \vee x)) = x \vee (z \wedge y)$	[161 :521]
590	$(x \wedge (y \vee x)) \vee y = x \vee y$	[9 $\rightarrow$ 430]
675	$(x \wedge y) \vee (((x \wedge y) \vee z) \wedge x) = x \wedge ((x \wedge y) \vee z)$	[414 $\rightarrow$ 60]
1481	$((x \vee y) \wedge (x \vee z)) \vee x = (x \vee y) \wedge (x \vee z)$	[364 $\rightarrow$ 68 :63, flip]

1510	$x \vee (y \wedge (x \vee z)) = x \vee (z \wedge y)$	[9 → 536]
1522,1521	$x \vee ((x \vee y) \wedge z) = x \vee (y \wedge z)$	[293 → 536]
1535,1534	$(x \wedge (y \vee z)) \vee z = z \vee (y \wedge x)$	[9 → 536]
1550	$x \vee (y \wedge z) = x \vee (z \wedge (x \vee y))$	[flip 1510]
1559,1558	$(x \wedge y) \vee (z \wedge x) = x \wedge ((x \wedge y) \vee z)$	[675 :1522]
1617,1616	$x \wedge (y \vee (z \wedge x)) = x \wedge (y \vee z)$	[255 :1559,1535]
1634,1633	$(x \wedge (y \vee z)) \vee y = y \vee (z \wedge x)$	[536 → 590 :6,322,1617,1535]
1645,1644	$x \vee (y \wedge (x \vee z)) = (x \vee z) \wedge (x \vee y)$	[1481 :1634]
1653	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	[1550 :1645]
1655	$\square$	[1653,21]

Problem 70C. Modularity in quasilattices.

$\{\mathrm{QLT}, (x\wedge y)\vee(z\wedge(x\vee y)) = (x\vee y)\wedge(z\vee(x\wedge y))\} \Rightarrow \{x\wedge(y\vee(x\wedge z)) = (x\wedge y)\vee(x\wedge z)\}$.

Proof (found by Otter 3.0.3+ on gyro at 775.31 seconds).

1	$x = x$	
3,2	$x \wedge x = x$	
4	$x \wedge y = y \wedge x$	
6,5	$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	
8,7	$x \vee x = x$	
9	$x \vee y = y \vee x$	
11,10	$(x \vee y) \vee z = x \vee (y \vee z)$	
12	$(x \wedge (y \vee z)) \vee (x \wedge y) = x \wedge (y \vee z)$	
14	$(x \vee (y \wedge z)) \wedge (x \vee y) = x \vee (y \wedge z)$	
16	$(x \wedge y) \vee (z \wedge (x \vee y)) = (x \vee y) \wedge (z \vee (x \wedge y))$	
17	$A \wedge (B \vee (A \wedge C)) \neq (A \wedge B) \vee (A \wedge C)$	
18	$(A \wedge B) \vee (A \wedge C) \neq A \wedge (B \vee (A \wedge C))$	[flip 17]
20	$x \wedge (y \wedge z) = y \wedge (x \wedge z)$	[4 → 5 :6]
22,21	$x \wedge (x \wedge y) = x \wedge y$	[2 → 5, flip]
23	$x \wedge (y \wedge z) = y \wedge (z \wedge x)$	[4 → 5]
26	$x \wedge (y \wedge z) = z \wedge (x \wedge y)$	[flip 23]
27	$x \vee (y \vee z) = y \vee (x \vee z)$	[9 → 10 :11]
29,28	$x \vee (x \vee y) = x \vee y$	[7 → 10, flip]
30	$x \vee (y \vee z) = y \vee (z \vee x)$	[9 → 10]
37,36	$x \wedge (y \wedge x) = x \wedge y$	[4 → 21]
40	$x \vee (y \vee x) = x \vee y$	[9 → 28]
43,42	$x \wedge (y \wedge (z \wedge x)) = x \wedge (y \wedge z)$	[5 → 36]
45,44	$x \wedge (y \wedge (x \wedge z)) = x \wedge (y \wedge z)$	[36 → 5 :6,6, flip]
49,48	$x \vee (y \vee (x \vee z)) = x \vee (y \vee z)$	[40 → 10 :11,11, flip]
52	$(x \wedge (y \vee z)) \vee (x \wedge z) = x \wedge (z \vee y)$	[9 → 12]
56	$((x \vee y) \wedge z) \vee (z \wedge x) = z \wedge (x \vee y)$	[4 → 12]
64	$(x \wedge (y \vee z)) \vee (y \wedge x) = x \wedge (y \vee z)$	[4 → 12]
68	$(x \wedge y) \vee (x \wedge (y \vee z)) = x \wedge (y \vee z)$	[9 → 12]

70	$(x \wedge (y \vee z)) \vee ((x \wedge y) \vee u) = (x \wedge (y \vee z)) \vee u$	[12 → 10, flip]
95	$x \vee (y \vee z) = x \vee (z \vee y)$	[9 → 27 :11]
100	$(x \vee (y \wedge (z \wedge u))) \wedge (x \vee u) = x \vee (u \wedge (y \wedge z))$	[23 → 14]
102	$(x \vee (y \wedge (z \wedge u))) \wedge (x \vee (y \wedge z)) = x \vee (y \wedge (z \wedge u))$	[5 → 14 :6]
110	$x \wedge (y \wedge ((x \wedge y) \vee x)) = x \wedge y$	[7 → 14 :6,8]
112	$(x \vee ((y \vee x) \wedge z)) \wedge (x \vee y) = x \vee ((y \vee x) \wedge z)$	[40 → 14]
118	$(x \vee (y \wedge z)) \wedge (y \vee x) = x \vee (y \wedge z)$	[9 → 14]
123,122	$(x \vee y) \wedge (x \vee (y \wedge z)) = x \vee (y \wedge z)$	[4 → 14]
139	$x \vee (y \vee z) = z \vee (y \vee x)$	[9 → 30]
175	$(x \wedge y) \vee (z \wedge (x \vee (y \wedge x))) = (x \vee (y \wedge x)) \wedge (z \vee (x \wedge y))$	[36 → 16 :37]
178	$x \vee (y \wedge x) = x \wedge (y \vee x)$	[2 → 16 :8,8,3]
189	$(x \vee (y \wedge x)) \wedge (z \vee (x \wedge y)) = (x \wedge y) \vee (z \wedge (x \vee (y \wedge x)))$	[flip 175]
192	$x \wedge (y \vee x) = x \vee (y \wedge x)$	[flip 178]
230,229	$(x \wedge y) \vee (y \wedge x) = x \wedge (y \wedge (y \vee (x \wedge y)))$	[36 → 178 :6]
239	$x \vee (x \wedge y) = x \wedge (y \vee x)$	[4 → 178]
242,241	$(x \wedge y) \vee y = y \wedge (x \vee y)$	[9 → 178]
261,260	$x \vee (x \wedge (y \vee x)) = x \vee (y \wedge x)$	[178 → 28]
266,265	$(x \wedge (y \vee x)) \vee z = x \vee ((y \wedge x) \vee z)$	[178 → 10]
275	$x \wedge (y \wedge (x \vee y)) = x \wedge (y \wedge (y \vee (x \wedge y)))$	[178 → 192 :6,22,37,230]
278	$x \vee (y \vee (x \wedge y)) = (x \vee y) \wedge (y \vee x)$	[40 → 192 :11,261, flip]
283	$(x \wedge y) \vee (x \wedge ((y \vee z) \wedge y)) = x \wedge (y \wedge (y \vee z))$	[12 → 192 :6,45,6,45, flip]
287,286	$x \vee (y \wedge x) = x \wedge (x \vee y)$	[9 → 192, flip]
289	$(x \vee y) \wedge y = y \wedge (y \vee x)$	[4 → 192 :287]
303	$x \vee (y \wedge (y \vee x)) = (x \vee y) \wedge (y \vee x)$	[278 :287]
305	$x \wedge (y \wedge (x \vee y)) = x \wedge (y \wedge (y \vee x))$	[275 :287,22]
311,310	$x \vee (x \wedge (y \vee x)) = x \wedge (x \vee y)$	[260 :287]
314	$(x \wedge y) \vee (y \wedge x) = x \wedge (y \wedge (y \vee x))$	[229 :287,22]
317	$x \wedge ((x \vee y) \wedge (z \vee (x \wedge y))) = (x \wedge y) \vee (z \wedge (x \wedge (x \vee y)))$	[189 :287,6,287]
318	$x \wedge (x \vee y) = x \wedge (y \vee x)$	[178 :287]
338,337	$(x \wedge y) \vee x = x \wedge (y \vee x)$	[9 → 239]
347,346	$x \wedge (y \wedge (y \vee x)) = x \wedge y$	[110 :338,45]
352,351	$(x \wedge y) \vee (y \wedge x) = x \wedge y$	[314 :347]
354,353	$x \wedge (y \wedge (x \vee y)) = x \wedge y$	[305 :347]
362,361	$x \vee (x \wedge y) = x \wedge (x \vee y)$	[239 → 28 :311, flip]
410,409	$(x \wedge y) \vee (x \wedge (y \wedge z)) = x \wedge (y \wedge ((x \wedge y) \vee z))$	[23 → 286 :6]
412,411	$(x \wedge y) \vee (x \wedge (z \wedge y)) = x \wedge (y \wedge ((x \wedge y) \vee z))$	[20 → 286 :6]
416,415	$x \wedge (y \wedge ((x \wedge y) \vee (y \vee z))) = x \wedge (y \wedge (y \vee z))$	[283 :412]
436,435	$x \wedge ((x \vee y) \wedge y) = y \wedge x$	[286 → 289 :6,43,242,6,22,354]
440,439	$(x \vee y) \wedge (y \vee x) = y \vee x$	[40 → 289 :11,8,3]
452	$x \vee (y \wedge (y \vee x)) = y \vee x$	[303 :440]
530	$x \vee (y \wedge (x \vee y)) = x \vee y$	[318 → 241 :266,49,338,29,3]
586,585	$((x \wedge y) \vee (x \wedge z)) \wedge (x \wedge (y \vee x)) = (x \wedge y) \vee (x \wedge z)$	[337 → 14]
699	$x \vee (y \wedge z) = (z \wedge y) \vee ((y \wedge z) \vee x)$	[351 → 139]
700	$x \vee (y \wedge z) = x \vee (z \wedge y)$	[351 → 95 :352]
705,704	$(x \wedge y) \vee ((y \wedge x) \vee z) = (x \wedge y) \vee z$	[351 → 10, flip]

707	$x \vee (y \wedge z) = (z \wedge y) \vee x$	[699 :705]
784	$(x \vee (y \wedge z)) \wedge (x \vee z) = x \vee (y \wedge z)$	[435 → 14 :436]
814,813	$(x \vee y) \wedge (z \wedge (y \vee x)) = (x \vee y) \wedge z$	[439 → 42 :37, flip]
818	$x \wedge (y \vee z) = (z \vee y) \wedge x$	[439 → 20 :814]
903,902	$x \vee ((y \vee x) \wedge y) = y \vee x$	[4 → 452]
932,931	$x \vee ((x \vee y) \wedge y) = x \vee y$	[4 → 530]
1003,1002	$x \vee (y \vee (x \wedge (x \vee z))) = x \vee (y \vee (x \wedge z))$	[361 → 48]
1112	$x \vee (y \vee (z \wedge u)) = y \vee ((u \wedge z) \vee x)$	[700 → 30]
1221	$x \vee ((y \wedge z) \vee u) = u \vee ((z \wedge y) \vee x)$	[707 → 30]
1311	$x \vee ((y \vee z) \wedge u) = ((z \vee y) \wedge u) \vee x$	[818 → 707]
1379	$(x \wedge (y \vee (x \wedge z))) \vee (x \wedge z) = x \wedge ((x \wedge z) \vee y)$	[21 → 52]
1383	$(x \wedge (y \vee z)) \vee (z \wedge x) = x \wedge (z \vee y)$	[4 → 52]
1455,1454	$x \vee (y \wedge (((z \wedge y) \vee x) \wedge z)) = (z \wedge y) \vee x$	[26 → 902]
1726	$(x \wedge y) \vee ((x \vee z) \wedge y) = y \wedge (x \vee z)$	[707 → 56]
1813,1812	$(x \wedge y) \vee (x \wedge ((x \wedge y) \vee z)) = x \wedge ((x \wedge y) \vee z)$	[21 → 68]
2347,2346	$(x \wedge (y \wedge (y \vee z))) \vee ((x \wedge y) \vee u) = (x \wedge (y \wedge (y \vee z))) \vee u$	[361 → 70 :362]
3311	$((x \wedge y) \vee z) \wedge (y \vee z) = z \vee (y \wedge x)$	[707 → 118]
3352	$x \vee (y \wedge z) = ((z \wedge y) \vee x) \wedge (y \vee x)$	[flip 3311]
3427,3426	$(x \vee y) \wedge (x \vee ((x \vee y) \wedge z)) = x \vee ((x \vee y) \wedge z)$	[28 → 122]
3497	$((x \vee y) \wedge x) \vee (x \vee (y \wedge z)) = x \vee (y \wedge z)$	[122 → 68 :123]
3499	$x \vee ((y \wedge z) \vee (x \wedge y)) = x \vee (y \wedge z)$	[122 → 64 :11,1003,123]
4860	$((x \wedge y) \vee z) \wedge (z \vee y) = (x \wedge y) \vee z$	[902 → 100 :1455]
5497	$x \vee ((x \vee (y \wedge z)) \wedge y) = x \vee (y \wedge z)$	[931 → 102 :3427,932]
5499	$((x \wedge y) \vee z) \wedge (z \vee (((x \wedge y) \vee z) \wedge x)) = (x \wedge y) \vee z$	[902 → 102 :903]
6221,6220	$(x \vee y) \wedge (y \vee ((x \vee y) \wedge z)) = y \vee ((x \vee y) \wedge z)$	[818 → 112]
6224	$x \vee (((y \wedge z) \vee x) \wedge y) = (y \wedge z) \vee x$	[5499 :6221]
7101,7100	$((x \vee y) \wedge x) \vee z = x \vee ((y \wedge x) \vee z)$	[818 → 265]
7180	$x \vee ((y \wedge x) \vee (y \wedge z)) = x \vee (y \wedge z)$	[3497 :7101,49]
7272	$(x \vee y) \wedge (y \vee (z \wedge x)) = y \vee (z \wedge x)$	[818 → 784]
7374	$((x \wedge y) \vee z) \wedge (y \vee z) = (x \wedge y) \vee z$	[707 → 3352, flip]
7393,7392	$(x \wedge y) \vee ((z \wedge y) \vee x) = x \vee (y \wedge z)$	[1221 → 3499]
7411,7410	$(x \vee y) \wedge ((z \wedge x) \vee y) = (z \wedge x) \vee y$	[818 → 4860]
7423,7422	$(((x \wedge y) \vee z) \wedge x) \vee z = z \vee (x \wedge y)$	[1311 → 5497]
7452,7451	$(x \wedge y) \vee ((z \wedge x) \vee y) = y \vee (x \wedge z)$	[1112 → 7180]
7458	$x \vee (y \wedge (z \wedge (z \vee x))) = (y \wedge z) \vee x$	[7272 → 1383 :6,11,410,416,7411]
7544,7543	$(x \wedge (y \wedge (y \vee z))) \vee z = (x \wedge y) \vee z$	[7374 → 1726 :6,2347,7411]
7558,7557	$x \vee (y \wedge ((y \vee x) \wedge z)) = x \vee (y \wedge z)$	[346 → 7392 :7544,7393,6, flip]
7586	$(x \wedge ((x \vee y) \wedge z)) \vee y = y \vee (x \wedge z)$	[6224 → 7458 :6,6,7423,7558]
7689	$x \vee (y \wedge (z \vee (y \wedge x))) = x \vee (y \wedge z)$	[317 → 7586 :11,7544,7452, flip]
7740	$x \vee (y \wedge ((x \wedge y) \vee z)) = x \vee (y \wedge z)$	[707 → 7689]
7759,7758	$(x \wedge (y \vee (x \wedge z))) \vee z = z \vee (x \wedge y)$	[9 → 7689]
7797,7796	$(x \wedge y) \vee (x \wedge z) = x \wedge ((x \wedge y) \vee z)$	[7740 → 585 :586,6,37,1813]
7802,7801	$x \wedge ((x \wedge y) \vee z) = x \wedge (y \vee (x \wedge z))$	[1379 :7797,7759, flip]
7807	$A \wedge (B \vee (A \wedge C)) \neq A \wedge (B \vee (A \wedge C))$	[18 :7797,7802]
7808	$\square$	[7807,1]

Problem 71. Majority polynomial single axiom (from [8]).

$$\left\{\begin{array}{l} p(x,y,y)=y \\ p(y,x,y)=y \\ p(y,y,x)=y \\ f(y)=y. \end{array}\right\} \Leftrightarrow \{p(p(x,y,y),u,p(p(x,y,y),f(y),z))=y\}.$$

($\Rightarrow$) is trivial. The proof of ($\Leftarrow$) follows.

Proof (found by Otter 3.0.3+ on gyro at 0.07 seconds).

1	$x=x$	
2	$p(A,A,B)=A,\ p(A,B,A)=A,\ p(B,A,A)=A,\ f(A)=A\ \rightarrow\ \Box$	
3	$p(p(x,y,y),z,p(p(x,y,y),f(y),u))=y$	
8,7	$p(p(x,y,y),z,y)=y$	[3 → 3]
10,9	$p(x,y,x)=x$	[7 → 7]
11	$p(x,y,p(p(z,x,x),f(x),u))=p(p(z,x,x),f(x),u)$	[3 → 7]
13	$p(A,A,B)=A,\ p(B,A,A)=A,\ f(A)=A\ \rightarrow\ \Box$	[2 :10 :1]
15,14	$p(x,y,p(x,f(x),z))=x$	[7 → 3 :8]
17,16	$p(x,y,y)=y$	[9 → 3 :10]
18	$p(A,A,B)=A,\ f(A)=A\ \rightarrow\ \Box$	[13 :17 :1]
19	$p(x,f(x),y)=x$	[11 :17,15,17, flip]
22,21	$f(x)=x$	[16 → 19]
24,23	$p(x,x,y)=x$	[19 :22]
25	$\Box$	[18 :24,22 :1,1]

With the preceding theorem we can construct a single axiom for any theory that admits a majority polynomial and can be axiomatized with absorption identities. The method and proof are given in [8]. Lattice theory qualifies, with majority polynomial $p(x,y,z)=(x\wedge y)\vee(z\wedge(x\vee y))$ and the four identities of McKenzie. Bundling the identities as $f(y)=p(p(L1(y),L2(y),u),p(L3(y),L4(y),u),v)$, then plugging this into the single axiom above, then expanding to obtain the single axiom in terms of $\wedge$ and $\vee$ produces a single axiom of weight 355 with 7 variables (see [8]). The motivation for the following three problems is to find a shorter single axiom for lattice theory. First we find a single axiom (in terms of a majority polynomial) that uses two obsorption identities instead of one, then we simplify McKenzie's absorption base.

Problem 71M. Majority polynomial single axiom with two absorption identities.

$$\left\{\begin{array}{l} p(x,y,y)=y \\ p(y,x,y)=y \\ p(y,y,x)=y \\ f(y)=y \\ g(y)=y \end{array}\right\} \Leftrightarrow \{p(x,p(y,z,p(g(y),y,u)),p(f(y),y,v))=y\}.$$

(⇒) is trivial. The proof of (⇐) follows.

Proof (found by Otter 3.0.3+ on gyro at 1.69 seconds).

1	$x = x$	
2	$p(A,A,B) = A,\ p(A,B,A) = A,\ p(B,A,A) = A,\ f(A) = A,\ g(A) = A \ \rightarrow\ \Box$	
3	$p(x,p(y,z,p(g(y),y,u)),p(f(y),y,v)) = y$	
5	$p(x,p(p(y,z,p(g(y),y,u)),v,y),p(f(p(y,z,p(g(y),y,u))),p(y,z,p(g(y),y,u)),w)) = p(y,z,p(g(y),y,u))$	[3 → 3]
10,9	$p(x,p(p(y,z,p(g(y),y,u)),v,y),y) = p(y,z,p(g(y),y,u))$	[3 → 5]
11	$p(x,p(y,z,p(g(y),y,u)),y) = p(y,v,p(g(y),y,w))$	[9 → 9]
12	$p(x,y,p(f(y),y,z)) = p(p(f(y),y,z),u,p(g(p(f(y),y,z)),p(f(y),y,z),v))$	[3 → 9]
13	$p(x,y,p(g(x),x,z)) = p(u,p(x,v,p(g(x),x,w)),x)$	[flip 11]
20	$p(x,y,p(g(x),x,z)) = p(x,u,p(g(x),x,v))$	[11 → 11]
23,22	$p(x,p(p(y,z,p(g(y),y,u)),v,y),p(y,w,p(g(y),y,v_6))) = p(y,z,p(g(y),y,u))$	[11 → 5]
24	$p(x,y,p(g(x),z,p(g(g(x)),g(x),u))) = p(x,v,p(g(x),x,w))$	[20 → 20]
25	$p(p(x,y,p(g(x),x,z)),u,x) = p(p(x,y,p(g(x),x,z)),v,p(g(p(x,y,p(g(x),x,z))),p(x,y,p(g(x),x,z)),w))$	[3 → 20]
27	$p(x,y,p(g(x),x,z)) = p(x,u,p(g(x),v,p(g(g(x)),g(x),w)))$	[flip 24]
28	$p(p(x,y,p(g(x),x,z)),u,p(g(p(x,y,p(g(x),x,z))),p(x,y,p(g(x),x,z)),v)) = p(p(x,y,p(g(x),x,z)),w,x)$	[flip 25]
55	$p(p(g(x),x,y),z,p(g(p(g(x),x,y)),p(g(x),x,y),u)) = p(v,p(x,w,p(g(x),x,v_6)),x)$	[11 → 13]
58	$p(x,p(y,z,p(g(y),y,u)),y) = p(p(g(y),y,v),w,p(g(p(g(y),y,v)),p(g(y),y,v),v_6))$	[flip 55]
64	$p(x,p(y,p(z,u,p(g(z),z,v)),z),p(f(z),z,w)) = z$	[13 → 3]
104	$p(x,p(y,p(z,u,p(g(z),z,v)),z),p(f(z),w,p(g(f(z)),v_6,p(g(g(f(z))),g(f(z)),v_7)))) = z$	[27 → 64]
123,122	$p(p(x,y,p(g(x),x,z)),u,p(g(p(x,y,p(g(x),x,z))),p(x,y,p(g(x),x,z)),v)) = p(x,y,p(g(x),x,z))$	[9 → 12 :23,10,10,10, flip]
124	$p(p(x,y,p(g(x),x,z)),u,x) = p(x,y,p(g(x),x,z))$	[28 :123, flip]
137	$p(x,y,p(p(f(x),x,z),u,p(g(p(f(x),x,z)),p(f(x),x,z),v))) = p(x,w,p(g(x),x,v_6))$	[12 → 20]
138	$p(x,y,p(p(f(x),x,z),u,p(g(p(f(x),x,z)),p(f(x),x,z),v))) = p(w,p(x,v_6,p(g(x),x,v_7)),x)$	[12 → 13]
139	$p(x,p(y,z,p(g(y),y,u)),p(p(f(y),y,v),w,p(g(p(f(y),y,v)),p(f(y),y,v),v_6))) = y$	[12 → 3]
141	$p(x,p(y,z,p(p(f(y),y,u),v,p(g(p(f(y),y,u)),p(f(y),y,u),w))),y) = p(y,v_6,p(g(y),y,v_7))$	[12 → 11]
147	$p(x,y,p(g(x),x,z)) = p(u,p(x,v,p(p(f(x),x,w),v_6,p(g(p(f(x),x,w)),p(f(x),x,w),v_7))),x)$	[flip 141]
149	$p(p(x,y,p(g(x),z,p(g(g(x)),g(x),u))),v,x) = p(x,y,p(g(x),x,p(g(g(x)),w,p(g(g(g(x))),g(g(x)),v_6))))$	[24 → 124]
150	$p(p(x,y,p(g(x),z,p(g(g(x)),g(x),u))),v,x) = p(x,y,p(g(x),x,p(g(g(x)),g(x),w)))$	[20 → 124]
161,160	$p(p(f(x),x,y),z,p(g(p(f(x),x,y)),p(f(x),x,y),u)) = x$	[64 → 124, flip]
163	$p(x,y,p(g(x),x,p(g(g(x)),z,p(g(g(g(x))),g(g(x)),u)))) = p(p(x,y,p(g(x),v,p(g(g(x)),g(x),w))),v_6,x)$	[flip 149]
164	$p(x,y,p(g(x),x,p(g(g(x)),g(x),z))) = p(p(x,y,p(g(x),u,p(g(g(x)),g(x),v))),w,x)$	[flip 150]
169	$p(x,y,p(g(x),x,z)) = p(u,p(x,v,x),x)$	[147 :161]
177,176	$p(x,p(y,z,p(g(y),y,u)),y) = y$	[139 :161]
179,178	$p(x,y,x) = x$	[138 :161,177]
181,180	$p(x,y,p(g(x),x,z)) = x$	[137 :161,179, flip]
192,191	$p(p(x,y,g(x)),z,x) = x$	[flip 163 :181,181,181]
194,193	$p(x,y,g(x)) = x$	[flip 164 :181,192,181, flip]
202,201	$p(x,y,y) = y$	[flip 169 :179,181]
203	$p(x,y,f(y)) = y$	[104 :181,202,181,194]
209	$p(g(x),x,y) = x$	[58 :181,202,181, flip]
211	$p(A,A,B) = A,\ f(A) = A,\ g(A) = A \ \rightarrow\ \Box$	[2 :179,202 :1,1]
213,212	$g(x) = x$	[201 → 193]
214	$p(A,A,B) = A,\ f(A) = A \ \rightarrow\ \Box$	[211 :213 :1]
216,215	$p(x,x,y) = x$	[209 :213]

217	$f(A) = A \;\rightarrow\; \square$	[214 :216 :1]
218	$f(x) = x$	[178 $\rightarrow$ 203]
220	$\square$	[218,217]

Problem 71N. A simpler absorptive base for lattice theory.

$$\left\{ \begin{array}{ll} \text{(L2)} & y \wedge (x \vee (y \vee z)) = y \\ \text{(L3)} & ((x \wedge y) \vee (y \wedge z)) \vee y = y \\ \text{(L4}') & ((y \vee x) \wedge (y \vee z)) \wedge y = y \end{array} \right\}.$$

It is sufficient to prove (L1) and (L4) of the McKenzie base.

Proof (found by Otter 3.0.3+ on gyro at 3.54 seconds).

1	$x = x$	
2	$B \vee (A \wedge (B \wedge C)) = B,\ ((A \vee B) \wedge (B \vee C)) \wedge B = B \;\rightarrow\; \square$	
3	$x \wedge (y \vee (x \vee z)) = x$	
5	$((x \wedge y) \vee (y \wedge z)) \vee y = y$	
8,7	$((x \vee y) \wedge (x \vee z)) \wedge x = x$	
9	$((x \wedge y) \vee y) \vee y = y$	[3 $\rightarrow$ 5]
11	$x \wedge (x \vee y) = x$	[5 $\rightarrow$ 3]
13	$((x \wedge y) \vee (y \wedge z)) \wedge y = (x \wedge y) \vee (y \wedge z)$	[5 $\rightarrow$ 11]
24,23	$(x \vee y) \wedge x = x$	[3 $\rightarrow$ 7]
27	$(x \vee (x \wedge y)) \vee x = x$	[7 $\rightarrow$ 5]
30,29	$x \vee (x \wedge y) = x$	[7 $\rightarrow$ 13 :24,8, flip]
31	$x \vee x = x$	[27 :30]
38,37	$x \wedge (y \vee x) = x$	[31 $\rightarrow$ 3]
40,39	$(x \wedge y) \vee y = y$	[9 $\rightarrow$ 23 :38, flip]
45	$(x \vee y) \vee x = x \vee y$	[23 $\rightarrow$ 29]
55	$x \vee (y \vee x) = y \vee x$	[37 $\rightarrow$ 39]
67	$(x \wedge y) \wedge (z \vee y) = x \wedge y$	[39 $\rightarrow$ 3]
69	$x \vee (y \wedge x) = x$	[39 $\rightarrow$ 45 :40]
106,105	$((x \vee y) \wedge (y \vee z)) \wedge y = y$	[55 $\rightarrow$ 7]
111	$B \vee (A \wedge (B \wedge C)) = B \;\rightarrow\; \square$	[2 :106 :1]
224	$(x \vee y) \vee (z \wedge y) = x \vee y$	[67 $\rightarrow$ 69]
677	$x \vee (y \wedge (x \wedge z)) = x$	[29 $\rightarrow$ 224 :30]
679	$\square$	[677,111]

Problem 72. Find a single axiom for LT shorter than the one obtained from Prob. 71.

We use the majority polynomial $p(x, y, z) = (x \wedge y) \vee (z \wedge (x \vee y))$, which is a variant of the one used above. With the majority polynomial single axiom of Prob. 71M and the absorptive 3-base of Prob. 71N, we bundle 2 of the identities as $f(y) = p(L3(y), u, L4'(y))$, and set $g(y) = L2(y)$. When the single axiom is expanded and written in terms of $\wedge$ and $\vee$, it has length 139 and 7 variables:

$$
\begin{array}{l}
(x \wedge ((y \wedge z) \vee ((((y \wedge (x_1 \vee (y \vee x_2))) \wedge y) \vee (u \wedge ((y \wedge (x_1 \vee (y \vee x_2))) \vee y))) \\
\wedge (y \vee z)))) \vee (((((v \wedge (((x_1 \wedge y) \vee (y \wedge x_2)) \vee y)) \vee ((((y \vee x_1) \wedge (y \vee x_2)) \wedge y) \\
\wedge (v \vee (((x_1 \wedge y) \vee (y \wedge x_2)) \vee y)))) \wedge y) \vee (w \wedge (((v \wedge (((x_1 \wedge y) \vee (y \wedge x_2)) \\
\vee y)) \vee ((((y \vee x_1) \wedge (y \vee x_2)) \wedge y) \wedge (v \vee (((x_1 \wedge y) \vee (y \wedge x_2)) \vee y)))) \vee y))) \\
\wedge (x \vee ((y \wedge z) \vee ((((y \wedge (x_1 \vee (y \vee x_2))) \wedge y) \vee (u \wedge ((y \wedge (x_1 \vee (y \vee x_2))) \vee y))) \\
\wedge (y \vee z))))) = y.
\end{array}
$$

Chapter 4

The Rule (gL)

Some references are [11, 9, 12].

4.1 Problems

Problem 1. The type is (2,0) with nullary e.

$$\left\{\begin{array}{l}(xy)(zu)=(xz)(yu)\\ \cdot\text{ is cancellataive}\\ ee=e\end{array}\right\}\Rightarrow\ \{(x(yz))((uv)w)=(x(uz))((yv)w)\}.$$

There is a model-theoretic reason, i.e. a second order proof, why this must be true.

Proof (found by Otter 3.0.3+ on gyro at 3.71 seconds).

1	$x\cdot y=x\cdot z\rightarrow y=z$	
4	$(x\cdot y)\cdot(z\cdot u)=(x\cdot z)\cdot(y\cdot u)$	
5	$e\cdot e=e$	
7	$(A\cdot(D\cdot C))\cdot((B\cdot E)\cdot F)\neq(A\cdot(B\cdot C))\cdot((D\cdot E)\cdot F)$	
8	$((x\cdot y)\cdot(z\cdot u))\cdot(v\cdot w)=((x\cdot z)\cdot v)\cdot((y\cdot u)\cdot w)$	[4 → 4]
10	$((x\cdot y)\cdot z)\cdot((u\cdot v)\cdot w)=((x\cdot u)\cdot(y\cdot v))\cdot(z\cdot w)$	[flip 8]
12	$(x\cdot e)\cdot(y\cdot e)=(x\cdot y)\cdot e$	[5 → 4, flip]
15,14	$(e\cdot x)\cdot(e\cdot y)=e\cdot(x\cdot y)$	[5 → 4, flip]
16	$(e\cdot x)\cdot e=e\cdot(x\cdot e)$	[5 → 12, flip]
27	$((e\cdot x)\cdot y)\cdot(e\cdot z)=(e\cdot(x\cdot e))\cdot(y\cdot z)$	[16 → 4, flip]
33,32	$((e\cdot x)\cdot y)\cdot((e\cdot z)\cdot u)=(e\cdot(x\cdot z))\cdot(y\cdot u)$	[14 → 4, flip]
72	$(e\cdot(x\cdot e))\cdot((e\cdot y)\cdot z)=e\cdot((x\cdot y)\cdot z)$	[14 → 27 :15, flip]
106,105	$(x\cdot(y\cdot e))\cdot((e\cdot z)\cdot u)=(x\cdot e)\cdot((y\cdot z)\cdot u)$	[72 → 72 :15,1, flip]
156	$(e\cdot(x\cdot y))\cdot((z\cdot u)\cdot v)=(e\cdot(z\cdot y))\cdot((x\cdot u)\cdot v)$	[14 → 10 :33]
236	$(x\cdot(y\cdot z))\cdot((u\cdot v)\cdot w)=(x\cdot(u\cdot z))\cdot((y\cdot v)\cdot w)$	[156 → 105 :106,1]
237	$\square$	[236,7]

Problem 1C. The type is (2,0) with nullary e.

$$\left\{ \begin{array}{l} (xy)(zu) = (xz)(yu) \\ ee = e \end{array} \right\} \;=\!\!(gL)\!\!\Rightarrow\; \{(x(yz))((uv)w) = (x(uz))((yv)w)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.76 seconds).

1	$(A \cdot (B \cdot C)) \cdot ((D \cdot E) \cdot F) \neq (A \cdot (D \cdot C)) \cdot ((B \cdot E) \cdot F)$	
3	$(x \cdot y) \cdot (z \cdot u) = (x \cdot z) \cdot (y \cdot u)$	
5,4	$e \cdot e = e$	
6	$((x \cdot y) \cdot (z \cdot u)) \cdot (v \cdot w) = ((x \cdot z) \cdot v) \cdot ((y \cdot u) \cdot w)$	[3 → 3]
10	$(x \cdot e) \cdot (y \cdot e) = (x \cdot y) \cdot e$	[4 → 3]
16	$(e \cdot x) \cdot e = e \cdot (x \cdot e)$	[4 → 10 :5,5, flip]
53,52	$((e \cdot x) \cdot y) \cdot (e \cdot z) = (e \cdot (x \cdot e)) \cdot (y \cdot z)$	[16 → 3]
67	$(x \cdot (y \cdot z)) \cdot ((u \cdot v) \cdot w) = (x \cdot (u \cdot z)) \cdot ((y \cdot v) \cdot w)$	[16 → 6 :53 :(gL) :(gL)]
68	$\square$	[67,1]

Problem 4B.

$$\{p(y,y,x) = p(x,z,z) = p(x,u,x) = x\} \;=\!\!(gL)\!\!\Rightarrow\; \{x = y\}.$$

Origin: These are the so-called Pixley polynomials characterizing the permutability and distributivity of congruences of an equational class of algebras.

Proof (found by Otter 3.0.3+ on gyro at 0.07 seconds).

1	$B \neq A$	
4,3	$p(x,x,y) = y$	
6,5	$p(x,y,y) = x$	
7	$p(x,y,x) = x$	
11,10	$p(x,y,z) = x$	[5 → 7 :(gL) :6,6]
12	$x = y$	[3 → 7 :(gL) :4,11,4]
13	$\square$	[12,1]

Problem 5.

$$\{x(yx) = y\} \;=\!\!(gL)\!\!\Rightarrow\; \{(xy)(zu) = (xz)(yu)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 20.49 seconds).

2	$x \cdot (y \cdot x) = y$	
3	$(A \cdot C) \cdot (B \cdot D) \neq (A \cdot B) \cdot (C \cdot D)$	
4	$(x \cdot (y \cdot x)) \cdot (z \cdot y) = z$	[2 → 2]
9	$(x \cdot y) \cdot x = y$	[2 → 2]
38	$(x \cdot (y \cdot x)) \cdot (z \cdot y) = (u \cdot z) \cdot u$	[9 → 4]
331	$(x \cdot (y \cdot z)) \cdot (u \cdot y) = (x \cdot u) \cdot z$	[(gL) 38]
333	$(x \cdot y) \cdot (z \cdot u) = (x \cdot z) \cdot (y \cdot u)$	[2 → 331]
334	$\square$	[333,3]

Problem 6. The type is (2,0) with nullary e.

$$\left\{\begin{array}{l} x(ex)=e \\ (xe)x=e \\ ee=e \end{array}\right\} =\!\!(gL)\!\!\Rightarrow \{(xy)(zu)=(xz)(yu)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 1.03 seconds).

2	$x\cdot(e\cdot x)=e$	
4	$(A\cdot C)\cdot(B\cdot D)\neq(A\cdot B)\cdot(C\cdot D)$	
7	$x\cdot(y\cdot x)=z\cdot(y\cdot z)$	[2 $\rightarrow$ 2 :(gL)]
8	$(x\cdot y)\cdot(z\cdot(x\cdot z))=u\cdot(y\cdot u)$	[7 $\rightarrow$ 7]
70	$(x\cdot y)\cdot(z\cdot u)=(x\cdot z)\cdot(y\cdot u)$	[(gL) 8]
71	$\Box$	[70,4]

Problem 8.

$$\{p(x,x,y)=p(x,y,x)=p(y,x,x)=x\} =\!\!(gL)\!\!\Rightarrow \{x=y\}.$$

This is a formal way of saying that no equational theory containing for example, the lattice theory will satisfy the rule $=\!\!(gL)\!\!\Rightarrow$. In particular, ternary Boolean algebras do not satisfy this rule.

Proof (found by Otter 3.0.3+ on gyro at 0.25 seconds).

1	$B\neq A$	
3	$p(x,x,y)=x$	
4	$p(x,y,x)=x$	
5	$p(x,y,y)=y$	
11	$p(x,y,z)=p(x,y,u)$	[3 $\rightarrow$ 3 :(gL)]
37	$p(x,y,z)=x$	[4 $\rightarrow$ 11, flip]
47	$p(x,y,z)=p(u,y,z)$	[5 $\rightarrow$ 5 :(gL)]
59	$p(x,y,z)=u$	[47 $\rightarrow$ 37]
60	$x=y$	[59 $\rightarrow$ 59]
61	$\Box$	[60,1]

Problem 9B.

$$\{x(y(yx))=yx, \text{cancellation}\} =\!\!(gL)\!\!\Rightarrow \{x(xy)=yx\}.$$

Proof (found by Otter 3.0.3+ on gyro at 1.42 seconds).

1	$A\cdot(A\cdot B)=B\cdot A \;\rightarrow\; \Box$
2	$x=x$
4	$x\cdot y=z,\; u\cdot y=z \;\rightarrow\; x=u$
5	$x\cdot(y\cdot(y\cdot x))=y\cdot x$

9	$(x \cdot (x \cdot y)) \cdot (y \cdot z) = y \cdot (x \cdot z)$	[5 $\rightarrow$ 5 :(gL)]
11	$(x \cdot x) \cdot y = x \cdot y$	[5 $\rightarrow$ 5 :(gL)]
23	$x \cdot x = x$	[11,4,2, flip]
37	$(x \cdot (x \cdot y)) \cdot (y \cdot x) = y \cdot x$	[23 $\rightarrow$ 9]
83	$x \cdot (x \cdot y) = y \cdot x$	[37,4,23, flip]
84	$\square$	[83,1]

Problem 11. An "esoteric" problem. The type of theory is (2) with just one binary operation, say $|$. Axioms assumed:

$$A = \left\{ \begin{array}{l} x \mid x = y \mid y \; (= e, \text{say}) \\ e \mid (e \mid x) = x \\ e \mid x = x \mid e. \end{array} \right\}$$

Prove or disprove $A \;\; =\!\!(gL)\!\!\Rightarrow$ {all Abelian group axioms satisfied by both left division \ and right-division /}. My [RP] conjecture: TRUE. In view of [1], it is enough to derive the following single identity from the axiom-set A using the rule (gL): $(x \mid y) \mid (((x \mid z) \mid (u \mid u)) \mid y) = z$.

Proof (found by Otter 3.0.3+ on gyro at 5.10 seconds).

1	$x = x$	
3,2	$x \mid x = e$	
5,4	$e \mid (e \mid x) = x$	
7,6	$(x \mid e) \mid e = x$	
8	$e \mid (x \mid e) = (e \mid x) \mid e$	
9	$(e \mid x) \mid e = e \mid (x \mid e)$	[flip 8]
10	$(A \mid B) \mid (((A \mid C) \mid (D \mid D)) \mid B) \neq C$	
11	$(A \mid B) \mid (((A \mid C) \mid e) \mid B) \neq C$	[copy,10 :3]
13	$(e \mid x) \mid (e \mid y) = e \mid (x \mid y)$	[6 $\rightarrow$ 9 :(gL) :3]
29	$(x \mid y) \mid (e \mid z) = (x \mid e) \mid (y \mid z)$	[4 $\rightarrow$ 13 :(gL) :3]
171	$(x \mid e) \mid (x \mid y) = y$	[2 $\rightarrow$ 29 :5, flip]
177	$(x \mid y) \mid (e \mid y) = x$	[2 $\rightarrow$ 29 :7]
207	$x \mid ((x \mid e) \mid y) = y$	[6 $\rightarrow$ 171]
261	$(x \mid y) \mid (z \mid y) = e \mid (z \mid x)$	[4 $\rightarrow$ 177 :(gL)]
325,324	$e \mid (((x \mid y) \mid e) \mid x) = y$	[207 $\rightarrow$ 261, flip]
345	$C \neq C$	[261 $\rightarrow$ 11 :325]
346	$\square$	[345,1]

Problem 12A.

$$\left\{ \begin{array}{l} x/e = x \\ e/(e/x) = x \\ x/x = e \end{array} \right\} =\!\!(gL)\!\!\Rightarrow \text{ Abelian group theory,}$$

with x/y as right division. It is enough if you derive one of the single axioms defining abelian groups with $x/y = x - y$. Here I [RP] have added the new axiom $x/x = e$ to kill the previous unwanted model $x + 3y \pmod 8$.

[[Note to RP: e/(e/x)=x is not needed; should we restate the problem?]]

Proof (found by Otter 3.0.3+ on gyro at 93.40 seconds).

2	$x/e = x$	
3	$x/x = e$	
10	$(A/B)/((A/C)/B) \neq C$	
11	$(x/e)/e = x$	$[2 \rightarrow 2]$
16	$x/(x/e) = e$	$[2 \rightarrow 3]$
18	$x/(y/y) = x$	$[3 \rightarrow 2]$
39	$(x/e)/y = x/y$	$[2 \rightarrow 11$:(gL)]
79	$((x/y)/z)/e = (x/z)/y$	$[11 \rightarrow 39$:(gL), flip]
244	$x/(x/y) = z/(z/y)$	$[16 \rightarrow 16$:(gL)]
570	$x/(x/y) = y$	$[18 \rightarrow 244$, flip]
799	$x/((x/y)/e) = y$	$[2 \rightarrow 570]$
9029	$(x/y)/((x/z)/y) = z$	$[79 \rightarrow 799]$
9030	$\square$	[9029,10]

Problem 12B.

$$\left\{ \begin{array}{l} x/(x/y) = y \\ x/x = e \end{array} \right\} =\!\!(gL)\!\!\Rightarrow \text{ Abelian group theory.}$$

Here, / is right division. It is sufficient to prove any of the single axioms for Abeilan groups in terms of right division.

Proof (found by Otter 3.0.3+ on gyro at 10.63 seconds).

2	$x/x = e$	
3	$x/(x/y) = y$	
8	$(A/((A/B)/C))/B \neq C$	
22	$x/e = x$	$[2 \rightarrow 3]$
25	$(x/e)/e = x$	$[22 \rightarrow 22]$
26	$(x/(x/y))/e = y$	$[3 \rightarrow 22]$
133	$(x/e)/y = x/y$	$[22 \rightarrow 25$:(gL)]
198	$((x/y)/z)/e = (x/z)/y$	$[25 \rightarrow 133$:(gL), flip]
2484	$(x/((x/y)/z))/y = z$	$[198 \rightarrow 26]$
2485	$\square$	[2484,8]

Problem 13B.

$$\left\{ \begin{array}{l} p(x, y, y) = x \\ p(x, x, y) = y \end{array} \right\} =\!\!(gL)\!\!\Rightarrow \{p(x, y, p(z, t, u)) = p(u, t, p(x, y, z))\}.$$

Proof (found by Otter 3.0.3+ on gyro at 480.90 seconds).

2	$p(x,y,y) = x$	
3	$p(x,x,y) = y$	
4	$p(A,B,p(C,D,E)) \neq p(E,D,p(A,B,C))$	
16	$p(x,x,y) = p(y,z,z)$	[2 → 3]
904	$p(x,y,z) = p(z,y,x)$	[(gL) 16]
1083	$p(x,x,p(y,z,u)) = p(u,z,y)$	[3 → 904]
1084	$p(p(x,y,z),u,u) = p(z,y,x)$	[2 → 904]
1088	$p(p(C,D,E),B,A) \neq p(E,D,p(A,B,C))$	[904 → 4]
2174	$p(x,y,p(z,u,y)) = p(x,u,z)$	[(gL) 1083]
2182	$p(p(x,y,z),x,u) = p(z,y,u)$	[(gL) 1084]
3622	$p(p(x,y,z),u,v) = p(z,y,p(v,u,x))$	[2174 → 2182]
3623	$\square$	[3622,1088]

Problem 1G. Group theory $=\!\!(gL)\!\!\Rightarrow$ Abelian group theory.

Proof (found by Otter 3.0.3+ on gyro at 13.57 seconds).

2	$e \cdot x = x$	
3	$x' \cdot x = e$	
4	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	
5	$B \cdot A \neq A \cdot B$	
6	$x'' = x$	
9	$e \cdot (e \cdot x) = x$	[2 → 2]
13	$x' \cdot (e \cdot x) = e$	[2 → 3]
16	$(x' \cdot x) \cdot y = y$	[3 → 2]
131	$x' \cdot (y \cdot x) = e \cdot (y \cdot e)$	[9 → 13 :(gL)]
228	$x' \cdot (x \cdot y) = y$	[4 → 16]
380	$e \cdot (x \cdot e) = x$	[131 → 228]
702	$x' \cdot (y \cdot x) = y$	[131 → 380]
1130	$x \cdot y = y \cdot x$	[702 → 228 :6]
1131	$\square$	[1130,5]

Problem 2G. The type is (2,0) with nullary e.

$$\{ex = xe = x\} \;=\!\!(gL)\!\!\Rightarrow\; \{xy = yx\}.$$

Proof (found by Otter 3.0.3+ on gyro at 31.16 seconds).

2	$e \cdot x = x$	
3	$x \cdot e = x$	
4	$B \cdot A \neq A \cdot B$	
5	$(e \cdot e) \cdot x = x$	[2 → 2]
7	$(x \cdot e) \cdot e = x$	[3 → 3]
79	$(x \cdot y) \cdot e = (e \cdot y) \cdot x$	[5 → 7 :(gL)]
355	$(e \cdot x) \cdot y = y \cdot x$	[3 → 79, flip]

814	$x \cdot y = y \cdot x$	[2 → 355]
815	$\square$	[814,4]

Problem 3G. The type is (2,0) with nullary e.

$$\{ex = xe = x\} \ =\!\!(gL)\!\!\Rightarrow \ \{(xy)z = x(yz)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 15.70 seconds).

2	$e \cdot x = x$	
3	$x \cdot e = x$	
4	$(A \cdot B) \cdot C \neq A \cdot (B \cdot C)$	
8	$(e \cdot x) \cdot e = x$	[2 → 3]
148	$(e \cdot x) \cdot y = x \cdot y$	[3 → 8 :(gL)]
195	$(x \cdot (y \cdot z)) \cdot e = (x \cdot y) \cdot z$	[8 → 148 :(gL), flip]
715	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	[3 → 195, flip]
716	$\square$	[715,4]

Problem 17C. The type of theory is (2,2,0) with two binary operations and nullary e.

$$\left\{ \begin{array}{ll} f(x,y) = f(y,x), & g(x,y) = g(y,x) \\ f(x,f(y,x)) = y, & g(x,g(y,x)) = y \\ f(e,e) = e), & g(e,e) = e) \end{array} \right\} \ =\!\!(gL)\!\!\Rightarrow \ \{f(x,y) = g(x,y)\}$$

Proof (found by Otter 3.0.3+ on gyro at 108.84 seconds).

2	$f(x,y) = f(y,x)$	
3	$f(x,f(y,x)) = y$	
4	$f(e,e) = e$	
5	$g(x,y) = g(y,x)$	
6	$g(x,g(y,x)) = y$	
7	$g(e,e) = e$	
8	$g(A,B) \neq f(A,B)$	
10	$f(e,e) = e$	
11	$f(x,f(y,x)) = y$	
12	$f(x,g(e,e)) = f(e,x)$	[7 → 2]
15	$f(x,f(x,y)) = y$	[2 → 3]
21	$g(x,g(x,y)) = y$	[5 → 6]
31	$f(x,f(x,g(y,z))) = g(z,y)$	[5 → 15]
87	$f(e,g(x,g(x,y))) = f(y,g(e,e))$	[21 → 12, flip]
370	$f(e,g(x,g(e,y))) = f(y,g(x,e))$	[4 → 87 :(gL) :10]
377	$f(x,g(x,e)) = e$	[6 → 370 :10, flip]
411	$f(x,g(x,y)) = f(z,g(z,y))$	[377 → 377 :(gL)]

424	$g(e,x) = f(x,e)$	[377 → 31, flip]
703	$f(x, g(x,y)) = y$	[424 → 411 :11]
788	$g(x,y) = f(x,y)$	[21 → 703, flip]
789	$\square$	[788,8]

Problem 19. The type is (2,0) with nullary e.

$$\left\{\begin{array}{l} (xe)e = x \\ e(ex) = x \\ ee = e \\ e(xe) = (ex)e \end{array}\right\} =\!\!(gL)\!\!\Rightarrow \{(xy)(zt) = (xz)(yt)\}$$

Proof (found by Otter 3.0.3+ on gyro at 8.58 seconds).

4	$e \cdot (e \cdot x) = x$	
6	$e \cdot e = e$	
8	$e \cdot (x \cdot e) = (e \cdot x) \cdot e$	
9	$(e \cdot x) \cdot e = e \cdot (x \cdot e)$	[flip 8]
10	$(A \cdot C) \cdot (B \cdot D) \neq (A \cdot B) \cdot (C \cdot D)$	
12	$(e \cdot x) \cdot (e \cdot y) = e \cdot (x \cdot y)$	[6 → 9 :(gL)]
22	$e \cdot (x \cdot (e \cdot y)) = (e \cdot x) \cdot y$	[4 → 12, flip]
28	$e \cdot (x \cdot (e \cdot (y \cdot z))) = (e \cdot y) \cdot (x \cdot z)$	[4 → 12 :(gL), flip]
246	$(x \cdot y) \cdot (z \cdot u) = (x \cdot z) \cdot (y \cdot u)$	[22 → 28 :(gL)]
247	$\square$	[246,10]

Problem 20. The type is (2,0) with nullary e. Let

$$\mathrm{A1} = \left\{\begin{array}{l} (xe)e = x \\ e(ex) = x \\ ee = e \\ e(xe) = (ex)e \end{array}\right\}, \qquad \mathrm{A2} = \mathrm{A1} \cup \{((xx)e)(xx) = e\}.$$

Then

$$\mathrm{A2} =\!\!(gL)\!\!\Rightarrow \left\{\begin{array}{l} (xy)(zt) = (xz)(yt) \\ ((xx)y)(xx) = y \end{array}\right\}$$

WWM was unable to get Otter to prove this theorem, but he sent to RP several (gL)-consequences of axioms A2, including

$$(x((xy)x))x = y \qquad (5)$$
$$x((x(yx))x) = y \qquad (6)$$

E-mail from RP to WWM, 4 November, 1994:

Here what we want is already obtained by Otter, namely the two identities (5) and (6). The only purpose of the Prob. 20 is to prove that any (gL)-algebra satisfying the axioms A2 must also satisfy all the identities true for all the three interpretations

$$a \cdot b = a - b, \;\; a \cdot b = -a + b, \;\text{ or } a \cdot b = -a - b$$

in Abelian groups. What you have obtained is a basis for the equational theory of algebras of type (2,0) satisfying all the identities common to above three operations in abelian groups. This is because the two identities (5) and (6) imply the two cancellation laws and make the algebra a quasigroup at the same time. David Kelly and myself do already have such a basis (last line in Kelly-RP table) but our set involved four variables. The above result shows that, modulo (gL), we can define the same equational theory by a set of *one*-variable sentences, i.e. the axioms A2. That is a *new* result and that was the intent of the problem 20.

Now coming to the actual problem posed as above, is $((xx)y)(xx) = y$ a (gL)-consequence? Unfortunately, no. Interpret $x.y$ as $x - y$ and e as 0, the zero in an Abelian group. This is a (gL)-algebra satisfying all the axioms (A2). This is also medial. However, $((x \cdot x) \cdot y) \cdot (x \cdot x) = (0 \cdot y) \cdot 0 = (-y) \cdot 0 = -y \neq y$ as claimed in the conclusion. No wonder Otter could not derive this. I will write a new proof for the following.

THEOREM. A (gL)-algebra of type (2,0) satisfies all the identities common to the three operations

$$a \cdot b = a - b, \;\; a \cdot b = -a + b, \;\text{ or } a \cdot b = -a - b$$

iff it satisfies the one variable identities

$$\mathrm{A2} = \left\{ \begin{array}{l} (xe)e = x \\ e(ex) = x \\ ee = e \\ e(xe) = (ex)e \\ ((xx)e)(xx) = e \end{array} \right\}.$$

The crucial step in the new proof will be the two identities (5) and (6).

From the above example, we can make the conjecture [Prob 20B below] that

$$\mathrm{A2} \;\overset{(gL)}{=\!\!\Longrightarrow}\; \{((xx)y)(xx) = (ey)e\},$$

because this identity is valid for all the three models. Thus this forms a test case. This was the original intention of the Prob. 20.

Problem 20A. A2 is defined in Prob. 20 above.

$$\mathrm{A2} \;\stackrel{(gL)}{\Longleftrightarrow}\; \left\{ \begin{array}{c} (x((xy)x))x = y \\ x((x(yx))x) = y \end{array} \right\}$$

Proof (found by Otter 3.0.3+ on gyro at 67.41 seconds).

1	$x = x$	
2	$(A\cdot((A\cdot B)\cdot A))\cdot A = B,\ A\cdot((A\cdot(B\cdot A))\cdot A) = B \ \rightarrow\ \square$	
4,3	$(x\cdot e)\cdot e = x$	
6,5	$e\cdot(e\cdot x) = x$	
8,7	$e\cdot e = e$	
9	$e\cdot(x\cdot e) = (e\cdot x)\cdot e$	
10	$(e\cdot x)\cdot e = e\cdot(x\cdot e)$	[flip 9]
11	$((x\cdot x)\cdot e)\cdot(x\cdot x) = e$	
13	$(e\cdot((e\cdot x)\cdot e))\cdot e = e\cdot((e\cdot(x\cdot e))\cdot e)$	[10 → 10]
14	$e\cdot((e\cdot x)\cdot e) = x\cdot e$	[5 → 10, flip]
15	$(e\cdot x)\cdot(e\cdot y) = e\cdot(x\cdot y)$	[7 → 10 :(gL)]
17	$(x\cdot e)\cdot(y\cdot e) = (x\cdot y)\cdot e$	[7 → 10 :(gL), flip]
18	$(e\cdot(x\cdot e))\cdot e = e\cdot x$	[3 → 10]
33	$(x\cdot y)\cdot(e\cdot z) = (x\cdot e)\cdot(y\cdot z)$	[7 → 15 :(gL)]
63	$((x\cdot e)\cdot y)\cdot z = ((x\cdot z)\cdot y)\cdot e$	[17 → 17 :(gL) :4]
65	$(x\cdot x)\cdot((x\cdot x)\cdot e) = e$	[11 → 17 :4,8]
85	$e\cdot((x\cdot y)\cdot z) = y\cdot((x\cdot e)\cdot z)$	[3 → 14 :(gL) :(gL)]
89,88	$e\cdot((e\cdot(x\cdot e))\cdot e) = x$	[3 → 14]
91,90	$(e\cdot((e\cdot x)\cdot e))\cdot e = x$	[13 :89]
169,168	$(x\cdot x)\cdot((x\cdot x)\cdot y) = y$	[5 → 65 :(gL) :6]
185	$(e\cdot(x\cdot y))\cdot e = y\cdot x$	[168 → 18 :(gL) :169]
216	$e\cdot((x\cdot y)\cdot e) = y\cdot x$	[185 → 185 :4]
548	$((x\cdot x)\cdot x)\cdot(e\cdot x) = e$	[11 → 33]
836	$(x\cdot((y\cdot y)\cdot y))\cdot y = x$	[548 → 33 :6,4]
1127	$((x\cdot y)\cdot z)\cdot u = ((x\cdot u)\cdot z)\cdot y$	[3 → 63 :(gL) :4]
1374	$(x\cdot((x\cdot e)\cdot x))\cdot x = e$	[85 → 836]
1459,1458	$(x\cdot((x\cdot y)\cdot x))\cdot x = y$	[90 → 1374 :(gL) :91]
1460	$B = B,\ A\cdot((A\cdot(B\cdot A))\cdot A) = B \ \rightarrow\ \square$	[2 :1459]
2999	$((x\cdot(e\cdot x))\cdot x)\cdot x = e$	[548 → 1127, flip]
3150	$x\cdot((x\cdot(e\cdot x))\cdot x) = e$	[2999 → 216 :8,8, flip]
3331,3330	$x\cdot((x\cdot(y\cdot x))\cdot x) = y$	[88 → 3150 :(gL) :89]
3332	$B = B \ \rightarrow\ \square$	[1460 :3331]
3333	$\square$	[3332,1]

Problem 20B. A2 is defined in Prob. 20 above.

$$\mathrm{A2} \;\stackrel{(gL)}{\Longleftrightarrow}\; \{((xx)y)(xx) = (ey)e\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.34 seconds).

2	$((A \cdot A) \cdot B) \cdot (A \cdot A) \neq (e \cdot B) \cdot e$	
3	$(x \cdot e) \cdot e = x$	
11	$((x \cdot x) \cdot e) \cdot (x \cdot x) = e$	
47	$((x \cdot x) \cdot y) \cdot (x \cdot x) = (e \cdot y) \cdot e$	[3 → 11 :(gL)]
48	□	[47,2]

Problem 22. Semilattice on a cubic curve.

$$\left\{ \begin{array}{l} f(x,x) = x \\ f(x, f(y,z)) = f(y, f(z,x)) \end{array} \right\} =\!\!(gL)\!\!\Rightarrow \{x = y\}.$$

Philosophy behind the problems 22, 23, and 8: There are no polynomially defined semilattice or lattice laws on say, the reals, rationals or the complex numbers. Hence, modulo the process (gL), a semilattice will be "crushed" into a singleton! The validity of the implication made in Problem 23 was proved by R. Padmanabhan, actually via Problem 8.

Proof (found by Otter 3.0.3+ on gyro at 1.25 seconds).

1	$B \neq A$	
3	$f(x,x) = x$	
5	$f(x, f(y,z)) = f(y, f(z,x))$	
8	$f(x, f(x,y)) = f(y,x)$	[3 → 5, flip]
10	$f(x, f(y, f(z,y))) = f(x,y)$	[3 → 5 :(gL), flip]
13	$f(x, f(y,x)) = f(y,x)$	[3 → 5]
60	$f(f(x,y),z) = f(y,z)$	[10 → 8 :(gL), flip]
118	$f(f(x,y),y) = y$	[3 → 60]
207,206	$f(f(x, f(y, f(z,y))), y) = y$	[10 → 118]
219,218	$f(x, f(y,x)) = x$	[10 → 118 :207, flip]
220	$f(x, f(f(y, f(z,x)), z)) = f(z,x)$	[5 → 118]
225,224	$f(x,y) = y$	[13 :219, flip]
226	$x = y$	[220 :225,225,225,225,225]
227	□	[226,1]

Problem 1L. The type is (2,0) with nullary e. Prove

$$\left\{ \begin{array}{l} x(yx) = y \\ ex = xe \\ ee = e \end{array} \right\} \begin{array}{c} =\!\!(gL)\!\!\Rightarrow \\ \Longleftarrow \end{array} \{(x((z(xy))(ey)))(ee) = z\},$$

i.e. your [WWM's] single identity for double inversion in abelian groups. The context is Theorem 3 of ??. Let e be an inflexion point on the curve Γ. Then the binary morphism xy of chord-tangent construction on Γ is completely characterized by the single identity. See the diagrams 3 and 4 of ??.

[[Question for RP: we have a similar problem using (xe)e=x instead of ex=xe. Should that one be included also?]]

Proof ($\Rightarrow$) found by Otter 3.0.3+ on gyro at 10.12 seconds.

1	$x = x$	
3,2	$e \cdot e = e$	
4	$e \cdot x = x \cdot e$	
6,5	$x \cdot (y \cdot x) = y$	
7	$(A \cdot ((C \cdot (A \cdot B)) \cdot (e \cdot B))) \cdot (e \cdot e) \neq C$	
8	$(A \cdot ((C \cdot (A \cdot B)) \cdot (e \cdot B))) \cdot e \neq C$	[copy,7 :3]
18,17	$(x \cdot y) \cdot x = y$	[5 $\rightarrow$ 5]
20,19	$e \cdot (e \cdot x) = x$	[4 $\rightarrow$ 5]
21	$x \cdot (x \cdot e) = e$	[4 $\rightarrow$ 5]
34	$x \cdot (x \cdot y) = y$	[19 $\rightarrow$ 21 :(gL) :20]
41,40	$(x \cdot y) \cdot y = x$	[17 $\rightarrow$ 34]
42	$x \cdot y = y \cdot x$	[5 $\rightarrow$ 34]
46	$(x \cdot y) \cdot z = z \cdot (y \cdot x)$	[42 $\rightarrow$ 42]
53	$(A \cdot ((C \cdot (B \cdot A)) \cdot (e \cdot B))) \cdot e \neq C$	[42 $\rightarrow$ 8]
114	$(x \cdot y) \cdot (z \cdot u) = (u \cdot z) \cdot (y \cdot x)$	[42 $\rightarrow$ 46]
204	$(x \cdot y) \cdot (z \cdot u) = (x \cdot z) \cdot (y \cdot u)$	[(gL) 114]
231	$C \neq C$	[204 $\rightarrow$ 53 :18,6,41]
232	$\square$	[231,1]

Proof ($\Leftarrow$) found by Otter 3.0.3+ on gyro at 1.10 seconds.

1	$e \cdot e = e,\ e \cdot A = A \cdot e,\ A \cdot (B \cdot A) = B \ \rightarrow\ \square$	
2	$x = x$	
4,3	$(x \cdot ((y \cdot (x \cdot z)) \cdot (e \cdot z))) \cdot (e \cdot e) = y$	
5	$(e \cdot (x \cdot (e \cdot e))) \cdot (e \cdot e) = y \cdot ((x \cdot (y \cdot z)) \cdot (e \cdot z))$	[3 $\rightarrow$ 3]
7	$x \cdot (((y \cdot (e \cdot e)) \cdot (x \cdot z)) \cdot (e \cdot z)) = y$	[3 $\rightarrow$ 5, flip]
10,9	$(e \cdot x) \cdot (e \cdot e) = e \cdot (x \cdot (e \cdot e))$	[5 $\rightarrow$ 3 :4]
12,11	$e \cdot (((x \cdot (e \cdot y)) \cdot (e \cdot y)) \cdot (e \cdot e)) = x$	[3 $\rightarrow$ 9, flip]
13	$(x \cdot (e \cdot ((x \cdot e) \cdot (e \cdot e)))) \cdot (e \cdot e) = e$	[9 $\rightarrow$ 3]
15	$e \cdot (e \cdot (e \cdot (e \cdot (e \cdot (e \cdot (e \cdot e)))))) = e$	[9 $\rightarrow$ 7 :10,10,10,10,10]
18,17	$e \cdot (((x \cdot y) \cdot y) \cdot (e \cdot e)) = x$	[11 $\rightarrow$ 11 :12]
19	$e \cdot (((x \cdot (e \cdot e)) \cdot y) \cdot y) = x$	[17 $\rightarrow$ 7 :18]
23	$((x \cdot e) \cdot x) \cdot (e \cdot e) = e$	[17 $\rightarrow$ 13]
26,25	$e \cdot (e \cdot e) = e$	[23 $\rightarrow$ 23]
28,27	$e \cdot e = e$	[15 :26,26,26]
29	$((x \cdot e) \cdot x) \cdot e = e$	[23 :28]
33	$e \cdot (((x \cdot e) \cdot y) \cdot y) = x$	[19 :28]
35	$e \cdot (((x \cdot y) \cdot y) \cdot e) = x$	[17 :28]
41	$x \cdot (((y \cdot e) \cdot (x \cdot z)) \cdot (e \cdot z)) = y$	[7 :28]
45	$e = e,\ e \cdot A = A \cdot e,\ A \cdot (B \cdot A) = B \ \rightarrow\ \square$	[1 :28]
46	$e \cdot ((e \cdot x) \cdot x) = (y \cdot e) \cdot y$	[29 $\rightarrow$ 33]
48,47	$e \cdot ((e \cdot x) \cdot x) = e$	[27 $\rightarrow$ 33]
49	$(x \cdot e) \cdot x = e$	[flip 46 :48]
52,51	$e \cdot (e \cdot x) = x$	[49 $\rightarrow$ 33]

59	$(e \cdot x) \cdot x = e$	[47 $\to$ 51 :28, flip]
61	$x \cdot (e \cdot x) = e$	[51 $\to$ 59]
64,63	$e \cdot (x \cdot e) = x$	[61 $\to$ 35 :52]
73,72	$(x \cdot y) \cdot y = x$	[35 :64]
80,79	$x \cdot (y \cdot x) = y$	[61 $\to$ 41 :73,52]
91	$e = e,\ e \cdot A = A \cdot e,\ B = B \ \to\ \square$	[45 :80]
102	$x \cdot y = y \cdot x$	[72 $\to$ 79]
221	$\square$	[91,2,102,2]

Problem 37 (corrected). The type is (3). (The hypotheses are called TQ.)

$$\left\{ \begin{array}{l} f(x,y,f(x,y,z)) = z \\ f(x,y,z) = f(x,z,y) \end{array} \right\} =\!(gL)\!\Rightarrow \{f(x,y,f(x,z,t)) = f(u,y,f(u,z,t))\}.$$

The first version of this problem had a different conclusion: $f(x,y,f(x,z,t)) = f(u,y,f(u,z,t))$. Otter was unable to prove the first version, then RP found the following counterexample. Define $f(x,y,z)$ as $ax+by+cz$ in an Abelian group. Such an algebra is always a (gL)-algebra. The first identity $f(x,y,f(x,y,z)) = z$ demands that $a + ac = 0$, $b + bc = 0$ and $c^2 = 1$. Thus $c = -1$ would satisfy all the conditions without the number a being 0. Now the second identity $f(x,y,z) = f(x,z,y)$ demands that $b = c$. Hence choose $b = c = -1$ and $a = 2$, say. Now, $f(x,y,f(z,x,t)) = 2x - y - (2z - x - t) = x - y - 2z - t$ which is not independent of x in the Abelian group of integers! Thus our goal is not a consequence of the assumptions. However, the following identity is true in this model: $f(x,y,f(x,z,t)) = f(u,y,f(u,z,t)$ because the LHS reduces to $2x - y - (2x - z - t) = z - y + t$ which is independent of x. So the corrected formulation of Problem 37 is given above.

Proof (found by Otter 3.0.3+ on gyro at 1.74 seconds).

2	$f(x,y,f(x,y,z)) = z$	
4	$f(A,B,f(A,C,D)) \neq f(E,B,f(E,C,D))$	
10	$f(x,y,f(x,y,z)) = f(u,v,f(u,v,z))$	[2 $\to$ 2]
151	$f(x,y,f(x,z,u)) = f(v,y,f(v,z,u))$	[(gL) 10]
152	$\square$	[151,4]

Problem 38 (corrected).

$$\left\{ \begin{array}{l} \text{TQ (see Prob. 37)} \\ p(z,y,v) = f(x,y,f(x,z,v)) \end{array} \right\} =\!(gL)\!\Rightarrow \{p(x,y,p(z,v,u)) = p(u,v,p(x,y,z))\}$$

Background. Guided by the intution gained through the model described for the previous problem, we suspect that $p(z,y,v) := f(x,y,f(x,z,v))$ must behave like the ternary Mal'cev polynomial $z - y + v$ in Abelian groups! Problems 37 and 38, taken together, construct a synthetic group law + on a quartic curve Γ such that if four points P, Q, R, S on Γ are collinear, then $P+Q+R+S=E$ under the group law.

(Note that by Prob. 13B, it is sufficient to prove $p(x,y,y)=p(y,y,x)=x$.)

Proof (found by Otter 3.0.3+ on gyro at 900.35 seconds).

2	$f(x,y,f(x,y,z))=z$	
3	$f(x,y,z)=f(x,z,y)$	
4	$p(z,y,v)=f(x,y,f(x,z,v))$	
5	$p(A,B,p(C,D,E))\neq p(E,D,p(A,B,C))$	
6	$f(x,y,f(x,z,y))=z$	[3 → 2]
7	$p(x,y,y)=x$	[6 → 4]
8	$p(x,x,y)=y$	[2 → 4]
14	$p(x,x,y)=p(y,z,z)$	[7 → 8]
64	$p(x,y,z)=p(z,y,x)$	[(gL) 14]
69	$p(x,y,p(z,u,u))=p(z,y,x)$	[7 → 64, flip]
71	$p(x,x,p(y,z,u))=p(u,z,y)$	[8 → 64]
213	$p(x,y,p(z,u,y))=p(x,u,z)$	[(gL) 71]
225	$p(x,y,p(z,u,y))=p(z,u,p(x,v,v))$	[69 → 213]
411	$p(x,y,p(z,u,v))=p(z,u,p(x,y,v))$	[(gL) 225]
414	$p(x,y,p(z,u,v))=p(v,u,p(x,y,z))$	[64 → 411]
415	$\square$	[414,5]

Problem 10G. Show that under (gL), two inversive groupoids with a common identity must be identical. That is,

$$\left\{\begin{array}{ll} f(e,x)=x, & F(e,x)=x \\ f(x,e)=x, & F(x,e)=x \\ f(g(x),x)=e, & F(G(x),x)=e \\ f(x,g(x))=e, & F(x,G(x))=e \end{array}\right\} \overset{(gL)}{=\!\!=\!\!\Rightarrow} \left\{\begin{array}{l} f(x,y)=F(x,y) \\ g(x)=G(x) \end{array}\right\}.$$

Note that this implies that two group laws with a commom identity are identical.

Proof (found by Otter 3.0.3+ on gyro at 33.82 seconds).

2	$f(e,x)=x$	
3	$f(x,e)=x$	
4	$f(g(x),x)=e$	
6	$F(e,x)=x$	
7	$F(x,e)=x$	
8	$F(G(x),x)=e$	
9	$F(x,G(x))=e$	
10	$f(A,B)=F(A,B),\ g(A)=G(A)\ \rightarrow\ \square$	
16,15	$f(e,f(e,x))=x$	[2 → 2]
22	$f(e,f(x,e))=x$	[2 → 3]
36,35	$f(e,F(e,x))=x$	[2 → 6]
51,50	$f(F(x,e),e)=x$	[3 → 7]

52	$f(e, F(x,e)) = x$	[2 → 7]
57	$f(g(x), F(x,e)) = e$	[7 → 4]
69	$f(g(x), f(x,e)) = e$	[3 → 4]
2004	$f(g(x), F(x,y)) = y$	[52 → 57 :(gL) :36]
2175	$f(g(x), e) = G(x)$	[9 → 2004]
2177,2176	$f(g(G(x)), e) = x$	[8 → 2004]
2188,2187	$G(f(g(x), e)) = x$	[2175 → 2175 :2177, flip]
2243	$g(x) = G(x)$	[7 → 2175 :51]
2775	$g(G(x)) = x$	[2175 → 2243 :2188]
2929,2928	$g(g(x)) = x$	[2243 → 2775]
3338	$f(g(x), f(x,y)) = y$	[22 → 69 :(gL) :16]
3457	$f(x,y) = F(x,y)$	[2004 → 3338 :2929]
3912	$\square$	[3457,10,2243]

Problem 14M3. The type is (3,0), with ternary f and nullary e.

$$\left\{ \begin{array}{l} f(w,x,y) = f(w,y,x) \\ f(e,e,e) = e \\ f(e,x,f(e,e,x)) = e \end{array} \right\} =\!\!(gL)\!\!\Rightarrow \{f(x, f(y,z,u), f(v,w,t)) = f(x, f(y,z,w), f(v,u,t)\}.$$

This is the ternary version of problem 6; note that the conclusion is the ternary medial law. (This theorem and versions for all higher arities follow easily from the binary version.)

Proof (found by Otter 3.0.3+ on gyro at 14.84 seconds).

1	$f(E_1, f(E_2,A,C), f(E_3,B,D)) \neq f(E_1, f(E_2,A,B), f(E_3,C,D))$	
6	$f(e,x,f(e,e,x)) = e$	
12	$f(x,y,f(z,u,y)) = f(x,v,f(z,u,v))$	[6 → 6 :(gL) :(gL) :(gL)]
45	$f(x, f(y,z,u), f(v,w,f(y,z,w))) = f(x, v_6, f(v,u,v_6))$	[12 → 12]
1765	$f(x, f(y,z,u), f(v,w,v_6)) = f(x, f(y,z,w), f(v,u,v_6))$	[(gL) 45]
1766	$\square$	[1765,1]

Problem 14S3. The type is (3,0), with ternary f and nullary e.

$$\left\{ \begin{array}{l} f(w,x,y) = f(w,y,x) \\ f(e,e,e) = e \\ f(e,x,f(e,x,y)) = y \end{array} \right\} =\!\!(gL)\!\!\Rightarrow \{f(w,x,f(w,x,y)) = y\}.$$

(The conclusion is the ternary Steiner law. Note that the binary version of this theorem is vacuously true.)

Proof (found by Otter 3.0.3+ on gyro at 2.72 seconds).

1	$f(A,B,f(A,B,C)) \neq C$
3	$f(x,y,z) = f(x,z,y)$

4	$f(e,e,e)=e$	
7,6	$f(e,x,f(e,x,y))=y$	
9,8	$f(e,x,f(e,y,x))=y$	[3 → 6]
12	$f(x,y,f(e,y,e))=f(x,e,e)$	[6 → 4 :(gL), flip]
21	$f(e,x,f(y,f(z,e,e),x))=f(z,u,f(y,u,e))$	[8 → 12 :(gL), flip]
367	$f(x,f(x,e,e),y)=f(e,e,y)$	[(gL) 21, flip]
415	$f(x,y,f(x,y,e))=e$	[367 → 21 :9, flip]
492	$f(x,y,f(x,y,z))=z$	[6 → 415 :(gL) :7]
494	$\square$	[492,1]

Problem 14S4. The type is (4,0), with 4-ary f and nullary e.

$$\left\{\begin{array}{l} f(v,w,x,y)=f(v,w,y,x) \\ f(e,e,e,e)=e \\ f(e,e,x,f(e,e,x,y))=y \end{array}\right\} =\!(gL)\!\Rightarrow \ \{f(v,w,x,f(v,w,x,y))=y\}.$$

The conclusion is the 4-ary Steiner law.

Proof (found by Otter 3.0.3+ on gyro at 4.22 seconds).

1	$f(D,A,B,f(D,A,B,C))\neq C$	
3	$f(x,y,z,u)=f(x,y,u,z)$	
4	$f(e,e,e,e)=e$	
6	$f(e,e,x,f(e,e,x,y))=y$	
11,10	$f(e,e,f(e,e,x,y),x)=y$	[3 → 6]
12	$f(x,y,z,f(e,e,z,e))=f(x,y,e,e)$	[6 → 4 :(gL) :(gL), flip]
18	$f(x,y,z,f(u,v,z,w))=f(x,y,v_6,f(u,v,v_6,w))$	[12 → 12 :(gL) :(gL) :(gL)]
29	$f(x,y,f(z,u,v,w),v)=f(x,y,v_6,f(z,u,v_6,w))$	[3 → 18]
33	$f(x,y,z,f(u,v,z,w))=f(x,y,f(u,v,v_6,w),v_6)$	[3 → 18]
119	$f(x,y,z,f(u,v,f(u,v,w,v_6),v_6))=f(x,y,z,w)$	[(gL) 29, flip]
146	$f(x,y,f(x,y,z,u),u)=z$	[119 → 10 :11, flip]
183	$f(x,y,z,f(x,y,z,u))=u$	[33 → 146]
185	$\square$	[183,1]

Problem 14S5. The type is (5,0), with 5-ary f and nullary e.

$$\left\{\begin{array}{l} f(u,v,w,x,y)=f(u,v,w,y,x) \\ f(e,e,e,e,e)=e \\ f(e,e,e,x,f(e,e,e,x,y))=y \end{array}\right\} =\!(gL)\!\Rightarrow \ \{f(u,v,w,x,f(u,v,w,x,y))=y\}.$$

The conclusion is the 5-ary Steiner law.

Proof (found by Otter 3.0.3+ on gyro at 3.93 seconds).

1	$f(E,D,A,B,f(E,D,A,B,C))\neq C$
3	$f(x,y,z,u,v)=f(x,y,z,v,u)$

4	$f(e,e,e,e,e) = e$	
7,6	$f(e,e,e,x,f(e,e,e,x,y)) = y$	
9,8	$f(e,e,e,x,f(e,e,e,y,x)) = y$	[3 → 6]
12	$f(x,y,z,u,f(e,e,e,u,e)) = f(x,y,z,e,e)$	[6 → 4 :(gL) :(gL) :(gL), flip]
21	$f(e,e,e,x,f(y,z,u,f(v,w,v_6,e,e),x)) = f(v,w,v_6,v_7,f(y,z,u,v_7,e))$	[8 → 12 :(gL) :(gL) :(gL), flip]
214	$f(x,y,z,f(x,y,z,e,e),u) = f(e,e,e,e,u)$	[(gL) 21, flip]
255	$f(x,y,z,u,f(x,y,z,u,e)) = e$	[214 → 21 :9, flip]
332	$f(x,y,z,u,f(x,y,z,u,v)) = v$	[6 → 255 :(gL) :7]
334	$\square$	[332,1]

Problem 14U. The type is (5,5,0), with nullary e.

$$\left\{\begin{array}{l} \text{full symmetey of } f \\ f(e,e,e,e,e) = e \\ f(e,e,e,x,f(e,e,e,x,y)) = y \\ \text{full symmetey of } g \\ g(e,e,e,e,e) = e \\ g(e,e,e,x,g(e,e,e,x,y)) = y \end{array}\right\} = (gL) \Rightarrow \ \{g(x,y,z,u,v) = f(x,y,z,u,v)\}.$$

By Problem 14S5, we may also assume the 5-ary Steiner laws for f and for g:

$$f(u,v,w,x,f(u,v,w,x,y)) = y$$
$$g(u,v,w,x,g(u,v,w,x,y)) = y.$$

Full symmetry of the operations causes an explosion in the Otter search space; to constrain the search, we incompletely specify symemtry with

$$f(u,v,w,x,y) = f(u,v,w,y,x)$$
$$g(u,v,w,x,y) = g(u,v,w,y,x)$$
$$g(x,y,z,u,w) = f(x,y,z,u,v) \rightarrow g(y,z,u,w,x) = f(y,z,u,v,x).$$

The Otter input file has several other techniques to constrain the search.

Proof (found by Otter 3.0.3+ on gyro at 400.44 seconds).

2	$g(x,y,z,u,w) = f(x,y,z,u,v) \ \rightarrow \ g(y,z,u,w,x) = f(y,z,u,v,x)$	
3	$f(u,v,w,x,y) = f(u,v,w,y,x)$	
4	$f(e,e,e,e,e) = e$	
5	$f(u,v,w,x,f(u,v,w,x,y)) = y$	
6	$g(u,v,w,x,y) = g(u,v,w,y,x)$	
7	$g(e,e,e,e,e) = e$	
8	$g(u,v,w,x,g(u,v,w,x,y)) = y$	
9	$g(A,B,C,D,E) = f(A,B,C,D,E) \ \rightarrow \ \square$	
11	$f(x,y,z,u,g(e,e,e,e,e)) = f(x,y,z,e,u)$	[7 → 3]
12	$f(e,e,e,e,g(e,e,e,e,e)) = e$	[7 → 4]
15	$f(x,y,z,e,g(u,v,w,v_6,g(u,v,w,v_6,v_7))) = f(x,y,z,v_7,g(e,e,e,e,e))$	[8 → 11, flip]
21	$f(x,y,z,e,g(u,v,w,v_6,g(u,v,w,v_7,v_6))) = f(x,y,z,v_7,g(e,e,e,e,e))$	[6 → 15]

25	$f(x,y,z,e,g(u,v,w,v_6,g(u,v,w,e,v_7))) = f(x,y,z,v_7,g(e,e,e,v_6,e))$	[(gL) 15]
46	$f(x,y,z,e,g(u,v,w,e,v_6)) = f(x,y,z,v_7,g(u,v,w,v_7,v_6))$	[21 $\rightarrow$ 25 :(gL) :(gL) :(gL) :(gL)]
53	$f(x,y,z,u,g(v,w,v_6,u,v_7)) = f(x,y,z,v_8,g(v,w,v_6,v_8,v_7))$	[46 $\rightarrow$ 46]
65	$f(e,e,e,x,g(e,e,e,x,e)) = e$	[12 $\rightarrow$ 46, flip]
70	$f(e,e,e,x,g(e,e,e,e,x)) = e$	[6 $\rightarrow$ 65]
71	$g(e,e,e,x,e) = f(e,e,e,x,e)$	[65 $\rightarrow$ 5, flip]
77	$f(x,y,z,u,g(v,w,v_6,v_7,u)) = f(x,y,z,v_8,g(v,w,v_6,v_7,v_8))$	[70 $\rightarrow$ 70 :(gL) :(gL) :(gL) :(gL) :(gL) :(gL) :(gL)]
83	$g(e,e,x,e,e) = f(e,e,x,e,e)$	[71,2]
100	$f(e,e,x,e,g(e,e,x,e,e)) = e$	[83 $\rightarrow$ 5]
116	$f(e,e,x,y,g(e,e,x,e,y)) = e$	[77 $\rightarrow$ 100]
117	$f(x,y,z,u,g(v,w,z,v_6,v_7)) = f(x,y,v_8,u,g(v,w,v_8,v_6,v_7))$	[100 $\rightarrow$ 100 :(gL) :(gL) :(gL) :(gL) :(gL) :(gL) :(gL)]
149	$f(e,e,x,y,g(e,e,x,y,e)) = e$	[65 $\rightarrow$ 117, flip]
171	$g(e,e,x,e,y) = f(e,e,x,y,e)$	[116 $\rightarrow$ 5, flip]
174	$f(x,y,z,u,g(v,w,z,u,v_6)) = f(x,y,v_7,v_8,g(v,w,v_7,v_8,v_6))$	[149 $\rightarrow$ 149 :(gL) :(gL) :(gL) :(gL) :(gL)]
182	$g(e,e,x,e,y) = f(e,e,x,e,y)$	[3 $\rightarrow$ 171]
221	$f(e,e,x,e,g(e,e,x,e,y)) = y$	[8 $\rightarrow$ 182, flip]
249	$f(e,e,x,y,g(e,e,x,y,z)) = z$	[174 $\rightarrow$ 221]
252	$g(e,e,x,y,z) = f(e,e,x,y,z)$	[8 $\rightarrow$ 249, flip]
256	$g(e,x,y,z,e) = f(e,x,y,z,e)$	[252,2]
269	$g(x,y,z,e,e) = f(x,y,z,e,e)$	[256,2]
272	$f(e,x,y,z,g(e,x,y,z,e)) = e$	[256 $\rightarrow$ 5]
280	$f(x,y,z,e,g(x,y,z,e,e)) = e$	[269 $\rightarrow$ 5]
300	$f(x,y,z,u,g(v,y,z,u,w)) = f(x,v_6,v_7,v_8,g(v,v_6,v_7,v_8,w))$	[272 $\rightarrow$ 272 :(gL) :(gL) :(gL)]
339	$f(x,y,z,u,g(x,y,z,u,e)) = e$	[53 $\rightarrow$ 280]
363	$g(x,y,z,u,e) = f(x,y,z,u,e)$	[339 $\rightarrow$ 5, flip]
375	$g(x,y,z,e,u) = f(x,y,z,e,u)$	[363,2]
408	$f(x,y,z,e,g(x,y,z,e,u)) = u$	[8 $\rightarrow$ 375, flip]
471	$f(x,y,z,u,g(x,y,z,u,v)) = v$	[300 $\rightarrow$ 408]
652	$g(x,y,z,u,v) = f(x,y,z,u,v)$	[8 $\rightarrow$ 471, flip]
653	$\square$	[652,9]

Problem 49. The type is (2,2,0) with $+$, $\cdot$, and e as the three operations.

$$\left\{\begin{array}{l} x+(y+z)=(x+y)+z \\ x+e=x \\ e+x=x \\ x\cdot(y\cdot x)=y \end{array}\right\} \overset{(gL)}{=\!\!\Longrightarrow} \{x+z=((e\cdot e)\cdot x)\cdot(z\cdot e)\}.$$

Comment. I [RP] am generalizing the uniqueness of the group law on a curve into a first order sentence (context: $+$ has a unit element e).

Proof (found by Otter 3.0.3+ on gyro at 11.29 seconds).

3	$x+e=x$
4	$e+x=x$
5	$x\cdot(y\cdot x)=y$

6	$((e \cdot e) \cdot A) \cdot (B \cdot e) \neq A + B$	
20	$(x \cdot y) \cdot x = y$	$[5 \rightarrow 5]$
28	$(x \cdot y) \cdot (e + x) = y$	$[4 \rightarrow 20]$
53	$((e \cdot e) \cdot (A + e)) \cdot (B \cdot e) \neq A + B$	$[3 \rightarrow 6]$
1231	$(x \cdot y) \cdot (z + x) = (u \cdot y) \cdot (z + u)$	$[28 \rightarrow 28$:(gL)]
1778	$((x \cdot y) \cdot (z + x)) \cdot (u \cdot y) = z + u$	$[1231 \rightarrow 20]$
1779	$\square$	[1778,53]

Problem 50A. The type is (2,2) with binary operations $\cdot$ and $\sharp$. Define the ternary Malcev terms as

$$M_1(x, y, z) = ((y \cdot y) \cdot x) \cdot (z \cdot y)$$
$$M_2(x, y, z) = ((y \sharp y) \sharp x) \sharp (z \sharp y).$$

Then

$$\left\{ \begin{array}{l} x \cdot (y \cdot x) = y \\ x \sharp (y \sharp x) = y \end{array} \right\} =\!\!(gL)\!\!\Rightarrow \{M_1(x, y, z) = M_2(x, y, z)\}.$$

We have been unable to get Otter to prove this in a single search without "loading the dice"; however, the result follows easily from the following problem.

Problem 50B. The type is (3,3) with two ternary operations M_1 and M_2.

$$\left\{ \begin{array}{l} M_1(x, y, y) = x,\ M_1(x, x, z) = z \\ M_2(x, y, y) = x,\ M_2(x, x, z) = z \end{array} \right\} =\!\!(gL)\!\!\Rightarrow \{M_1(x, y, z) = M_2(x, y, z)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 92.05 seconds).

2	$M_1(x, y, y) = x$	
3	$M_1(x, x, z) = z$	
4	$M_2(x, y, y) = x$	
5	$M_2(x, x, z) = z$	
6	$M_2(A, B, C) \neq M_1(A, B, C)$	
16	$M_1(x, x, y) = M_1(y, z, z)$	$[2 \rightarrow 3]$
31	$M_1(x, y, M_2(y, z, z)) = x$	$[4 \rightarrow 2]$
38	$M_2(x, x, y) = M_2(y, z, z)$	$[4 \rightarrow 5]$
52	$M_1(x, y, z) = M_1(z, y, x)$	[(gL) 16]
53	$M_1(x, y, M_2(z, z, u)) = M_1(u, y, x)$	$[5 \rightarrow 52$, flip]
61	$M_2(x, y, z) = M_2(z, y, x)$	[(gL) 38]
66	$M_2(x, x, M_2(y, z, u)) = M_2(u, z, y)$	$[5 \rightarrow 61]$
72	$M_1(x, y, M_2(z, z, u)) = M_1(u, y, M_2(v, v, x))$	$[5 \rightarrow 53]$
420	$M_2(x, y, M_2(z, x, u)) = M_2(u, y, z)$	[(gL) 66]
517	$M_2(x, y, M_2(y, x, z)) = z$	$[4 \rightarrow 420]$
938	$M_1(x, y, M_2(z, u, v)) = M_1(v, y, M_2(z, u, x))$	[(gL) 72]
965	$M_1(x, y, M_2(y, x, z)) = z$	$[31 \rightarrow 938$, flip]
990	$M_2(x, y, z) = M_1(x, y, z)$	$[517 \rightarrow 965$, flip]
991	$\square$	[990,6]

Problem 55. The type is (2).

$$\left\{ \begin{array}{l} \text{cancellation} \\ x(y(zu)) = ((xy)z)u \end{array} \right\} =\!(gL)\!\Rightarrow \ \{xy = yx\}.$$

This and the next few problems generalize Prob. 42, CS $=\!(gL)\!\Rightarrow$ $xy = yx$, to weaker types of associativity.

Proof (found by Otter 3.0.3+ on gyro at 15.92 seconds).

1	$B \cdot A = A \cdot B \ \rightarrow \ \square$	
2	$x = x$	
3	$x \cdot y = z,\ x \cdot u = z \ \rightarrow \ y = u$	
4	$x \cdot y = z,\ u \cdot y = z \ \rightarrow \ x = u$	
5	$x \cdot (y \cdot (z \cdot u)) = ((x \cdot y) \cdot z) \cdot u$	
6	$((x \cdot y) \cdot z) \cdot u = x \cdot (y \cdot (z \cdot u))$	[flip 5]
7	$((((x \cdot y) \cdot z) \cdot u) \cdot v) \cdot w = x \cdot ((y \cdot (z \cdot u)) \cdot (v \cdot w))$	[$6 \rightarrow 6$]
8	$((x \cdot (y \cdot (z \cdot u))) \cdot v) \cdot w = ((x \cdot y) \cdot z) \cdot (u \cdot (v \cdot w))$	[$6 \rightarrow 6$]
14	$((x \cdot y) \cdot z) \cdot (u \cdot v) = x \cdot (((y \cdot z) \cdot u) \cdot v)$	[$6 \rightarrow 6$]
38	$((x \cdot y) \cdot z) \cdot (u \cdot v) = x \cdot ((y \cdot (z \cdot u)) \cdot v)$	[$6 \rightarrow 7$:(gL)]
42	$x \cdot (((y \cdot z) \cdot u) \cdot v) = x \cdot ((y \cdot (z \cdot u)) \cdot v)$	[$14 \rightarrow 38$]
53	$((x \cdot y) \cdot z) \cdot u = (x \cdot (y \cdot z)) \cdot u$	[42,3,2, flip]
63	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	[53,4,2, flip]
111	$((x \cdot (y \cdot (z \cdot u))) \cdot v) \cdot w = ((x \cdot y) \cdot v) \cdot (u \cdot (z \cdot w))$	[(gL) 8]
125	$(x \cdot (y \cdot z)) \cdot u = (x \cdot y) \cdot (z \cdot u)$	[$63 \rightarrow 63$]
273	$(x \cdot y) \cdot (z \cdot (u \cdot v)) = (x \cdot u) \cdot (z \cdot (y \cdot v))$	[$8 \rightarrow 111$:(gL)]
453	$(x \cdot y) \cdot z = y \cdot (x \cdot z)$	[125,3,273, flip]
540	$x \cdot y = y \cdot x$	[453,4,63]
541	$\square$	[540,1]

Problem 55A. The type is (2).

$$\left\{ \begin{array}{l} \text{cancellation} \\ x(y(z(uv))) = (((xy)z)u)v \end{array} \right\} =\!(gL)\!\Rightarrow \ \{xy = yx\}.$$

Proof (found by Otter 3.0.3+ on gyro at 13.78 seconds).

1	$B \cdot A = A \cdot B \ \rightarrow \ \square$	
3	$x \cdot y = z,\ x \cdot u = z \ \rightarrow \ y = u$	
4	$x \cdot y = z,\ u \cdot y = z \ \rightarrow \ x = u$	
5	$x \cdot (y \cdot (z \cdot (u \cdot v))) = (((x \cdot y) \cdot z) \cdot u) \cdot \vee$	
6	$(((x \cdot y) \cdot z) \cdot u) \cdot v = x \cdot (y \cdot (z \cdot (u \cdot v)))$	[flip 5]
7	$((x \cdot (y \cdot (z \cdot (u \cdot v)))) \cdot w) \cdot v_6 = ((x \cdot y) \cdot z) \cdot (u \cdot (v \cdot (w \cdot v_6)))$	[$6 \rightarrow 6$]
8	$(x \cdot (y \cdot (z \cdot (u \cdot v)))) \cdot w = (x \cdot y) \cdot (z \cdot (u \cdot (v \cdot w)))$	[$6 \rightarrow 6$]
25	$((x \cdot (y \cdot (z \cdot (u \cdot v)))) \cdot w) \cdot v_6 = ((x \cdot y) \cdot w) \cdot (u \cdot (v \cdot (z \cdot v_6)))$	[(gL) 7]
32	$(x \cdot (y \cdot (z \cdot (u \cdot v)))) \cdot w = (x \cdot y) \cdot (w \cdot (u \cdot (v \cdot z)))$	[25,4,8, flip]

33	$(x \cdot (y \cdot (z \cdot u))) \cdot v = (x \cdot (y \cdot (v \cdot u))) \cdot z$	[25,4,7 :(gL)]
37	$(x \cdot y) \cdot (z \cdot (u \cdot (v \cdot w))) = (x \cdot v) \cdot (z \cdot (u \cdot (y \cdot w)))$	[7 → 25 :(gL)]
56	$x \cdot (y \cdot (z \cdot (u \cdot v))) = x \cdot (v \cdot (z \cdot (u \cdot y)))$	[8 → 32 :(gL)]
61	$x \cdot (y \cdot (z \cdot u)) = u \cdot (y \cdot (z \cdot x))$	[56,3,33]
208	$x \cdot y = y \cdot x$	[37,4,61]
209	$\square$	[208,1]

Problem 56. The type is (2).

$$\left\{\begin{array}{l}\text{cancellation}\\ x(y(zu)) = (xy)(zu)\end{array}\right\} \stackrel{(gL)}{=\!\Rightarrow} \{xy = yx\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.43 seconds).

1	$B \cdot A = A \cdot B \;\rightarrow\; \square$	
3	$x \cdot y = z,\; x \cdot u = z \;\rightarrow\; y = u$	
4	$x \cdot y = z,\; u \cdot y = z \;\rightarrow\; x = u$	
5	$x \cdot (y \cdot (z \cdot u)) = (x \cdot y) \cdot (z \cdot u)$	
6	$(x \cdot y) \cdot (z \cdot u) = x \cdot (y \cdot (z \cdot u))$	[flip 5]
13	$(x \cdot y) \cdot (z \cdot u) = x \cdot ((y \cdot z) \cdot u)$	[6 → 6 :(gL)]
15	$(x \cdot y) \cdot (z \cdot u) = x \cdot (z \cdot (y \cdot u))$	[(gL) 6]
42	$(x \cdot y) \cdot z = y \cdot (x \cdot z)$	[15,3,13, flip]
91	$x \cdot y = y \cdot x$	[42,4,15]
92	$\square$	[91,1]

Problem 57. The type is (2).

$$\left\{\begin{array}{l}\text{cancellation}\\ x(y(zu)) = (x(yz))u\end{array}\right\} \stackrel{(gL)}{=\!\Rightarrow} \{xy = yx\}.$$

Counterexample. The clauses

$(x \cdot y) \cdot (z \cdot u) = (x \cdot z) \cdot (y \cdot u)$ % medial law
$x \cdot (y \cdot (z \cdot u)) = (x \cdot (y \cdot z)) \cdot u$
$B \cdot A \neq A \cdot B$

```
properties(_*_, quasigroup).
```

have the following model (found by MACE 1.0.0 on gyro at 4.19 seconds).

```
*:
     | 0 1 2
   --+------
   0 | 0 2 1
   1 | 1 0 2
   2 | 2 1 0
B: 1
A: 0
```

Problem 57A. The type is (2).

$$\left\{\begin{array}{l}\text{cancellation}\\ (x(yz))u=(xy)(zu)\end{array}\right\} =\!\!(gL)\!\!\Rightarrow \{xy=yx\}.$$

Proof (found by Otter 3.0.3+ on gyro at 1.97 seconds).

1	$B\cdot A=A\cdot B \;\rightarrow\; \Box$	
3	$x\cdot y=z,\; x\cdot u=z \;\rightarrow\; y=u$	
4	$x\cdot y=z,\; u\cdot y=z \;\rightarrow\; x=u$	
5	$(x\cdot(y\cdot z))\cdot u=(x\cdot y)\cdot(z\cdot u)$	
8	$((x\cdot(y\cdot z))\cdot u)\cdot v=((x\cdot y)\cdot z)\cdot(u\cdot v)$	[5 → 5]
24	$((x\cdot(y\cdot z))\cdot u)\cdot v=((x\cdot y)\cdot u)\cdot(z\cdot v)$	[(gL) 8]
29	$(x\cdot(y\cdot z))\cdot u=(x\cdot z)\cdot(y\cdot u)$	[5 → 24 :(gL)]
32	$(x\cdot y)\cdot(z\cdot u)=(x\cdot z)\cdot(y\cdot u)$	[8 → 24 :(gL)]
67	$x\cdot(y\cdot z)=x\cdot(z\cdot y)$	[29,4,5]
183	$x\cdot y=y\cdot x$	[67,3,32]
184	$\Box$	[183,1]

Problem 58. The type is (2,1,0) with nullary e.

$$\left\{\begin{array}{l}x'x=e\\ ex=x\\ xe=x\end{array}\right\} =\!\!(gL)\!\!\Rightarrow \left\{\begin{array}{l}xy=yx\\ (xy)z=x(yz)\end{array}\right\}.$$

Proof. This follows from Probs. 2G and 3G.

4.2 Sample Figures

Here are some sample figures. We'll probably delete some and add others.

Figure 4.1 is an example of a figure.
Figure 4.2 is an example of a figure.
Figure 4.3 is an example of a figure.
Figure 4.4 is an example of a figure.
Figure 4.5 is an example of a figure.
Figure 4.6 is an example of a figure.
Figure 4.7 is an example of a figure.
Figure 4.8 is an example of a figure.

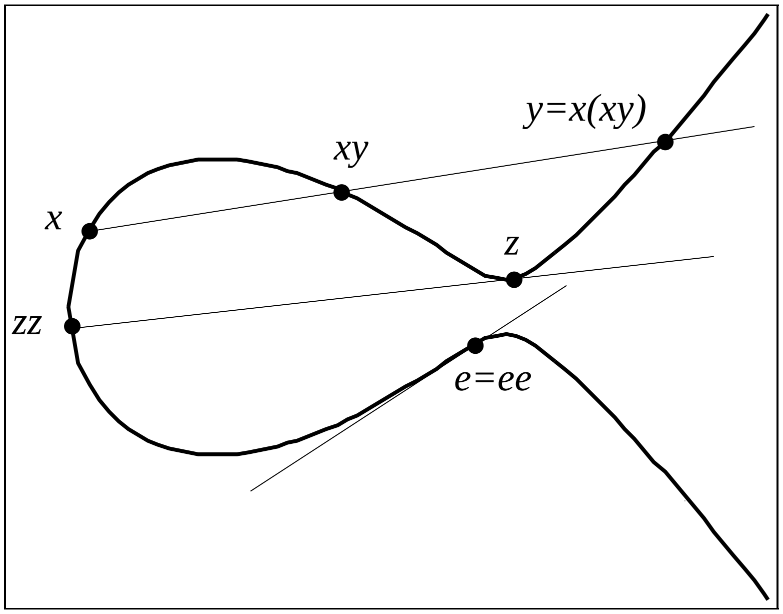

Figure 4.1: omega-chord

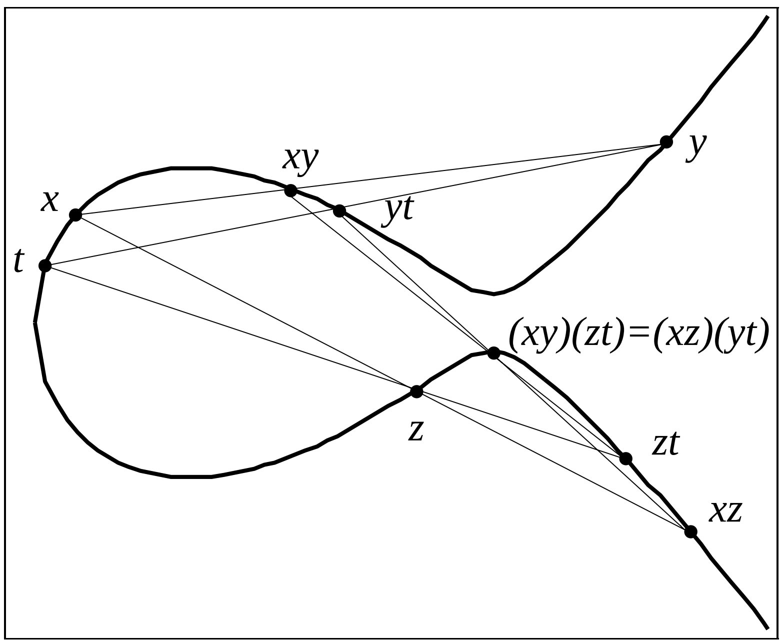

Figure 4.2: omega-medial

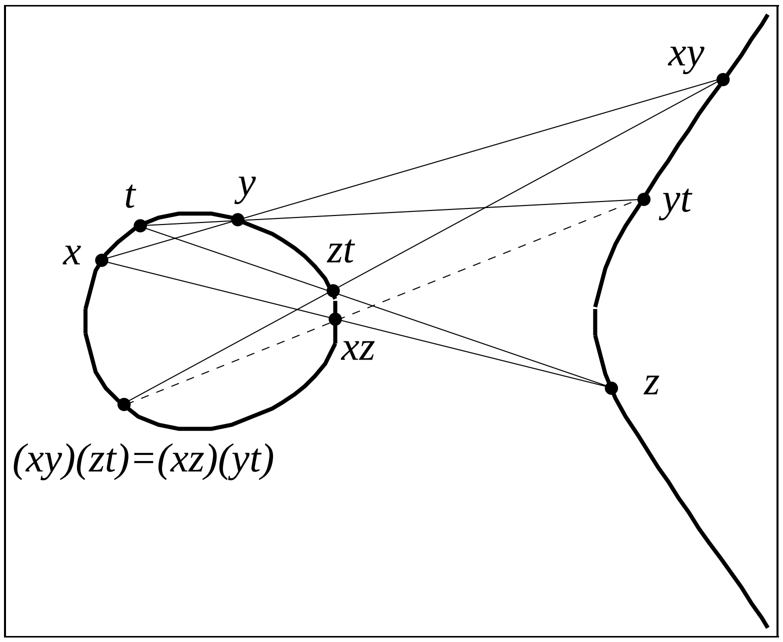

Figure 4.3: egg-medial

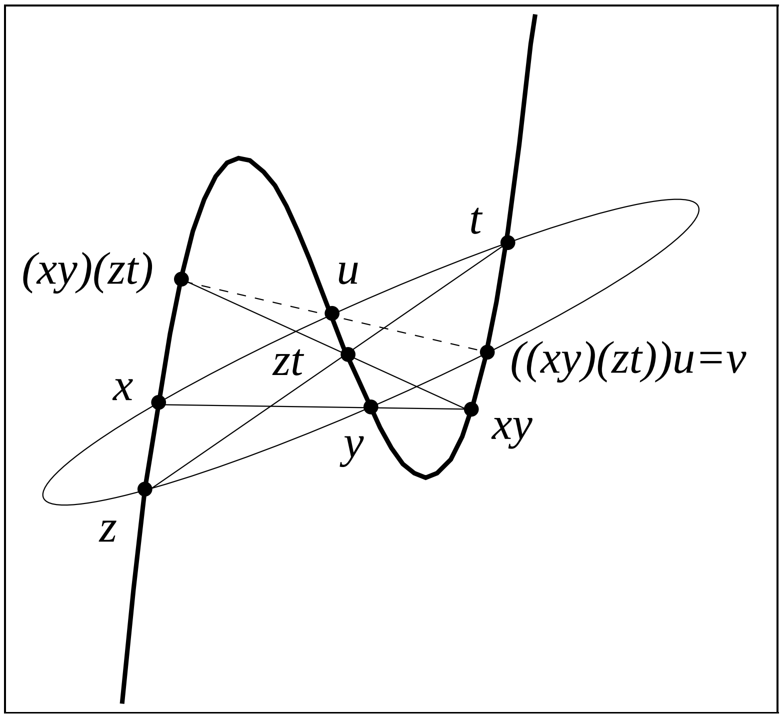

Figure 4.4: s-conic

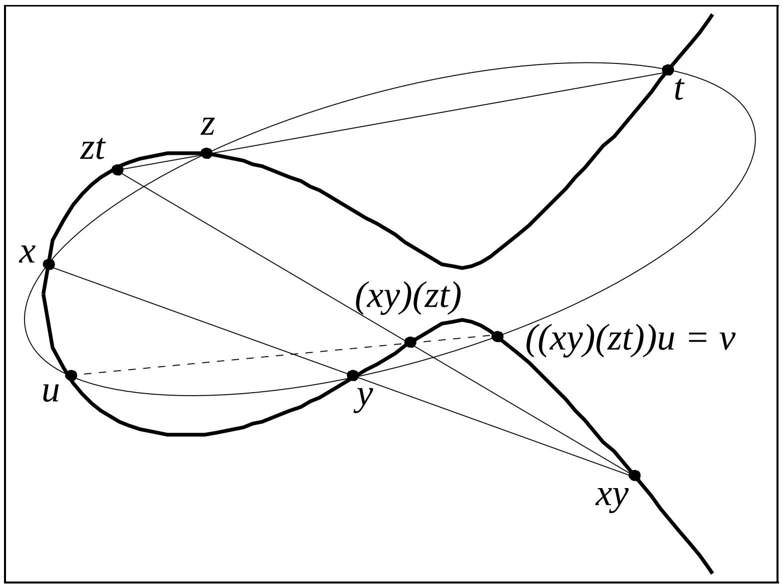

Figure 4.5: omega-conic

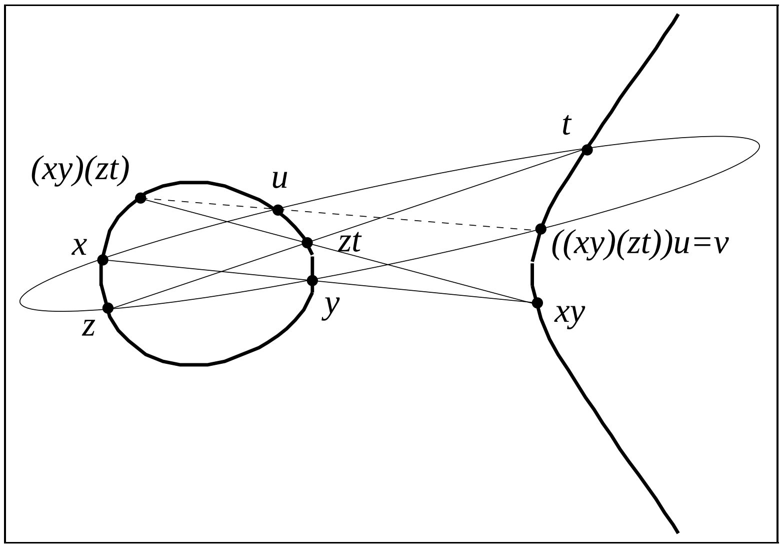

Figure 4.6: egg-conic

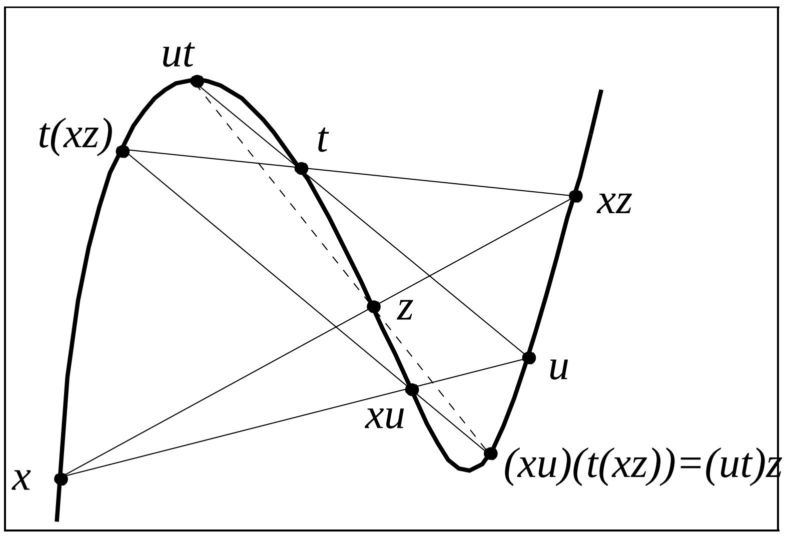

Figure 4.7: 2-independent

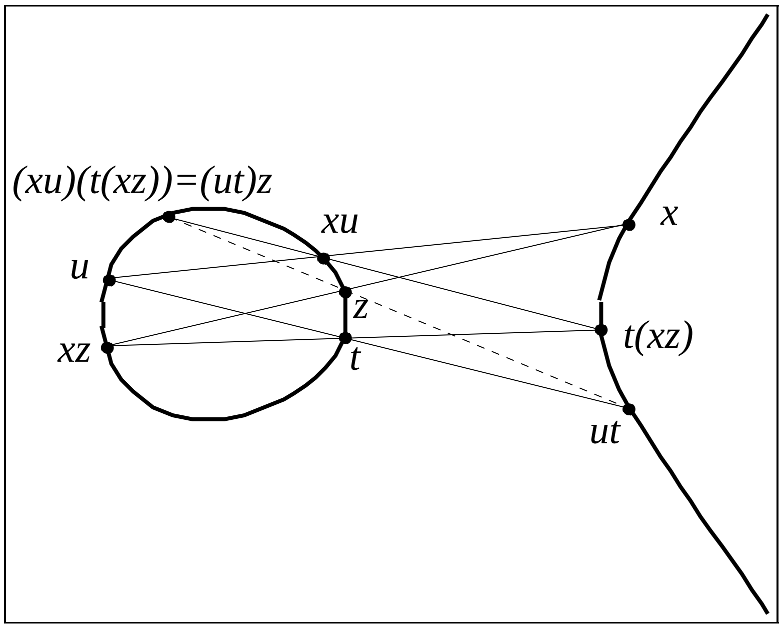

Figure 4.8: egg-independent

Chapter 5

Quasigroups

This is the chapter on quasigroups.

Problem 4A.

$$\{x(y(yx)) = yx, \text{cancellation}\} \Rightarrow \{x(xy) = yx\}.$$

Open. (True if one further assumes that '$\cdot$' is a quasigroup, but this stronger assumption is not equational. See the next problem.)

I [WWM] think this is false, because in an unconstrained Otter search, the given clauses just keep getting larger. MACE is useless here, because a finite calcellative operation is a quasigruop.

Problem 4AQ.

$$\{x(y(yx)) = yx, \text{quasigroup}\} \Rightarrow \{x(xy) = yx\}.$$

This was known to be true.

Proof (found by Otter 3.0.3+ on gyro at 0.24 seconds).

1	$x = x$	
2	$x \cdot y = z,\ u \cdot y = z \ \rightarrow\ x = u$	
3	$x \cdot (y \cdot (y \cdot x)) = y \cdot x$	
6,5	$x \cdot j(x, y) = y$	
7	$A \cdot (A \cdot B) = B \cdot A \ \rightarrow\ \square$	
8	$B \cdot A = A \cdot (A \cdot B) \ \rightarrow\ \square$	[flip 7]
13	$j(x, y) \cdot (x \cdot y) = y$	[5 $\rightarrow$ 3 :6]
15	$j(x, x \cdot y) = y$	[2,3,13, flip]
17	$j(j(x, y), x \cdot y) \cdot y = x \cdot y$	[13 $\rightarrow$ 13]
45	$j(j(x, y), x \cdot y) = x$	[2,1,17, flip]
49	$j(x, y \cdot (y \cdot x)) = y$	[15 $\rightarrow$ 45]
57	$x \cdot y = y \cdot (y \cdot x)$	[49 $\rightarrow$ 5]
58	$\square$	[57,8]

Problem 30.

$$\{((xx)y)(z(xy)) = z\} \Leftrightarrow \left\{ \begin{array}{l} xx = x \\ (xy)x = y \end{array} \right\}.$$

Status: True. First proved by Padmanabhan.

Proof ($\Rightarrow$) found by Otter 3.0.3+ on gyro at 0.21 seconds.

1	$x = x$	
3,2	$((x \cdot x) \cdot y) \cdot (z \cdot (x \cdot y)) = z$	
4	$A \cdot A = A,\ (A \cdot B) \cdot A = B \ \rightarrow\ \Box$	
7	$((x \cdot x) \cdot (y \cdot z)) \cdot x = (y \cdot y) \cdot z$	[2 $\rightarrow$ 2]
9	$(x \cdot x) \cdot (y \cdot (y \cdot y)) = x \cdot (y \cdot y)$	[2 $\rightarrow$ 7, flip]
18,17	$(x \cdot (y \cdot y)) \cdot x = (y \cdot y) \cdot (y \cdot y)$	[9 $\rightarrow$ 7]
19	$((x \cdot x) \cdot (x \cdot x)) \cdot (y \cdot (x \cdot x)) = y \cdot y$	[9 $\rightarrow$ 2]
22	$x \cdot x = ((y \cdot y) \cdot (y \cdot y)) \cdot (x \cdot (y \cdot y))$	[flip 19]
28,27	$(x \cdot x) \cdot (x \cdot x) = (x \cdot x) \cdot x$	[17 $\rightarrow$ 7 :18]
30,29	$x \cdot x = x$	[22 :28,3]
34,33	$(x \cdot y) \cdot x = y$	[17 :30,30,30,30]
35	$A = A,\ B = B \ \rightarrow\ \Box$	[4 :30,34]
36	$\Box$	[35,1,1]

Proof ($\Leftarrow$) found by Otter 3.0.3+ on gyro at 0.02 seconds.

3,2	$x \cdot x = x$	
4	$(x \cdot y) \cdot x = y$	
6	$((A \cdot A) \cdot B) \cdot (C \cdot (A \cdot B)) \neq C$	
7	$(A \cdot B) \cdot (C \cdot (A \cdot B)) \neq C$	[copy,6 :3]
8	$x \cdot (y \cdot x) = y$	[4 $\rightarrow$ 4]
10	$\Box$	[8,7]

Problem 31. Prove

$$\{(((xx)y)x)(zy) = z\} \Leftrightarrow \left\{ \begin{array}{l} xx = x \\ (xy)x = y \end{array} \right\}.$$

Status: True. First proved by Barry Wolk.

Proof ($\Rightarrow$) found by Otter 3.0.3+ on gyro at 0.26 seconds.

1	$x = x$	
3,2	$(((x \cdot x) \cdot y) \cdot x) \cdot (z \cdot y) = z$	
4	$A \cdot A = A,\ (A \cdot B) \cdot A = B \ \rightarrow\ \Box$	
5	$((((x \cdot x) \cdot x) \cdot y) \cdot (((x \cdot x) \cdot x) \cdot x)) \cdot (z \cdot y) = z$	[2 $\rightarrow$ 2]
7	$(((x \cdot x) \cdot (y \cdot z)) \cdot x) \cdot y = ((u \cdot u) \cdot z) \cdot u$	[2 $\rightarrow$ 2]
8	$((x \cdot x) \cdot y) \cdot x = (((z \cdot z) \cdot (u \cdot y)) \cdot z) \cdot u$	[flip 7]

9	$((x\cdot x)\cdot y)\cdot x = ((z\cdot z)\cdot y)\cdot z$	[7 → 7]
19,18	$((x\cdot x)\cdot x)\cdot(y\cdot x) = y$	[9 → 5 :3]
29,28	$(((x\cdot y)\cdot(x\cdot y))\cdot(x\cdot y))\cdot x = (y\cdot y)\cdot y$	[18 → 18]
31,30	$((x\cdot x)\cdot x)\cdot(((y\cdot y)\cdot z)\cdot y) = (x\cdot x)\cdot z$	[9 → 18]
33	$(x\cdot x)\cdot x = ((y\cdot y)\cdot x)\cdot y$	[2 → 18 :29]
37,36	$(((x\cdot x)\cdot(y\cdot z))\cdot x)\cdot y = (z\cdot z)\cdot z$	[18 → 2]
38	$((x\cdot x)\cdot y)\cdot((x\cdot x)\cdot x) = ((z\cdot z)\cdot y)\cdot z$	[18 → 9]
39	$((x\cdot x)\cdot y)\cdot((x\cdot x)\cdot x) = (y\cdot y)\cdot y$	[18 → 8 :37]
40	$(((x\cdot x)\cdot y)\cdot((x\cdot x)\cdot x))\cdot(z\cdot y) = z$	[18 → 2]
42	$((x\cdot x)\cdot y)\cdot x = ((z\cdot z)\cdot y)\cdot((z\cdot z)\cdot z)$	[flip 38]
43	$(x\cdot x)\cdot x = ((y\cdot y)\cdot x)\cdot((y\cdot y)\cdot y)$	[flip 39]
44	$((x\cdot x)\cdot((y\cdot y)\cdot y))\cdot x = (y\cdot y)\cdot((y\cdot y)\cdot y)$	[18 → 33, flip]
46	$((x\cdot x)\cdot(((y\cdot y)\cdot y)\cdot y))\cdot x = (y\cdot y)\cdot y$	[2 → 33 :19, flip]
49,48	$(x\cdot x)\cdot x = x\cdot x$	[33 → 18 :31]
54,53	$((x\cdot x)\cdot(y\cdot y))\cdot x = y\cdot y$	[46 :49,49,49]
55	$(x\cdot x)\cdot(x\cdot x) = x\cdot x$	[44 :49,54,49, flip]
58,57	$((x\cdot x)\cdot y)\cdot(x\cdot x) = y\cdot y$	[43 :49,49, flip]
60,59	$((x\cdot x)\cdot y)\cdot x = y\cdot y$	[42 :49,58]
62,61	$(x\cdot x)\cdot(y\cdot x) = y$	[40 :49,58]
63	$((x\cdot y)\cdot(x\cdot y))\cdot x = y\cdot y$	[36 :60,49]
72,71	$x\cdot x = x$	[55 :62, flip]
76,75	$(x\cdot y)\cdot x = y$	[63 :72,72]
77	$A = A,\ B = B \ \rightarrow\ \Box$	[4 :72,76]
78	$\Box$	[77,1,1]

Proof (⇐) found by Otter 3.0.3+ on gyro at 0.05 seconds.

3,2	$x\cdot x = x$	
5,4	$(x\cdot y)\cdot x = y$	
6	$(((A\cdot A)\cdot B)\cdot A)\cdot(C\cdot B) \neq C$	
7	$B\cdot(C\cdot B) \neq C$	[copy,6 :3,5]
8	$x\cdot(y\cdot x) = y$	[4 → 4]
10	$\Box$	[8,7]

Problem 32.

$$\left\{\begin{array}{l} xx = x \\ (xy)x = y \end{array}\right\} \Leftrightarrow \{(xy)((zz)(xy)) = z\}.$$

Status: Open Problem. I [RP] could neither prove it nor able to get a counterexample.

Counterexample. The clauses

$(x\cdot y)\cdot((z\cdot z)\cdot(x\cdot y)) = z$
$A\cdot A \neq A$

have the following model (found by MACE 1.0.0 on gyro at 0.70 seconds).

```
*:
     | 0 1 2 3 4
   --+----------
   0 | 1 3 2 4 0
   1 | 4 2 0 3 1
   2 | 0 4 3 1 2
   3 | 2 1 4 0 3
   4 | 3 0 1 2 4
A: 0
```

Problem 33.

$$\{((xx)y)(z(yx)) = z\} \Leftrightarrow \left\{ \begin{array}{l} xy = yx \\ xx = x \\ (xy)x = y \end{array} \right\}.$$

Status: True.

Proof ($\Rightarrow$) found by Otter 3.0.3+ on gyro at 0.17 seconds.

1	$x = x$	
3,2	$((x \cdot x) \cdot y) \cdot (z \cdot (y \cdot x)) = z$	
4	$B \cdot A = A \cdot B,\ A \cdot A = A,\ (A \cdot B) \cdot A = B\ \rightarrow\ \Box$	
5	$x \cdot (y \cdot ((x \cdot ((z \cdot z) \cdot z)) \cdot (z \cdot z))) = y$	[2 $\rightarrow$ 2]
9	$x \cdot (x \cdot ((y \cdot y) \cdot y)) = (y \cdot y) \cdot y$	[2 $\rightarrow$ 5]
11	$((x \cdot y) \cdot (x \cdot y)) \cdot (x \cdot y) = x \cdot y$	[2 $\rightarrow$ 9 :3, flip]
14,13	$(x \cdot x) \cdot (x \cdot x) = (x \cdot x) \cdot x$	[2 $\rightarrow$ 9]
16,15	$((x \cdot x) \cdot x) \cdot ((x \cdot x) \cdot x) = x \cdot x$	[13 $\rightarrow$ 2]
20,19	$((x \cdot x) \cdot x) \cdot (x \cdot x) = x \cdot x$	[13 $\rightarrow$ 13 :14,16,14, flip]
26,25	$(x \cdot x) \cdot x = x \cdot x$	[19 $\rightarrow$ 2 :20, flip]
27	$(x \cdot x) \cdot (y \cdot (x \cdot x)) = y$	[19 $\rightarrow$ 2 :14,26,26,14,26]
34,33	$(x \cdot y) \cdot (x \cdot y) = x \cdot y$	[11 :26]
43	$x \cdot y = (y \cdot y) \cdot x$	[2 $\rightarrow$ 27 :34,34]
44	$(x \cdot x) \cdot y = y \cdot x$	[flip 43]
45	$((x \cdot (y \cdot y)) \cdot (y \cdot y)) \cdot (z \cdot x) = z$	[27 $\rightarrow$ 2 :34]
50,49	$(x \cdot (y \cdot y)) \cdot (y \cdot y) = x$	[27 $\rightarrow$ 43 :34, flip]
53	$x \cdot (y \cdot x) = y$	[45 :50]
60,59	$(x \cdot y) \cdot x = y$	[53 $\rightarrow$ 53]
64,63	$(x \cdot y) \cdot y = x$	[43 $\rightarrow$ 53 :34]
65	$B \cdot A = A \cdot B,\ A \cdot A = A\ \rightarrow\ \Box$	[4 :60 :1]
67,66	$x \cdot x = x$	[53 $\rightarrow$ 2 :64]
68	$B \cdot A = A \cdot B\ \rightarrow\ \Box$	[65 :67 :1]
75	$x \cdot y = y \cdot x$	[44 :67]
76	$\Box$	[75,68]

Proof ($\Leftarrow$) found by Otter 3.0.3+ on gyro at 0.06 seconds.

1	$x = x$	
3,2	$x \cdot x = x$	
4	$(x \cdot y) \cdot x = y$	
6	$x \cdot y = y \cdot x$	
7	$((A \cdot A) \cdot B) \cdot (C \cdot (B \cdot A)) \neq C$	
8	$(A \cdot B) \cdot (C \cdot (B \cdot A)) \neq C$	[copy,7 :3]
10,9	$x \cdot (y \cdot x) = y$	[4 → 4]
15	$C \neq C$	[6 → 8 :10]
16	$\square$	[15,1]

Problem 34. Type (2,1) with one binary and one unary.

$$\{(g(x) \cdot y) \cdot (z \cdot (y \cdot x)) = z\} \Leftrightarrow \left\{ \begin{array}{l} x \cdot y = y \cdot x \\ g(x) = x \\ (x \cdot y) \cdot x = y \end{array} \right\}.$$

Status: True. Proved by Padmanabhan and Barry Wolk. Comment: This shows that the equational theory of any finitely based subvariety of Stein Quasigroups is one-based. The unary $g(x)$ packs all the remaining identities true in the subvariety in question. For example, if $g(x) = x \cdot x$, then we obtain Problem 33.

Proof ($\Rightarrow$) found by Otter 3.0.3+ on gyro at 1.18 seconds.

1	$x = x$	
2	$(g(x) \cdot y) \cdot (z \cdot (y \cdot x)) = z$	
4	$B \cdot A = A \cdot B,\ g(A) = A,\ (A \cdot B) \cdot A = B \ \rightarrow\ \square$	
5	$(g(x \cdot (y \cdot z)) \cdot (g(z) \cdot y)) \cdot (u \cdot x) = u$	[2 → 2]
8,7	$(g(x \cdot y) \cdot z) \cdot z = g(y) \cdot x$	[2 → 2]
10,9	$g(x) \cdot (g(y \cdot z) \cdot x) = g(y) \cdot g(z)$	[7 → 7 :8, flip]
13	$(g(x) \cdot g(y \cdot z)) \cdot (g(y) \cdot g(z)) = g(x)$	[9 → 2]
15	$(g(g(x)) \cdot g(y)) \cdot g(z) = g(z) \cdot g(y \cdot x)$	[13 → 2]
18,17	$g(x \cdot y) \cdot x = g(y)$	[7 → 5]
19	$g(g(x)) \cdot g(y \cdot x) = g(y)$	[17 → 17]
25,24	$g(x \cdot (y \cdot z)) = g(x) \cdot (g(z) \cdot y)$	[2 → 17, flip]
35	$g(x) \cdot (y \cdot (z \cdot (z \cdot x))) = y$	[17 → 2]
47	$g(x) \cdot (g(x) \cdot g(y)) = g(g(y))$	[19 → 13]
48	$g(x) \cdot (y \cdot (g(x \cdot z) \cdot g(z))) = y$	[19 → 2]
58	$(g(x) \cdot y) \cdot y = (g(x) \cdot g(z)) \cdot g(z)$	[35 → 7 :25,18]
67	$(g(x) \cdot g(y)) \cdot g(y) = (g(x) \cdot z) \cdot z$	[flip 58]
78	$g(x) \cdot g(y) = g(y) \cdot x$	[47 → 7 :10]
92,91	$g(x \cdot y) \cdot g(y) = g(x)$	[19 → 78, flip]
94	$g(x) \cdot (y \cdot g(x)) = y$	[48 :92]
118	$g(x) \cdot (g(x) \cdot g(y \cdot z)) = g(g(z)) \cdot g(y)$	[15 → 94]
126	$g(g(x)) \cdot g(y) = g(z) \cdot (g(z) \cdot g(y \cdot x))$	[flip 118]
130,129	$g(g(x)) \cdot (y \cdot g(x)) = y$	[94 → 35]
136,135	$x \cdot (x \cdot g(y \cdot z)) = g(z) \cdot y$	[94 → 7]

138,137	$g(x \cdot g(y)) = g(x) \cdot g(y)$	[94 → 17, flip]
140,139	$(g(x) \cdot g(y)) \cdot g(y) = g(x)$	[94 → 13 :138,130]
143	$(g(g(x)) \cdot y) \cdot y = g(x)$	[94 → 2]
147,146	$g(x) \cdot (y \cdot x) = y$	[94 → 2 :138,140]
152	$g(g(x)) \cdot g(y) = g(x) \cdot y$	[126 :136]
155,154	$(g(x) \cdot y) \cdot y = g(x)$	[67 :140, flip]
156	$g(x \cdot y) = g(x) \cdot g(y)$	[9 :147]
166,165	$g(g(x)) = g(x)$	[143 :155]
193,192	$g(x) \cdot g(y) = g(x) \cdot y$	[152 :166]
219,218	$g(x \cdot y) = g(x) \cdot y$	[156 :193]
263	$x \cdot (x \cdot y) = g(y)$	[146 → 154]
286,285	$g(x) = x$	[263 → 35 :219,147]
292,291	$(x \cdot y) \cdot x = y$	[263 → 2 :286,286]
295	$x \cdot y = y \cdot x$	[263 → 2 :286,286]
308	$B \cdot A = A \cdot B,\ A = A,\ B = B \ \rightarrow\ \square$	[4 :286,292]
359	$\square$	[308,295,1,1]

Proof (⇐) found by Otter 3.0.3+ on gyro at 0.04 seconds.

1	$x = x$	
2	$x \cdot y = y \cdot x$	
4,3	$g(x) = x$	
5	$(x \cdot y) \cdot x = y$	
7	$(g(A) \cdot B) \cdot (C \cdot (B \cdot A)) \neq C$	
8	$(A \cdot B) \cdot (C \cdot (B \cdot A)) \neq C$	[copy,7 :4]
10,9	$x \cdot (y \cdot x) = y$	[5 → 5]
15	$C \neq C$	[2 → 8 :10]
16	$\square$	[15,1]

Chapter 6

Semigroups

6.1 A Conjecture of Padmanabhan

E-mail from RP to WWM, 7 May, 1993:

> Let me now state more problems for Otter, all in one area: this time it is semigroups. These are based on an uproved conjecture of mine. So, any negative solution will disprove my conjecture once and for all. But every positive solution, apart from being a new theorem in Mathematics will generalize the corresponding result from group theory. It will thus strengthen the conjecture and also may pave way for the general proof. So much for the preamble.
>
> Groups are of type (2,1,0) with an associative multiplication, an inverse and an identity element. The semigroup of a group is, of course, cancellative: $ab = ac \;\Rightarrow\; a'ab = a'ac \;\Rightarrow\; b = c$. The cancellative law, as a first order property, is within the language of type (2). This is the backdrop of my conjecture.
>
> *Conjecture.* Let A be a set of axioms of type (2) and let a be an axiom of the same type. If every group satisfying A also satisfies a, then every cancellative semigroup satisfying A must satisfy a as well. To put it more formally, let GT be the axioms of group theory and CS be the associative law and the two cancellation laws. Then
>
> $$\text{if } (A, \mathrm{GT} \Rightarrow a) \text{ then } (A, \mathrm{CS} \Rightarrow a).$$
>
> From the computational point of view, this conjecture says that if one uses the luxury of the richer language of {associativity, inverse and identity} to derive a from A, then one can do the same thing within the limited language of just one binary operation which is associative and cancellative. Over the past ten years, I have mentioned this problem to many famous group theorists (including S.I. Adjan, B.H. Neumann, Sevrin, Narain Gupta and many others). No one believed

it right away and tried to give counter examples. But none of them really worked and the problem is still open. I too believe that this may not true and the only way is to find a counter-example of a rather difficult nature. I can see the very useful role Otter can play here.

For our purpose, all we need to do is to pick up "good" A's in the language of one binary operation implying a non-trivial a in groups and try to prove it under the milder hypothesis of just CS. Over the years, I have collected some examples.

Problem 39.

$$\{\text{CS},\ xyzyx = yxzxy\} \Rightarrow \{xyyx = yxxy\}.$$

In groups, simply substitute $z = e$ to derive the RHS. Status: True due to a second order reason: such semigroups can be imbedded in groups satisfying the same identity and hence OK. The imbedding theorem was proved by A.I. Malcev, B.H. Neumann and others around the 1950's. A first order proof was given by R. Padmanabhan.

Proof (found by Otter 3.0.3+ on gyro at 9.90 seconds).

1	$A \cdot B \cdot B \cdot A = B \cdot A \cdot A \cdot B \ \rightarrow\ \square$	
2	$x = x$	
3	$x \cdot y = z,\ x \cdot u = z \ \rightarrow\ y = u$	
6,5	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
7	$x \cdot y \cdot z \cdot y \cdot x = y \cdot x \cdot z \cdot x \cdot y$	
11	$x \cdot y \cdot z \cdot u \cdot y \cdot x = y \cdot x \cdot z \cdot u \cdot x \cdot y$	[5 → 7 :6]
17	$x \cdot y \cdot z \cdot y \cdot x \cdot u = y \cdot x \cdot z \cdot x \cdot y \cdot u$	[7 → 5 :6,6,6,6,6,6,6]
22	$x \cdot y \cdot z \cdot u \cdot \vee \cdot y \cdot x = y \cdot x \cdot z \cdot u \cdot \vee \cdot x \cdot y$	[5 → 11 :6]
262	$x \cdot y \cdot x \cdot z \cdot z \cdot x = x \cdot y \cdot z \cdot x \cdot x \cdot z$	[22,3,17]
329	$x \cdot y \cdot z \cdot z \cdot y = x \cdot z \cdot y \cdot y \cdot z$	[262,3,2]
376	$x \cdot y \cdot y \cdot x = y \cdot x \cdot x \cdot y$	[329,3,2]
377	$\square$	[376,1]

Problem 40.

$$\{\text{Group Theory},\ z[x,y] = [x,y]z\} \Rightarrow \{xyyx = yxxy\}$$

Here $[x, y]$ stands for the commutator $x'y'xy$. In other words, the identity in question says that the commutators commute. These are the so-called meta-abelian groups. Status: True. This is also a Corollory of Problem 39.

Proof (found by Otter 3.0.3+ on gyro at 8.26 seconds).

1	$A \cdot (B \cdot (B \cdot A)) \neq B \cdot (A \cdot (A \cdot B))$
4,3	$e \cdot x = x$

8,7	$x' \cdot x = e$	
12,11	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	
16,15	$x'' = x$	
18,17	$(x \cdot y)' = y' \cdot x'$	
20,19	$x' \cdot (x \cdot y) = y$	
22,21	$x \cdot (x' \cdot y) = y$	
23	$h(x,y) = x' \cdot (y' \cdot (x \cdot y))$	
24	$x \cdot h(y,z) = h(y,z) \cdot x$	
25	$x' \cdot (y' \cdot (x \cdot y)) = h(x,y)$	[flip 23]
26	$h(x,y) \cdot z = z \cdot h(x,y)$	[flip 24]
28	$x \cdot (y \cdot h(z,u)) = h(z,u) \cdot (x \cdot y)$	[11 → 24]
39	$x' \cdot (y' \cdot (x \cdot (y \cdot z))) = z \cdot h(x,y)$	[23 → 26 :12,12,12]
66,65	$h(h(x,y),z) = e$	[26 → 25 :20,8, flip]
81	$x \cdot h(x,y) = y' \cdot (x \cdot y)$	[25 → 21]
85	$x' \cdot (y \cdot x) = y \cdot h(y,x)$	[flip 81]
203	$x \cdot (y \cdot h(y,x)) = y \cdot x$	[85 → 21]
206	$h(x,y) = h(y', x \cdot y)$	[85 → 25 :16,18,12,20,22]
208	$h(x', y \cdot x) = h(y,x)$	[flip 206]
247	$x \cdot (h(y,x) \cdot y) = y \cdot x$	[24 → 203]
251	$h(x,y) \cdot (y \cdot x) = x \cdot y$	[28 → 203]
264	$x \cdot h(y,x) = x \cdot h(x', y)$	[85 → 39 :16,20]
383	$h(x' \cdot y', x \cdot y) = e$	[251 → 208 :18,66]
546	$h(x,y) = h(y,x')$	[264 → 21 :16,22]
559	$h(x \cdot y, y \cdot x) = e$	[383 → 546 :18,16,16, flip]
775	$x \cdot (y \cdot (y \cdot x)) = y \cdot (x \cdot (x \cdot y))$	[559 → 247 :4,12,12]
776	$\square$	[775,1]

Problem 41.

$$\{\text{CS},\ xyzyx = yxzxy\} \ =\!\!(gL)\!\!\Rightarrow\ \{xy = yx\}.$$

Status: Open Problem. My [RP's] intution tells me that this must be true.

Proof (found by Otter 3.0.3+ on gyro at 3.08 seconds).

1	$A \cdot B = B \cdot A \ \rightarrow\ \square$	
2	$x = x$	
3	$x \cdot y = z,\ x \cdot u = z \ \rightarrow\ y = u$	
6,5	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	
7	$x \cdot (y \cdot (z \cdot (y \cdot x))) = y \cdot (x \cdot (z \cdot (x \cdot y)))$	
14	$x \cdot (y \cdot (z \cdot (u \cdot (v \cdot y)))) = x \cdot (z \cdot (y \cdot (u \cdot (y \cdot v))))$	[7 → 5 :(gL) :6]
18	$x \cdot (y \cdot (z \cdot (y \cdot u))) = y \cdot (x \cdot (z \cdot (u \cdot y)))$	[14,3,2]
27	$x \cdot (y \cdot (z \cdot u)) = x \cdot (y \cdot (u \cdot z))$	[18,3,2 :(gL)]
48	$x \cdot (y \cdot z) = y \cdot (x \cdot z)$	[(gL) 18]
93	$x \cdot (y \cdot z) = x \cdot (z \cdot y)$	[27,3,48]
131	$x \cdot y = y \cdot x$	[93,3,48]

132	$\square$	[131,1]

Problem 42.

$$\{\mathrm{CS}\} \;=\!(gL)\!\Rightarrow\; \{xy = yx\}.$$

Status: Open problem. Conjecture: I [RP] have no idea.

Proof (found by Otter 3.0.3+ on gyro at 5.79 seconds).

2	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	
3	$B \cdot A = A \cdot B \ \rightarrow\ \square$	
4	$x \cdot y = u,\ x \cdot z = u \ \rightarrow\ y = z$	
5	$y \cdot x = u,\ z \cdot x = u \ \rightarrow\ y = z$	
6	$(((x \cdot y) \cdot z) \cdot u) \cdot v = (x \cdot (y \cdot z)) \cdot (u \cdot v)$	[2 $\rightarrow$ 2]
11	$(x \cdot (y \cdot z)) \cdot u = (x \cdot y) \cdot (z \cdot u)$	[2 $\rightarrow$ 2]
432	$(((x \cdot y) \cdot z) \cdot u) \cdot v = (x \cdot u) \cdot ((y \cdot z) \cdot v)$	[(gL) 6]
433	$((x \cdot y) \cdot z) \cdot u = x \cdot (u \cdot (y \cdot z))$	[432,5,11, flip]
1205	$x \cdot y = y \cdot x$	[433,4,432]
1206	$\square$	[1205,3]

Problem 43.

$$\{\mathrm{CS},\ xy^2 = y^2x\} \Rightarrow \{(xy)^4 = x^4y^4\}.$$

Status: True. I [RP] have a proof (unpublished).

In fact, we can derive a stronger conclusion, $xyxyzyxu = zxxxyyyu$, even without the cancellation laws.

Proof (found by Otter 3.0.3+ on gyro at 203.65 seconds).

1	$A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \neq A \cdot A \cdot A \cdot A \cdot B \cdot B \cdot B \cdot B$	
4,3	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
5	$x \cdot y \cdot y = y \cdot y \cdot x$	
8	$x \cdot y \cdot z \cdot y \cdot z = y \cdot z \cdot y \cdot z \cdot x$	[3 $\rightarrow$ 5 :4,4]
13	$x \cdot x \cdot y \cdot z = y \cdot x \cdot x \cdot z$	[5 $\rightarrow$ 3 :4,4,4]
14	$x \cdot y \cdot y \cdot z = y \cdot y \cdot x \cdot z$	[flip 13]
48	$x \cdot y \cdot x \cdot y \cdot z \cdot u = z \cdot x \cdot y \cdot x \cdot y \cdot u$	[8 $\rightarrow$ 3 :4,4,4,4,4,4,4]
50	$x \cdot y \cdot z \cdot y \cdot z \cdot u = y \cdot z \cdot y \cdot z \cdot x \cdot u$	[flip 48]
78	$x \cdot y \cdot z \cdot z \cdot y \cdot u = y \cdot y \cdot x \cdot z \cdot z \cdot u$	[14 $\rightarrow$ 14]
3988	$x \cdot y \cdot y \cdot y \cdot z \cdot z \cdot z \cdot u = y \cdot z \cdot y \cdot z \cdot x \cdot z \cdot y \cdot u$	[78 $\rightarrow$ 50]
4008	$x \cdot y \cdot x \cdot y \cdot z \cdot y \cdot x \cdot u = z \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot u$	[flip 3988]
4009	$\square$	[4008,1]

Problem 44.

$$\{\mathrm{CS},\ xy^2 = y^2x\}\ {=}(gL){\Rightarrow}\ \{xy = yx\}.$$

Status: Open.

This is true. See Prob. 42.

Problem 45.

$$\{\mathrm{CS},\ xy^3 = y^3x\} \Rightarrow \{(xy)^9 = x^9y^9\}.$$

Status: Open. Remark. The statement

$$\{\mathrm{GT},\ xy^3 = y^3x\} \Rightarrow \{(xy)^9 = x^9y^9\}$$

is true. It was first proved by Narain Gupta in 1969.

In fact, we prove this for ordinary semigroups. (This is a difficult problem for Otter, and we use a very specialized search strategy; it might be worth while to include a description of how the strategy was developed.)

Proof (found by Otter 3.0.3+ on gyro at 354.80 seconds).

2	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
3	$x \cdot y \cdot y \cdot y = y \cdot y \cdot y \cdot x$	
4	$p(A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B)$	
5	$-p(A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot B \cdot B \cdot B \cdot B \cdot B \cdot B \cdot B \cdot B \cdot B)$	
6	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
8	$x \cdot y \cdot z \cdot y \cdot z \cdot y \cdot z = y \cdot z \cdot y \cdot z \cdot y \cdot z \cdot x$	[2 $\rightarrow$ 3 :6,6,6,6,6,6,6]
9	$x \cdot y \cdot z \cdot z \cdot z = z \cdot z \cdot z \cdot x \cdot y$	[2 $\rightarrow$ 3]
12	$x \cdot x \cdot x \cdot y \cdot z = y \cdot x \cdot x \cdot x \cdot z$	[3 $\rightarrow$ 2 :6,6,6,6,6]
15	$x \cdot x \cdot x \cdot y = y \cdot x \cdot x \cdot x$	[3 $\rightarrow$ 2 :6]
16	$x \cdot y \cdot y \cdot y \cdot z = y \cdot y \cdot y \cdot x \cdot z$	[3 $\rightarrow$ 2 :6,6,6,6,6]
20	$x \cdot y \cdot z \cdot u \cdot y \cdot z \cdot u \cdot y \cdot z \cdot u = y \cdot z \cdot u \cdot y \cdot z \cdot u \cdot y \cdot z \cdot u \cdot x$	[2 $\rightarrow$ 8 :6,6,6,6,6,6]
22	$x \cdot y \cdot z \cdot u \cdot z \cdot u \cdot z \cdot u = z \cdot u \cdot z \cdot u \cdot z \cdot u \cdot x \cdot y$	[2 $\rightarrow$ 8]
40	$p(A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A)$	[8 $\rightarrow$ 4]
42	$x \cdot y \cdot x \cdot y \cdot x \cdot y \cdot z = z \cdot x \cdot y \cdot x \cdot y \cdot x \cdot y$	[8 $\rightarrow$ 2 :6]
65	$p(A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A)$	[8 $\rightarrow$ 40 :6,6,6,6,6,6]
69	$p(A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot A)$	[8 $\rightarrow$ 65]
92	$p(B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot A \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B)$	[8 $\rightarrow$ 69 :6,6,6,6,6,6,6,6,6,6,6]
98	$x \cdot x \cdot x \cdot y \cdot z \cdot u \cdot u \cdot u = u \cdot x \cdot x \cdot x \cdot u \cdot u \cdot y \cdot z$	[9 $\rightarrow$ 12]
144	$p(B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot A \cdot A \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B)$	[12 $\rightarrow$ 92]
174	$p(B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot A \cdot A \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B)$	[12 $\rightarrow$ 144]
176	$p(B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot A \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B)$	[12 $\rightarrow$ 174]
180	$p(A \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A)$	[8 $\rightarrow$ 176 :6,6,6,6,6,6,6,6,6,6,6]
272	$p(A \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot A \cdot B)$	[22 $\rightarrow$ 180]
296	$p(A \cdot A \cdot B \cdot A \cdot A \cdot B \cdot A \cdot A \cdot B \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B)$	[20 $\rightarrow$ 272 :6,6,6,6]
351	$p(A \cdot A \cdot B \cdot A \cdot A \cdot B \cdot A \cdot A \cdot B \cdot B \cdot B \cdot A \cdot B \cdot B \cdot A \cdot B \cdot B \cdot A)$	[20 $\rightarrow$ 296]
390	$p(A \cdot A \cdot B \cdot A \cdot A \cdot B \cdot A \cdot A \cdot A \cdot B \cdot B \cdot B \cdot B \cdot B \cdot A \cdot B \cdot B \cdot A)$	[16 $\rightarrow$ 351]

449	$p(A\cdot A\cdot B\cdot A\cdot A\cdot B\cdot B\cdot B\cdot B\cdot B\cdot B\cdot A\cdot B\cdot B\cdot A\cdot A\cdot A\cdot A)$	[15 → 390 :6,6,6,6,6,6,6,6]
687	$p(A\cdot A\cdot B\cdot A\cdot A\cdot A\cdot B\cdot B\cdot A\cdot A\cdot A\cdot A\cdot B\cdot B\cdot B\cdot B\cdot B\cdot B)$	[42 → 449 :6,6,6,6,6,6]
2438	$p(A\cdot A\cdot A\cdot A\cdot A\cdot A\cdot A\cdot A\cdot A\cdot B\cdot B\cdot B\cdot B\cdot B\cdot B\cdot B\cdot B\cdot B)$	[98 → 687 :6,6,6,6,6,6,6,6]
2439	$\square$	[2438,5]

Problem 46.

$$\{\text{CS},\ xy^n = y^n x\} \Rightarrow \{(xy)^n = (yx)^n\}.$$

Status: True. This theorem is outside the scope of Otter, because it requires induction. However, Otter easily proves the statment for $n = 2, 3, 4, 5$, and it is easy to construct a hand proof of the general theorem by looking at the Otter proofs. For example, here is an Otter proof for $n = 4$.

Proof (found by Otter 3.0.3+ on gyro at 1.32 seconds).

1	$x = x$	
2	$x\cdot y = z,\ x\cdot u = z\ \rightarrow\ y = u$	
5,4	$(x\cdot y)\cdot z = x\cdot y\cdot z$	
6	$x\cdot y\cdot y\cdot y\cdot y = y\cdot y\cdot y\cdot y\cdot x$	
7	$B\cdot A\cdot B\cdot A\cdot B\cdot A\cdot B\cdot A = A\cdot B\cdot A\cdot B\cdot A\cdot B\cdot A\cdot B\ \rightarrow\ \square$	
9	$x\cdot y\cdot z\cdot y\cdot z\cdot y\cdot z\cdot y\cdot z = y\cdot z\cdot y\cdot z\cdot y\cdot z\cdot y\cdot z\cdot x$	[4 → 6 :5,5,5,5,5,5]
145	$x\cdot y\cdot x\cdot y\cdot x\cdot y\cdot x\cdot y = y\cdot x\cdot y\cdot x\cdot y\cdot x\cdot y\cdot x$	[9,2,1]
146	$\square$	[145,7]

Proof for all n.

$$\begin{array}{rll} xy^n & = y^n x & \text{hypothesis} \\ x(xw)^n & = (xw)^n x & \text{set } y = xw \\ & = x(wx)^n & \text{change power notation} \\ (xw)^n & = (wx)^n & \text{cancel } x \end{array}$$

E-mail from RP to WWM, 11 May, 1993:

> The proof of Problem 42.
>
> $$\{\text{CS}\}\ =\!\!(gL)\!\!\Rightarrow\ \{xy = yx\}$$
>
> is especially excellent. This means that not only a group law, but even a cancellative semigroup law on a projective curve must be commutatiave! This is an absolutely new result.
>
> Now the commutative law has only two variables and $=\!\!(gL)\!\!\Rightarrow$ is an inference rule. Believing in the cosmic orderliness of mathematics, one can ask the question whether the two cancellation laws along with the set of all two-variable consequences of associativity will already (gL)-imply $xy = yx$. Let us formalize:

Problem 48.

$$\left\{\begin{array}{l} x(yx)=(xy)x\\ x(xy)=(xx)y\\ \text{cancellation}\end{array}\right\} =\!(gL)\!\Rightarrow \{xy=yx\}$$

Status: OPEN.

Proof (found by Otter 3.0.3+ on gyro at 1.24 seconds).

1	$B \cdot A = A \cdot B \ \rightarrow\ \square$	
2	$x = x$	
3	$x \cdot y = z,\ x \cdot u = z \ \rightarrow\ y = u$	
4	$x \cdot y = z,\ u \cdot y = z \ \rightarrow\ x = u$	
5	$(x \cdot x) \cdot y = x \cdot (x \cdot y)$	
6	$(x \cdot y) \cdot x = x \cdot (y \cdot x)$	
12	$(x \cdot x) \cdot (y \cdot z) = x \cdot ((x \cdot y) \cdot z)$	[6 → 5 :(gL)]
13	$x \cdot ((y \cdot z) \cdot u) = x \cdot (y \cdot (z \cdot u))$	[5 → 12 :(gL), flip]
14	$(x \cdot x) \cdot (y \cdot z) = x \cdot (y \cdot (x \cdot z))$	[5 → 12 :(gL)]
19	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	[13,3,2, flip]
22	$(x \cdot y) \cdot z = y \cdot (x \cdot z)$	[14,3,12, flip]
38	$x \cdot y = y \cdot x$	[22,4,19]
39	$\square$	[38,1]

Problem 60A. The type is (2).

$$\left\{\begin{array}{l} (xy)z = x(yz)\\ xyzu = yzux\end{array}\right\} \Rightarrow \left\{\begin{array}{l} (xy)^3 = x^3y^3\\ x^3y^3 = y^3x^3\end{array}\right\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.90 seconds).

1	$A \cdot B \cdot A \cdot B \cdot A \cdot B \neq A \cdot A \cdot A \cdot B \cdot B \cdot B$	
4,3	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
5	$x \cdot y \cdot z \cdot u = y \cdot z \cdot u \cdot x$	
7	$x \cdot y \cdot z \cdot u \cdot \vee \cdot w = y \cdot w \cdot z \cdot u \cdot \vee \cdot x$	[5 → 5 :4,4]
10	$x \cdot y \cdot z \cdot u \cdot v = z \cdot u \cdot \vee \cdot x \cdot y$	[3 → 5]
11	$x \cdot y \cdot z \cdot u \cdot \vee \cdot w = w \cdot x \cdot z \cdot u \cdot \vee \cdot y$	[flip 7]
13	$x \cdot y \cdot z \cdot u \cdot v = u \cdot \vee \cdot x \cdot y \cdot z$	[flip 10]
120	$x \cdot y \cdot z \cdot u \cdot \vee \cdot w = z \cdot x \cdot \vee \cdot w \cdot y \cdot u$	[13 → 11]
121	$\square$	[120,1]

Proof (found by Otter 3.0.3+ on gyro at 0.14 seconds).

1	$A \cdot A \cdot A \cdot B \cdot B \cdot B \neq B \cdot B \cdot B \cdot A \cdot A \cdot A$	
4,3	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
5	$x \cdot y \cdot z \cdot u = y \cdot z \cdot u \cdot x$	

6	$x \cdot y \cdot z \cdot u = u \cdot x \cdot y \cdot z$	[flip 5]
20	$x \cdot y \cdot z \cdot u = z \cdot u \cdot x \cdot y$	[5 $\longrightarrow$ 6]
30	$x \cdot y \cdot z \cdot u \cdot \vee \cdot w = u \cdot \vee \cdot w \cdot z \cdot x \cdot y$	[6 $\longrightarrow$ 20 :4,4]
31	$\square$	[30,1]

Problem 60B. The type is (2).

$$\left\{ \begin{array}{l} (xy)z = x(yz) \\ xyzuv = yzuvx \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} (xy)^4 = x^4y^4 \\ x^4y^4 = y^4x^4 \end{array} \right\}.$$

Proof (found by Otter 3.0.3+ on gyro at 28.90 seconds).

1	$A \cdot B \cdot A \cdot B \cdot A \cdot B \cdot A \cdot B \neq A \cdot A \cdot A \cdot A \cdot B \cdot B \cdot B \cdot B$	
4,3	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
5	$x \cdot y \cdot z \cdot u \cdot v = y \cdot z \cdot u \cdot \vee \cdot x$	
6	$x \cdot y \cdot z \cdot u \cdot v = v \cdot x \cdot y \cdot z \cdot u$	[flip 5]
7	$x \cdot y \cdot z \cdot u \cdot \vee \cdot w \cdot v_6 \cdot v_7 = y \cdot z \cdot v_7 \cdot u \cdot \vee \cdot w \cdot v_6 \cdot x$	[5 $\longrightarrow$ 5 :4,4,4]
12	$x \cdot y \cdot z \cdot u \cdot \vee \cdot w \cdot v_6 \cdot v_7 = v_7 \cdot x \cdot y \cdot u \cdot \vee \cdot w \cdot v_6 \cdot z$	[flip 7]
32	$x \cdot y \cdot z \cdot u \cdot \vee \cdot w = z \cdot u \cdot \vee \cdot w \cdot y \cdot x$	[6 $\longrightarrow$ 5 :4]
35	$x \cdot y \cdot z \cdot u \cdot \vee \cdot w = w \cdot \vee \cdot x \cdot y \cdot z \cdot u$	[flip 32]
1170	$x \cdot y \cdot z \cdot u \cdot \vee \cdot w \cdot v_6 \cdot v_7 = z \cdot x \cdot \vee \cdot v_6 \cdot v_7 \cdot u \cdot y \cdot w$	[35 $\longrightarrow$ 12 :4]
1171	$\square$	[1170,1]

Proof (found by Otter 3.0.3+ on gyro at 0.18 seconds).

1	$A \cdot A \cdot A \cdot A \cdot B \cdot B \cdot B \cdot B \neq B \cdot B \cdot B \cdot B \cdot A \cdot A \cdot A \cdot A$	
4,3	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
5	$x \cdot y \cdot z \cdot u \cdot v = y \cdot z \cdot u \cdot \vee \cdot x$	
6	$x \cdot y \cdot z \cdot u \cdot v = v \cdot x \cdot y \cdot z \cdot u$	[flip 5]
24	$x \cdot y \cdot z \cdot u \cdot v = u \cdot \vee \cdot x \cdot y \cdot z$	[5 $\longrightarrow$ 6]
39	$x \cdot y \cdot z \cdot u \cdot \vee \cdot w \cdot v_6 \cdot v_7 = v \cdot w \cdot v_6 \cdot v_7 \cdot u \cdot x \cdot y \cdot z$	[6 $\longrightarrow$ 24 :4,4,4]
40	$\square$	[39,1]

Problem 60N. The type is (2).

$$\left\{ \begin{array}{l} (xy)z = x(yz) \\ x_1x_2\cdots x_n = x_2\cdots x_nx_1 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \forall m \geq \lceil n/2 \rceil, x^my^m = y^mx^m \\ \forall m \geq n-1, (xy)^m = x^my^m \end{array} \right\}.$$

This is true. The proof is left as an exercise for the reader.

Problem 60C. The type is (2).

$$\left\{\begin{array}{l}\text{cancellation}\\(xy)z = x(yz)\\xyzu = yzux\end{array}\right\} \Rightarrow \{xy = yx\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.05 seconds).

1	$A \cdot B = B \cdot A \;\rightarrow\; \square$	
4	$x \cdot y = z,\; u \cdot y = z \;\rightarrow\; x = u$	
5	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
7	$x \cdot y \cdot z \cdot u = y \cdot z \cdot u \cdot x$	
9	$x \cdot y = y \cdot x$	[7,4,5]
10	$\square$	[9,1]

Problem 60M. The type is (2).

$$\left\{\begin{array}{l}\text{cancellation}\\(xy)z = x(yz)\\x_1x_2\cdots x_n = x_2\cdots x_nx_1\end{array}\right\} \Rightarrow \{xy = yx\}.$$

This is true. The proof is left as an exercise for the reader.

Problem 61. The type is (2).

$$\left\{\begin{array}{l}\text{cancellation}\\(xy)z = x(yz)\\xyz = zyx\end{array}\right\} \Rightarrow \{xy = yx\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.05 seconds).

1	$A \cdot B = B \cdot A \;\rightarrow\; \square$	
4	$x \cdot y = z,\; u \cdot y = z \;\rightarrow\; x = u$	
5	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
7	$x \cdot y \cdot z = z \cdot y \cdot x$	
8	$x \cdot y = y \cdot x$	[7,4,5]
9	$\square$	[8,1]

Problem 62A. Let $a, b, c, d, a_0, b_0, c_0, d_0$ be nullary constants.

$$\left\{\begin{array}{l}\text{GT}\\aa_0 = bb_0\\ca_0 = db_0\\ac_0 = bd_0\end{array}\right\} \Rightarrow \{cc_0 = dd_0\}$$

(A classical result of Malcev (1939) says that this doen not hold for CS.)

Proof (found by Otter 3.0.3+ on gyro at 0.33 seconds).

1	$d \cdot d_0 \neq c \cdot c_0$	
4,3	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	
15	$x' \cdot (x \cdot y) = y$	
18,17	$x \cdot (x' \cdot y) = y$	
23	$a \cdot a_0 = b \cdot b_0$	
24	$b \cdot b_0 = a \cdot a_0$	[flip 23]
26	$c \cdot a_0 = d \cdot b_0$	
27	$d \cdot b_0 = c \cdot a_0$	[flip 26]
29	$a \cdot c_0 = b \cdot d_0$	
30	$b \cdot d_0 = a \cdot c_0$	[flip 29]
32	$b \cdot (b_0 \cdot x) = a \cdot (a_0 \cdot x)$	[24 $\rightarrow$ 3 :4, flip]
34	$d \cdot (b_0 \cdot x) = c \cdot (a_0 \cdot x)$	[27 $\rightarrow$ 3 :4, flip]
44	$b' \cdot (a \cdot c_0) = d_0$	[30 $\rightarrow$ 15]
62	$b' \cdot (a \cdot (a_0 \cdot x)) = b_0 \cdot x$	[32 $\rightarrow$ 15]
97,96	$b' \cdot (a \cdot x) = b_0 \cdot ({a_0}' \cdot x)$	[17 $\rightarrow$ 62]
106	$b_0 \cdot ({a_0}' \cdot c_0) = d_0$	[44 :97]
114	$d \cdot d_0 = c \cdot c_0$	[106 $\rightarrow$ 34 :18]
116	$\square$	[114,1]

Problem 62G. Let $a, b, c, d, a_0, b_0, c_0, d_0$ be nullary constants.

$$\left\{ \begin{array}{l} \text{CS} \\ aa_0 = bb_0 \\ ca_0 = db_0 \\ ac_0 = bd_0 \end{array} \right\} \overset{(gL)}{=\!\!=\!\!\Rightarrow} \{cc_0 = dd_0\}.$$

(This is often called Ore's left principle. This is useful for embedding an integral domain into a field. The idea behind this problem is that (gL) is more powerful that CS in the context of GT.)

Proof (found by Otter 3.0.3+ on gyro at 56.55 seconds).

1	$d \cdot d_0 = c \cdot c_0 \;\rightarrow\; \square$	
3	$x \cdot y = z,\; x \cdot u = z \;\rightarrow\; y = u$	
4	$x \cdot y = z,\; u \cdot y = z \;\rightarrow\; x = u$	
5	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	
6	$b \cdot b_0 = a \cdot a_0$	
7	$d \cdot b_0 = c \cdot a_0$	
8	$b \cdot d_0 = a \cdot c_0$	
10	$((x \cdot (y \cdot z)) \cdot u) \cdot v = ((x \cdot y) \cdot z) \cdot (u \cdot v)$	[5 $\rightarrow$ 5]
18	$(x \cdot y) \cdot (z \cdot u) = x \cdot ((y \cdot z) \cdot u)$	[5 $\rightarrow$ 5]
28	$(x \cdot a) \cdot a_0 = x \cdot (b \cdot b_0)$	[6 $\rightarrow$ 5]
38	$(x \cdot c) \cdot a_0 = x \cdot (d \cdot b_0)$	[7 $\rightarrow$ 5]
49	$(b \cdot d_0) \cdot x = a \cdot (c_0 \cdot x)$	[8 $\rightarrow$ 5]
267	$((x \cdot (y \cdot z)) \cdot u) \cdot v = ((x \cdot y) \cdot u) \cdot (z \cdot v)$	[(gL) 10]
341	$(x \cdot y) \cdot (z \cdot u) = (x \cdot z) \cdot (y \cdot u)$	[10 $\rightarrow$ 267 :(gL)]

2729	$(x \cdot y) \cdot z = x \cdot (z \cdot y)$	[18,4,341, flip]
2897	$x \cdot y = y \cdot x$	[2729,3,18]
3403	$c_0 \cdot a = b \cdot d_0$	[2897,4,49, flip]
3889	$((b \cdot d_0) \cdot x) \cdot a_0 = (c_0 \cdot x) \cdot (b \cdot b_0)$	[3403 $\rightarrow$ 28 :(gL)]
4246	$(c_0 \cdot c) \cdot (b \cdot x) = (b \cdot d_0) \cdot (d \cdot x)$	[38 $\rightarrow$ 3889 :(gL), flip]
5599	$d \cdot d_0 = c_0 \cdot c$	[4246,4,2897]
7792	$d \cdot d_0 = c \cdot c_0$	[2897 $\rightarrow$ 5599]
7793	$\square$	[7792,1]

Problem 60Y. The type is (2).

$$\left\{\begin{array}{l} \text{cancellation} \\ (xy)z = x(yz) \\ x^4 y = y z^4 \end{array}\right\} \Rightarrow \{x^4 y = y\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.41 seconds).

2	$B \cdot A \cdot A \cdot A \cdot A = B \ \rightarrow \ \square$	
3	$x = x$	
4	$x \cdot y = z, \ x \cdot u = z \ \rightarrow \ y = u$	
7,6	$(x \cdot y) \cdot z = x \cdot y \cdot z$	
8	$x \cdot x \cdot x \cdot x \cdot y = y \cdot z \cdot z \cdot z \cdot z$	
12	$x \cdot x \cdot x \cdot x = y \cdot y \cdot y \cdot y$	[8,4,3]
21	$x \cdot x \cdot x \cdot x \cdot x \cdot x = x \cdot x$	[12,4,6 :7,7, flip]
24	$x \cdot x \cdot x \cdot x \cdot x = x$	[21,4,3, flip]
30	$x \cdot y \cdot y \cdot y \cdot y = x$	[12 $\rightarrow$ 24]
32	$\square$	[30,2]

Problem 60Z. The type is (2).

$$\left\{\begin{array}{l} \text{cancellation} \\ (xy)z = x(yz) \\ x^n y = y z^n \end{array}\right\} \Rightarrow \{x^n y = y\}.$$

This is true. The proof is left as an exercise for the reader.

Chapter 7

Groups

7.1 Self-Dual Bases for Group Theory

Consider a language of type (2,1,0) or (2,1). Let the *dual* of a term t, written $d(t)$, be its mirror image, i.e., be obtained by reversing the arguments of all occurrences of the binary operation. A set S of identities is *self-dual* if $t_1 = t_2 \in S \Rightarrow d(t_1) = d(t_2)$ (modulo renaming of variables) $\in S$.

Problems 64A and 64B arose from a challenge to find an independent self-dual axiomatization, of size two, for group theory. (It is known that a self-dual single axiom for group theory does not exist.) Problems 65A and 65B are the analogous problems for Abelian group theory. The self-dual axiomatizations of these problems were found by techniques similar to those presented by McCune in [4]. See [10] for a solution to the general problem of independent self-dual n-bases, $\forall n > 1$, for groups and Abelian groups.

Problem 64A. The type is (2,1). Show that the self-dual pair

$$((x \cdot y) \cdot z) \cdot (y \cdot z)' = x$$
$$(z \cdot y)' \cdot (z \cdot (y \cdot x)) = x$$

is a basis for group theory.

Proof (found by Otter 3.0.3+ on gyro at 0.38 seconds).

1	$x = x$	
2	$A \cdot (B \cdot B') = A,\ B \cdot B' = A \cdot A',\ (A \cdot B) \cdot C = A \cdot (B \cdot C) \ \rightarrow\ \Box$	
3	$((x \cdot y) \cdot z) \cdot (y \cdot z)' = x$	
6,5	$(x \cdot y)' \cdot (x \cdot (y \cdot z)) = z$	
7	$(x \cdot y) \cdot ((z \cdot u)' \cdot y)' = (x \cdot z) \cdot u$	[3 → 3]
8	$x \cdot (y \cdot (z \cdot y)')' = x \cdot z$	[3 → 3]
12	$(x \cdot y) \cdot z = (x \cdot u) \cdot ((y \cdot z)' \cdot u)'$	[flip 7]
31	$(x \cdot (y \cdot x)')' = y$	[8 → 5 :6, flip]
37	$((x \cdot y)' \cdot x)' = y$	[31 → 31]

40,39	$x \cdot (y \cdot ((x \cdot y)' \cdot z)) = z$	[31 → 5]
42,41	$(x' \cdot (y \cdot z)')' = y \cdot (z \cdot x)$	[5 → 37]
57,56	$(x \cdot y) \cdot (z \cdot ((y \cdot z)' \cdot u)) = x \cdot u$	[7 → 31 :42]
78	$((x \cdot y) \cdot z) \cdot ((y \cdot z)' \cdot (x' \cdot u)) = u$	[3 → 39]
83,82	$(x \cdot y) \cdot (z' \cdot y)' = x \cdot z$	[39 → 7 :57]
85	$(x \cdot y) \cdot y' = x$	[39 → 3 :57]
93,92	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	[12 :83]
103,102	$x \cdot (y \cdot y') = x$	[85 :93]
106	$x \cdot (x' \cdot y) = y$	[78 :93,93,93,40]
131	$B \cdot B' = A \cdot A' \;\rightarrow\; \Box$	[2 :103,93 :1,1]
158	$x \cdot x' = y \cdot y'$	[102 → 106]
159	$\Box$	[158,131]

Problem 64B. The type is (2,1). Show that the self-dual axiomatization of group theory in Prob. 64A is independent.

It is known that a single axiom for groups in terms of product and inverse must have at least 3 occurrences of inverse [3]. We can also give a simple model in which one is true and the other false.

The clauses

$$((x \cdot y) \cdot z) \cdot (y \cdot z)' = x$$
$$(C \cdot B)' \cdot (C \cdot (B \cdot A)) \neq A$$

have the following model (found by MACE 1.0.0 on gyro at 0.02 seconds).

```
*:
      | 0 1
    --+----
    0 | 0 0
    1 | 1 1
@:
        0 1
    -------
        0 0
C: 0
B: 0
A: 1
```

Problem 65A. The type is (2,1). Show that the self-dual pair

$$(z \cdot (x \cdot y)) \cdot (y \cdot z)' = x$$
$$(z \cdot y)' \cdot ((y \cdot x) \cdot z) = x.$$

is a basis for Abelian group theory.

Proof (found by Otter 3.0.3+ on gyro at 1.50 seconds).

1	$x = x$	
2	$A \cdot (B \cdot B') = A,\ B \cdot B' = A \cdot A',\ (A \cdot B) \cdot C = A \cdot (B \cdot C),\ B \cdot A = A \cdot B \ \rightarrow\ \square$	
3	$(x \cdot (y \cdot z)) \cdot (z \cdot x)' = y$	
5	$(x \cdot y)' \cdot ((y \cdot z) \cdot x) = z$	
7	$(x \cdot y) \cdot ((z \cdot u)' \cdot x)' = u \cdot (y \cdot z)$	[3 $\rightarrow$ 3]
17	$((x \cdot y)' \cdot y)' \cdot z = z \cdot x$	[3 $\rightarrow$ 5]
23	$x \cdot (y \cdot (y \cdot z)')' = z \cdot x$	[5 $\rightarrow$ 3]
26	$x \cdot y = y \cdot (z \cdot (z \cdot x)')'$	[flip 23]
28	$((x \cdot y) \cdot (z \cdot x)') \cdot z = y$	[5 $\rightarrow$ 17, flip]
42,41	$((x \cdot (y \cdot z))' \cdot y)' = z \cdot x$	[28 $\rightarrow$ 28, flip]
59	$x \cdot ((y \cdot (z \cdot x)') \cdot z) = y$	[28 $\rightarrow$ 7 :42, flip]
71,70	$(x \cdot (x \cdot y)')' = y$	[5 $\rightarrow$ 59]
72	$x \cdot y = y \cdot x$	[26 :71]
80	$(((x \cdot (y \cdot z)') \cdot y) \cdot (u \cdot z)) \cdot x' = u$	[59 $\rightarrow$ 3]
113	$(x \cdot y)' \cdot ((x \cdot z) \cdot y) = z$	[72 $\rightarrow$ 5]
130	$(x \cdot (y \cdot x)')' = y$	[72 $\rightarrow$ 70]
136,135	$((x \cdot y)' \cdot z')' = (y \cdot z) \cdot x$	[5 $\rightarrow$ 70]
158,157	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	[3 $\rightarrow$ 130 :136]
191,190	$(x \cdot y)' \cdot (x \cdot (z \cdot y)) = z$	[113 :158]
209	$x \cdot (y \cdot x') = y$	[80 :158,158,158,191,158]
251	$A \cdot (B \cdot B') = A,\ B \cdot B' = A \cdot A' \ \rightarrow\ \square$	[2 :158 :1,72]
270	$x \cdot (x' \cdot y) = y$	[72 $\rightarrow$ 209]
278,277	$(x \cdot y')' = y \cdot x'$	[209 $\rightarrow$ 70]
291,290	$x \cdot (y \cdot y') = x$	[130 :278,158]
303	$B \cdot B' = A \cdot A' \ \rightarrow\ \square$	[251 :291 :1]
323	$x \cdot x' = y \cdot y'$	[290 $\rightarrow$ 270]
324	$\square$	[323,303]

Problem 65B. The type is (2,1). Show that the self-dual axiomatization of Abelian group theory in Prob. 65A is independent.

We give a simple model in which one is true and the other false.

The clauses

$$(z \cdot (x \cdot y)) \cdot (y \cdot z)' = x$$
$$(C \cdot B)' \cdot ((B \cdot A) \cdot C) \neq A$$

have the following model (found by MACE 1.0.0 on gyro at 0.33 seconds).

```
*:
      | 0 1 2
    --+------
    0 | 0 1 2
    1 | 2 0 1
    2 | 1 2 0
@:
```

```
         0 1 2
     ---------
         0 2 1
  C: 0
  B: 0
  A: 1
```

Chapter 8

TC and RC

Reidemeister Closure Condition (RC)

$$\left\{ \begin{array}{l} z_1 \cdot x = z_2 \cdot y \\ z_3 \cdot x = z_4 \cdot y \\ z_1 \cdot u = z_2 \cdot v \end{array} \right\} \Rightarrow \{z_3 \cdot u = z_4 \cdot v\}.$$

Thomsen Closure Condition (TC)

$$\left\{ \begin{array}{l} z_1 \cdot x = z_2 \cdot y \\ z_1 \cdot z = z_3 \cdot y \end{array} \right\} \Rightarrow \{z_2 \cdot z = z_3 \cdot x\}.$$

Problem 80. TC $\Rightarrow$ RC.

Counterexample. The clauses

$$\begin{array}{l} z_1 \cdot x \neq u | z_2 \cdot y \neq u | z_1 \cdot z \neq w | z_3 \cdot y \neq w | z_2 \cdot z = z_3 \cdot x \\ C_4 \cdot A = C_3 \cdot B \\ C_2 \cdot A = C_1 \cdot B \\ C_4 \cdot F = C_3 \cdot E \\ C_2 \cdot F \neq C_1 \cdot E \end{array}$$

have the following model (found by MACE 1.0.0 on gyro at 3.87 seconds).

```
*:
      | 0 1 2
    --+------
    0 | 0 0 0
    1 | 0 0 0
    2 | 1 1 2
C4: 0
A: 0
C3: 0
B: 0
```

```
C2: 2
C1: 2
F: 0
E: 2
```

Problem 81. RC $\Rightarrow$ TC.
MACE 1.0.0 could not be used to search for a counterexample, because the hypothesis RC is too complex, and Otter did not find a proof. [[Note to RP: What is the status of this?]]

Problem 82.

$$\left\{\begin{array}{l}\text{TC}\\ \text{cancellation}\end{array}\right\} \Rightarrow \text{RC}.$$

We did not find a proof with Otter, and with MACE we found no counterexamlpes of orders 2, 3, or 4 (5 is too big for MACE). [[Note to WWM: Otter exhausts space; not sure if complete.]]

Problem 83.

$$\left\{\begin{array}{l}\text{TC}\\ \text{quasigroup}\end{array}\right\} \Rightarrow \text{RC}.$$

Proof (found by Otter 3.0.3+ on gyro at 22.22 seconds).

1	$x = x$	
2	$x \cdot y = z,\ u \cdot v = z,\ x \cdot w = v_6,\ v_7 \cdot v = v_6 \ \rightarrow\ u \cdot w = v_7 \cdot y$	
3	$x \cdot y = z,\ x \cdot u = z \ \rightarrow\ y = u$	
4	$x \cdot y = z,\ u \cdot y = z \ \rightarrow\ x = u$	
6,5	$g(x) \cdot (x \cdot y) = y$	
7	$(x \cdot y) \cdot h(y) = x$	
10,9	$C_4 \cdot A = C_3 \cdot B$	
11	$C_2 \cdot A = C_1 \cdot B$	
14,13	$C_4 \cdot F = C_3 \cdot E$	
15	$C_2 \cdot F = C_1 \cdot E \ \rightarrow\ \Box$	
17	$((C_4 \cdot x) \cdot y) \cdot A = ((C_3 \cdot B) \cdot y) \cdot x$	[9,2,1,7,7]
42	$g(C_4) \cdot (C_3 \cdot E) = F$	[13 $\rightarrow$ 5]
133	$g(g(C_4)) \cdot F = C_3 \cdot E$	[42 $\rightarrow$ 5]
164	$g(g(C_4)) = C_4$	[133,4,13, flip]
175	$C_3 \cdot (g(C_4) \cdot (x \cdot E)) = x \cdot F$	[133,2,1,5,1]
182	$C_4 \cdot (g(C_4) \cdot x) = x$	[164 $\rightarrow$ 5]
210	$x \cdot h(g(C_4) \cdot x) = C_4$	[182 $\rightarrow$ 7]
247,246	$h(g(C_4) \cdot x) = g(x) \cdot C_4$	[210 $\rightarrow$ 5, flip]
258	$x \cdot (g(x) \cdot C_4) = C_4$	[210 :247]
261,260	$g(g(x)) = x$	[258,4,5]
263,262	$h(x) = g(C_4 \cdot x) \cdot C_4$	[258,3,7]

274	$(x \cdot y) \cdot (g(C_4 \cdot y) \cdot C_4) = x$	[7 :263]
281,280	$x \cdot (g(x) \cdot y) = y$	[260 $\rightarrow$ 5]
319	$x \cdot B = C_2 \cdot (g(C_1) \cdot (x \cdot A))$	[280,2,1,1,11]
405,404	$((C_3 \cdot E) \cdot x) \cdot B = ((C_3 \cdot B) \cdot x) \cdot E$	[175,2,280,9,17 :281,14,6, flip]
407	$(x \cdot (g(C_4 \cdot y) \cdot C_4)) \cdot y = x$	[274,4,1, flip]
435,434	$x \cdot (g(C_4 \cdot (y \cdot x)) \cdot C_4) = g(y)$	[5 $\rightarrow$ 274]
530	$C_2 \cdot F = C_1 \cdot E$	[407,2,319,1,13 :10,405,435,261]
532	$\square$	[530,15]

Problem 84.

$$\left\{ \begin{array}{l} \mathrm{TC} \\ e \cdot x = x \\ x \cdot e = x \end{array} \right\} \Rightarrow \mathrm{RC}.$$

Proof (found by Otter 3.0.3+ on gyro at 2.19 seconds).

1	$x = x$	
2	$x \cdot y = z,\ u \cdot v = z,\ x \cdot w = v_6,\ v_7 \cdot v = v_6 \ \rightarrow\ u \cdot w = v_7 \cdot y$	
3	$e \cdot x = x$	
5	$x \cdot e = x$	
7	$C_4 \cdot A = C_3 \cdot B$	
9	$C_2 \cdot A = C_1 \cdot B$	
11	$C_4 \cdot F = C_3 \cdot E$	
13	$C_2 \cdot F = C_1 \cdot E \ \rightarrow\ \square$	
15	$x \cdot y = y \cdot x$	[2,3,5,3,5]
40	$C_4 \cdot C_1 = C_2 \cdot C_3$	[2,15,9,15,7, flip]
77	$E \cdot C_1 = C_2 \cdot F$	[2,11,15,40,1]
207	$C_2 \cdot F = C_1 \cdot E$	[15 $\rightarrow$ 77, flip]
209	$\square$	[207,13]

Problem 85.

$$\left\{ \begin{array}{l} \mathrm{TC} \\ x \cdot e = x \end{array} \right\} \Rightarrow \mathrm{RC}.$$

Counterexample. The clauses

$x = x$
$z_1 \cdot x \neq u | z_2 \cdot y \neq u | z_1 \cdot z \neq w | z_3 \cdot y \neq w | z_2 \cdot z = z_3 \cdot x$
$x \cdot e = x$
$C_4 \cdot A = C_3 \cdot B$
$C_2 \cdot A = C_1 \cdot B$
$C_4 \cdot F = C_3 \cdot E$
$C_2 \cdot F \neq C_1 \cdot E$

have the following model (found by MACE 1.0.0 on gyro at 1.47 seconds).

```
*:
      | 0 1 2
    --+------
    0 | 0 0 0
    1 | 1 1 2
    2 | 2 2 1
e: 0
C4: 0
A: 0
C3: 0
B: 0
C2: 1
C1: 1
F: 0
E: 2
```

Problem 86.

$$\left\{ \begin{array}{l} \mathrm{TC} \\ (x \cdot e) \cdot e = x \\ e \cdot (e \cdot x) = x \end{array} \right\} \Rightarrow \mathrm{RC}.$$

Proof (found by Otter 3.0.3+ on gyro at 51.56 seconds).

1	$x = x$	
2	$x \cdot y = z,\ u \cdot v = z,\ x \cdot w = v_6,\ v_7 \cdot v = v_6 \ \rightarrow\ u \cdot w = v_7 \cdot y$	
4,3	$(x \cdot e) \cdot e = x$	
6,5	$e \cdot (e \cdot x) = x$	
7	$C_4 \cdot A = C_3 \cdot B$	
9	$C_2 \cdot A = C_1 \cdot B$	
11	$C_4 \cdot F = C_3 \cdot E$	
13	$C_2 \cdot F = C_1 \cdot E \ \rightarrow\ \square$	
20	$((e \cdot x) \cdot e) \cdot (e \cdot (y \cdot e)) = y \cdot x$	[2,1,3,5,1]
22	$x \cdot y = ((e \cdot y) \cdot e) \cdot (e \cdot (x \cdot e))$	[2,5,1,1,3]
39	$((e \cdot E) \cdot e) \cdot A = ((e \cdot B) \cdot e) \cdot F$	[2,11,20,7,20]
60	$((e \cdot E) \cdot e) \cdot (e \cdot (C_1 \cdot e)) = C_2 \cdot F$	[2,20,9,1,39, flip]
70	$C_2 \cdot F = C_1 \cdot E$	[22 $\rightarrow$ 60 :6,4,4,6, flip]
72	$\square$	[70,13]

Problem 87.

$$\left\{ \begin{array}{l} \mathrm{TC} \\ ((x \cdot e) \cdot e) \cdot e = x \\ e \cdot (e \cdot (e \cdot x)) = x \end{array} \right\} \Rightarrow \mathrm{RC}.$$

Proof (found by Otter 3.0.3+ on gyro at 67.33 seconds).

1	$x = x$	
2	$x \cdot y = z,\ u \cdot v = z,\ x \cdot w = v_6,\ v_7 \cdot v = v_6 \ \rightarrow\ u \cdot w = v_7 \cdot y$	
4,3	$((x \cdot e) \cdot e) \cdot e = x$	
6,5	$e \cdot (e \cdot (e \cdot x)) = x$	
7	$C_4 \cdot A = C_3 \cdot B$	
9	$C_2 \cdot A = C_1 \cdot B$	
11	$C_4 \cdot F = C_3 \cdot E$	
13	$C_2 \cdot F = C_1 \cdot E \ \rightarrow\ \square$	
14	$(((x \cdot y) \cdot e) \cdot e) \cdot z = (((x \cdot z) \cdot e) \cdot e) \cdot y$	[2,1,3,1,3]
18,17	$(((C_2 \cdot x) \cdot e) \cdot e) \cdot A = (((C_1 \cdot B) \cdot e) \cdot e) \cdot x$	[2,1,3,9,3]
33	$x \cdot (e \cdot (e \cdot y)) = ((y \cdot e) \cdot e) \cdot (e \cdot (e \cdot (x \cdot e)))$	[2,5,1,5,3]
36	$(((e \cdot x) \cdot e) \cdot e) \cdot (e \cdot (e \cdot (y \cdot e))) = y \cdot x$	[2,1,3,5,1]
293	$(((e \cdot E) \cdot e) \cdot e) \cdot A = (((e \cdot B) \cdot e) \cdot e) \cdot F$	[2,11,36,7,36]
418,417	$(((e \cdot E) \cdot e) \cdot e) \cdot (e \cdot (e \cdot (C_1 \cdot B))) = (((C_2 \cdot F) \cdot e) \cdot e) \cdot B$	[2,36,17,36,293 :4,6,4, flip]
423	$(((C_2 \cdot F) \cdot e) \cdot e) \cdot B = (((C_1 \cdot B) \cdot e) \cdot e) \cdot E$	[2,33,17,293,293 :4,6,18,418, flip]
515	$C_2 \cdot F = C_1 \cdot E$	[2,3,14,3,423 :4,4, flip]
517	$\square$	[515,13]

Problem 88.

$$\left\{ \begin{array}{l} \text{TC} \\ \text{semilattice} \end{array} \right\} \Rightarrow \{x = y\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.40 seconds).

1	$x = x$	
2	$x \cdot y = z,\ u \cdot v = z,\ x \cdot w = v_6,\ v_7 \cdot v = v_6 \ \rightarrow\ u \cdot w = v_7 \cdot y$	
4,3	$x \cdot x = x$	
5	$x \cdot y = y \cdot x$	
7,6	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	
8	$B = A \ \rightarrow\ \square$	
22	$x \cdot (x \cdot y) = x \cdot y$	[2,6,3,6,1 :7,4]
25,24	$x \cdot (y \cdot x) = y \cdot x$	[2,5,3,6,1 :7,4]
39,38	$x \cdot y = y$	[2,5,3,5,22 :7,25,4]
40	$x = y$	[2,5,1,3,22 :39,39,39]
41	$\square$	[40,8]

Problem 89.

$$\left\{ \begin{array}{l} \text{RC} \\ \text{semilattice} \end{array} \right\} \Rightarrow \{x = y\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.52 seconds).

1	$x = x$	
2	$x \cdot y = z,\ u \cdot v = z,\ w \cdot y = v_6,\ v_7 \cdot v = v_6,\ x \cdot v_8 = v_9,\ u \cdot v_{10} = v_9\ \rightarrow\ w \cdot v_8 = v_7 \cdot v_{10}$	
4,3	$x \cdot x = x$	
5	$x \cdot y = y \cdot x$	
7,6	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$	
8	$B = A\ \rightarrow\ \Box$	
9	$x \cdot (y \cdot x) = y \cdot x$	[2,5,3,5,3,1,5 :7,4]
16,15	$x \cdot y = y$	[2,9,9,9,3,1,9 :4,4,7,4, flip]
17	$x = y$	[2,9,9,5,5,5,9 :16,16,16]
18	$\Box$	[17,8]

Problem 90.

$$\left\{\begin{array}{l} \mathrm{TC} \\ m(x,y,z) = ((y \cdot y) \cdot x) \cdot (z \cdot y) \\ x \cdot (y \cdot x) = y \end{array}\right\} \Rightarrow \{m(x,y,m(z,u,v)) = m(m(x,y,z),u,v)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 15.37 seconds).

1	$x = x$	
2	$x \cdot y = z,\ u \cdot v = z,\ x \cdot w = v_6,\ v_7 \cdot v = v_6\ \rightarrow\ u \cdot w = v_7 \cdot y$	
4,3	$m(x,y,z) = ((y \cdot y) \cdot x) \cdot (z \cdot y)$	
5	$x \cdot (y \cdot x) = y$	
7	$m(A,B,m(C,D,E)) = m(m(A,B,C),D,E)\ \rightarrow\ \Box$	
8	$((B \cdot B) \cdot A) \cdot ((((D \cdot D) \cdot C) \cdot (E \cdot D)) \cdot B) = ((D \cdot D) \cdot (((B \cdot B) \cdot A) \cdot (C \cdot B))) \cdot (E \cdot D)\ \rightarrow\ \Box$	[copy,7 :4,4,4,4]
14	$(x \cdot y) \cdot x = y$	[5 → 5]
17	$(x \cdot ((y \cdot z) \cdot u)) \cdot y = (x \cdot z) \cdot u$	[2,14,14,1,14, flip]
19	$x \cdot (y \cdot (z \cdot u)) = ((u \cdot x) \cdot y) \cdot z$	[2,14,5,5,14]
784	$x \cdot (((y \cdot z) \cdot u) \cdot v) = (y \cdot ((((v \cdot w) \cdot x) \cdot z) \cdot u)) \cdot w$	[2,1,5,5,17]
1734,1733	$(x \cdot ((((y \cdot z) \cdot u) \cdot v) \cdot w)) \cdot z = (x \cdot (u \cdot (v \cdot y))) \cdot w$	[19 → 17]
1752	$x \cdot (((y \cdot z) \cdot u) \cdot v) = (y \cdot (x \cdot (z \cdot v))) \cdot u$	[784 :1734]
1753	$\Box$	[1752,8]

Problem 91.

$$\left\{\begin{array}{l} \mathrm{TC} \\ x \cdot (y \cdot x) = y \end{array}\right\} \Rightarrow \{(x * y) * (z * u) = (x * z) * (y * u)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.09 seconds).

2	$x \cdot y = z,\ u \cdot v = z,\ x \cdot w = v_6,\ v_7 \cdot v = v_6\ \rightarrow\ u \cdot w = v_7 \cdot y$	
3	$x \cdot (y \cdot x) = y$	
5	$(A \cdot C) \cdot (B \cdot D) = (A \cdot B) \cdot (C \cdot D)\ \rightarrow\ \Box$	
11	$(x \cdot y) \cdot x = y$	[3 → 3]
19	$(x \cdot y) \cdot (z \cdot u) = (x \cdot z) \cdot (y \cdot u)$	[2,3,11,3,11]
20	$\Box$	[19,5]

Problem 92.

$$\left\{\begin{array}{l}\mathrm{RC}\\ x\cdot y=y\cdot x\\ x\cdot(y\cdot x)=y\end{array}\right\}\Rightarrow\{((x\cdot y)\cdot z)\cdot(u\cdot x)=y\cdot(z\cdot u)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 5.46 seconds).

2	$x\cdot y=z,\ u\cdot v=z,\ w\cdot y=v_6,\ v_7\cdot v=v_6,\ x\cdot v_8=v_9,\ u\cdot v_{10}=v_9\ \rightarrow\ w\cdot v_8=v_7\cdot v_{10}$	
3	$x\cdot y=y\cdot x$	
4	$x\cdot(y\cdot x)=y$	
6	$((A\cdot B)\cdot C)\cdot(D\cdot A)=B\cdot(C\cdot D)\ \rightarrow\ \square$	
68	$(x\cdot y)\cdot x=y$	$[4\rightarrow 4]$
70	$x\cdot(x\cdot y)=y$	$[3\rightarrow 4]$
284	$((x\cdot y)\cdot z)\cdot(u\cdot x)=y\cdot(z\cdot u)$	[2,70,70,68,4,4,70]
286	$\square$	[284,6]

Problem 93.

$$\left\{\begin{array}{l}\mathrm{RC}\\ x\cdot e=x\\ e\cdot x=x\end{array}\right\}\Rightarrow\{(x\cdot y)\cdot z=x\cdot(y\cdot z)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.14 seconds).

1	$x=x$	
2	$x\cdot y=z,\ u\cdot v=z,\ w\cdot y=v_6,\ v_7\cdot v=v_6,\ x\cdot v_8=v_9,\ u\cdot v_{10}=v_9\ \rightarrow\ w\cdot v_8=v_7\cdot v_{10}$	
3	$x\cdot e=x$	
5	$e\cdot x=x$	
7	$(A\cdot B)\cdot C=A\cdot(B\cdot C)\ \rightarrow\ \square$	
8	$(x\cdot y)\cdot z=x\cdot(y\cdot z)$	[2,3,5,3,1,1,5]
10	$\square$	[8,7]

Problem 94.

$$\left\{\begin{array}{l}\mathrm{RC}\\ x\cdot e=x\\ x\cdot(x\cdot y)=y\end{array}\right\}\Rightarrow\{(x\cdot(e\cdot y))\cdot(e\cdot c)=x\cdot(e\cdot(y\cdot(e\cdot c)))\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.17 seconds).

1	$x=x$	
2	$x\cdot y=z,\ u\cdot v=z,\ w\cdot y=v_6,\ v_7\cdot v=v_6,\ x\cdot v_8=v_9,\ u\cdot v_{10}=v_9\ \rightarrow\ w\cdot v_8=v_7\cdot v_{10}$	
3	$x\cdot e=x$	
5	$x\cdot(x\cdot y)=y$	
7	$(A\cdot(e\cdot B))\cdot(e\cdot C)=A\cdot(e\cdot(B\cdot(e\cdot C)))\ \rightarrow\ \square$	
12	$(x\cdot(y\cdot z))\cdot u=x\cdot(y\cdot(z\cdot u))$	[2,3,5,3,1,1,5]
14	$\square$	[12,7]

Problem 95.

$$\left\{\begin{array}{l}\mathrm{TC}\\ e\cdot e=e\\ (x\cdot e)\cdot e=x\\ x\cdot(x\cdot y)=y\end{array}\right\}\Rightarrow\{(x\cdot(((x\cdot y)\cdot z)\cdot(y\cdot e)))\cdot(e\cdot e)=z\}.$$

The conclusion is a single axiom for Abelian groups in terms of $-x-y$.

Proof (found by Otter 3.0.3+ on gyro at 1.11 seconds).

1	$x=x$	
2	$x\cdot y=z,\ u\cdot v=z,\ x\cdot w=v_6,\ v_7\cdot v=v_6\ \rightarrow\ u\cdot w=v_7\cdot y$	
4,3	$e\cdot e=e$	
6,5	$(x\cdot e)\cdot e=x$	
7	$x\cdot(y\cdot x)=y$	
9	$(A\cdot(((A\cdot B)\cdot C)\cdot(B\cdot e)))\cdot(e\cdot e)=C\ \rightarrow\ \square$	
10	$(A\cdot(((A\cdot B)\cdot C)\cdot(B\cdot e)))\cdot e=C\ \rightarrow\ \square$	[copy,9 :4]
30	$e\cdot x=x\cdot e$	[2,5,3,7,1 :4,6]
39,38	$(x\cdot y)\cdot x=y$	[7 $\rightarrow$ 7]
50	$x\cdot y=y\cdot x$	[2,30,5,30,5 :6,6]
55,54	$(x\cdot y)\cdot y=x$	[2,38,38,5,38 :6, flip]
180,179	$(((x\cdot y)\cdot z)\cdot u)\cdot x=z\cdot(u\cdot y)$	[2,7,38,50,7]
195	$C=C\ \rightarrow\ \square$	[50 $\rightarrow$ 10 :180,39,55]
196	$\square$	[195,1]

Problem 96.

$$\left\{\begin{array}{l}e\cdot e=e\\ (x\cdot e)\cdot e=x\\ x\cdot(x\cdot y)=y\end{array}\right\}\ =\!(gL)\!\Rightarrow\ \{(x\cdot(((x\cdot y)\cdot z)\cdot(y\cdot e)))\cdot(e\cdot e)=z\}.$$

The conclusion is a single axiom for Abelian groups in terms of $-x-y$.

Proof (found by Otter 3.0.3+ on gyro at 11.31 seconds).

1	$x=x$	
3,2	$e\cdot e=e$	
5,4	$(x\cdot e)\cdot e=x$	
7,6	$x\cdot(y\cdot x)=y$	
8	$(A\cdot(((A\cdot B)\cdot C)\cdot(B\cdot e)))\cdot(e\cdot e)\neq C$	
9	$(A\cdot(((A\cdot B)\cdot C)\cdot(B\cdot e)))\cdot e\neq C$	[copy,8 :3]
11,10	$(x\cdot y)\cdot x=y$	[6 $\rightarrow$ 6]
12	$e\cdot x=x\cdot e$	[4 $\rightarrow$ 6]
14,13	$x\cdot(x\cdot e)=e$	[4 $\rightarrow$ 10]
17,16	$e\cdot(e\cdot x)=x$	[12 $\rightarrow$ 12 :5]
28	$(A\cdot(((A\cdot B)\cdot C)\cdot(e\cdot B)))\cdot e\neq C$	[12 $\rightarrow$ 9]
30	$x\cdot(x\cdot y)=z\cdot(z\cdot y)$	[13 $\rightarrow$ 13 :(gL)]

34,33	$x \cdot (x \cdot y) = y$	[13 → 30 :14,17, flip]
41	$x \cdot y = y \cdot x$	[6 → 30 :34]
45	$(x \cdot y) \cdot z = z \cdot (y \cdot x)$	[41 → 41]
118	$(x \cdot y) \cdot (z \cdot u) = (u \cdot z) \cdot (y \cdot x)$	[41 → 45]
123	$(A \cdot ((e \cdot B) \cdot (C \cdot (A \cdot B)))) \cdot e \neq C$	[45 → 28]
214	$(x \cdot y) \cdot (z \cdot u) = (x \cdot z) \cdot (y \cdot u)$	[(gL) 118]
238	$C \neq C$	[214 → 123 :7,7,11]
239	$\square$	[238,1]

Problem 97.

$$\left\{ \begin{array}{l} \mathrm{TC} \\ e \cdot x = x \\ x \cdot e = x \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x \cdot y = y \cdot x \\ (x \cdot y) \cdot z = x \cdot (y \cdot z) \end{array} \right\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.55 seconds).

1	$x = x$	
2	$x \cdot y = z,\ u \cdot v = z,\ x \cdot w = v_6,\ v_7 \cdot v = v_6 \ \rightarrow\ u \cdot w = v_7 \cdot y$	
3	$B \cdot A = A \cdot B,\ (A \cdot B) \cdot C = A \cdot (B \cdot C) \ \rightarrow\ \square$	
4	$e \cdot x = x$	
7,6	$x \cdot e = x$	
8	$x \cdot (y \cdot z) = y \cdot (x \cdot z)$	[2,4,1,4,1]
9	$x \cdot y = y \cdot x$	[2,4,6,4,6]
32,31	$(x \cdot y) \cdot z = y \cdot (x \cdot z)$	[2,6,9,8,9 :7, flip]
45	$\square$	[3 :32 :9,8]

Problem 98.

$$\left\{ \begin{array}{l} \mathrm{TC} \\ p(x,y) = e \cdot (y \cdot x) \\ x \cdot (y \cdot x) = y \\ x \cdot e = e \cdot x \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} p(x,y) = p(y,x) \\ p(p(x,y),z) = p(x,p(y,z)) \end{array} \right\}.$$

Proof (found by Otter 3.0.3+ on gyro at 2.09 seconds).

1	$x = x$	
2	$x \cdot y = z,\ u \cdot v = z,\ x \cdot w = v_6,\ v_7 \cdot v = v_6 \ \rightarrow\ u \cdot w = v_7 \cdot y$	
4,3	$p(x,y) = e \cdot (y \cdot x)$	
5	$x \cdot (y \cdot x) = y$	
7	$x \cdot e = e \cdot x$	
8	$p(A,B) = p(B,A),\ p(p(A,B),C) = p(A,p(B,C)) \ \rightarrow\ \square$	
9	$e \cdot (B \cdot A) = e \cdot (A \cdot B),\ e \cdot ((e \cdot (C \cdot B)) \cdot A) = e \cdot (C \cdot (e \cdot (B \cdot A))) \ \rightarrow\ \square$	[copy,8 :4,4,4,4,4,4, flip]
17,16	$(x \cdot y) \cdot x = y$	[5 → 5]
19	$x \cdot y = y \cdot x$	[2,5,7,1,7 :17]

115	$x \cdot (x \cdot y) = y$	[16 $\rightarrow$ 19, flip]
117	$(x \cdot y) \cdot y = x$	[5 $\rightarrow$ 19, flip]
119	$x \cdot (y \cdot z) = x \cdot (z \cdot y)$	[2,115,115,5,115]
243,242	$(x \cdot (y \cdot z)) \cdot u = y \cdot (x \cdot (z \cdot u))$	[2,117,115,5,117, flip]
348	$\square$	[9 :243 :119,1]

Chapter 9

Problems not yet placed in the proper chapter

Problem 99. Binary overlay with Steiner law.

$$\left\{\begin{array}{l} x*(y*x)=y \\ g(x,e)=f(x,e) \\ g(e,x)=f(e,x) \end{array}\right\} =\!\!(gL)\!\!\Rightarrow \{g(x,y)=f(x,y)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 337.79 seconds).

1	$g(A,B)\neq f(A,B)$	
3	$x\cdot(y\cdot x)=y$	
5	$g(x,e)=f(x,e)$	
6	$g(e,x)=f(e,x)$	
8,7	$(x\cdot y)\cdot x=y$	[3 → 3]
11	$g(x,e)\cdot(y\cdot f(x,e))=y$	[5 → 3]
21	$g(e,x)\cdot(y\cdot f(e,x))=y$	[6 → 3]
129	$g(x,y)\cdot(z\cdot f(x,u))=g(v,y)\cdot(z\cdot f(v,u))$	[11 → 11 :(gL) :(gL)]
3809	$g(x,y)\cdot(z\cdot f(x,y))=z$	[21 → 129, flip]
4013	$g(x,y)=f(x,y)$	[3809 → 3 :8, flip]
4014	$\square$	[4013,1]

Problem 100. Ternary overlay with Steiner law.

$$\left\{\begin{array}{l} x*(y*x)=y \\ g(x,e,e)=f(x,e,e) \\ g(e,x,e)=f(e,x,e) \\ g(e,e,x)=f(e,e,x) \end{array}\right\} =\!\!(gL)\!\!\Rightarrow \{g(x,y,z)=f(x,y,z)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 2825.77 seconds).

1	$g(A,B,C) \neq f(A,B,C)$	
4,3	$x \cdot (y \cdot x) = y$	
5	$g(x,e,e) = f(x,e,e)$	
6	$g(e,x,e) = f(e,x,e)$	
7	$g(e,e,x) = f(e,e,x)$	
8	$(x \cdot y) \cdot x = y$	[3 → 3]
12	$(g(x,e,e) \cdot y) \cdot f(x,e,e) = y$	[5 → 8]
24	$(g(e,x,e) \cdot y) \cdot f(e,x,e) = y$	[6 → 8]
40	$(g(e,e,x) \cdot y) \cdot f(e,e,x) = y$	[7 → 8]
119	$(g(x,y,z) \cdot u) \cdot f(x,v,w) = (g(v_6,y,z) \cdot u) \cdot f(v_6,v,w)$	[12 → 12 :(gL) :(gL) :(gL) :(gL)]
265	$(g(x,y,z) \cdot u) \cdot f(v,y,w) = (g(x,v_6,z) \cdot u) \cdot f(v,v_6,w)$	[24 → 24 :(gL) :(gL) :(gL) :(gL)]
2982	$(g(x,e,y) \cdot z) \cdot f(x,e,y) = z$	[40 → 119, flip]
4853	$(g(x,y,z) \cdot u) \cdot f(x,y,z) = u$	[2982 → 265, flip]
5196	$g(x,y,z) = f(x,y,z)$	[4853 → 8 :4]
5197	$\square$	[5196,1]

Problem 101. Binary overlay for quasigroup.

$$\left\{ \begin{array}{l} \cdot \text{ is a quasigroup} \\ g(x,e) = f(x,e) \\ g(e,x) = f(e,x) \end{array} \right\} \overset{(gL)}{\Longrightarrow} \{g(x,y) = f(x,y)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 101.59 seconds).

1	$g(A,B) = f(A,B) \;\rightarrow\; \square$	
3	$x \cdot y = z,\ x \cdot u = z \;\rightarrow\; y = u$	
7	$(x/y) \cdot y = x$	
9	$g(x,e) = f(x,e)$	
10	$g(e,x) = f(e,x)$	
19	$(x/g(y,e)) \cdot f(y,e) = x$	[9 → 7]
27	$(x/g(e,y)) \cdot f(e,y) = x$	[10 → 7]
68	$(x/g(y,z)) \cdot f(y,u) = (x/g(v,z)) \cdot f(v,u)$	[19 → 19 :(gL) :(gL)]
89	$(x/g(y,z)) \cdot f(y,z) = x$	[68 → 27]
626	$g(x,y) = f(x,y)$	[89,3,7]
627	$\square$	[626,1]

Problem 102. Ternary overlay for quasigroup.

$$\left\{ \begin{array}{l} \cdot \text{ is a quasigroup} \\ g(x,e,e) = f(x,e,e) \\ g(e,x,e) = f(e,x,e) \\ g(e,e,x) = f(e,e,x) \end{array} \right\} \overset{(gL)}{\Longrightarrow} \{g(x,y) = f(x,y)\}.$$

Proof (found by Otter 3.0.3+ on gyro at 75.51 seconds).

1	$g(A,B,C)=f(A,B,C) \;\rightarrow\; \Box$	
3	$x\cdot y=z,\ x\cdot u=z \;\rightarrow\; y=u$	
5	$(x/y)\cdot y=x$	
7	$g(x,e,e)=f(x,e,e)$	
8	$g(e,x,e)=f(e,x,e)$	
9	$g(e,e,x)=f(e,e,x)$	
14	$(x/g(y,e,e))\cdot f(y,e,e)=x$	[7 $\rightarrow$ 5]
22	$(x/g(e,y,e))\cdot f(e,y,e)=x$	[8 $\rightarrow$ 5]
30	$(x/g(e,e,y))\cdot f(e,e,y)=x$	[9 $\rightarrow$ 5]
50	$(x/g(y,z,u))\cdot f(y,v,w)=(x/g(v_6,z,u))\cdot f(v_6,v,w)$	[14 $\rightarrow$ 14 :(gL) :(gL) :(gL) :(gL)]
89	$(x/g(y,z,e))\cdot f(y,z,e)=x$	[50 $\rightarrow$ 22]
149	$(x/g(y,z,u))\cdot f(v,w,u)=(x/g(y,z,v_6))\cdot f(v,w,v_6)$	[30 $\rightarrow$ 30 :(gL) :(gL) :(gL) :(gL)]
464	$(x/g(y,z,u))\cdot f(y,z,u)=x$	[89 $\rightarrow$ 149, flip]
683	$g(x,y,z)=f(x,y,z)$	[464,3,5]
684	$\Box$	[683,1]

Problem 103.

$$\left\{\begin{array}{l} x-(y-x)=x \\ x-(x-y)=y-(y-x) \\ (x-y)-z=(x-z)-(y-z) \\ x\cdot y=x-(x-y) \end{array}\right\} \Rightarrow \{(x\cdot y)\cdot z=x\cdot(y\cdot z)\}.$$

[[from kalman/c1.out]]

Proof (found by Otter 3.0.3+ on gyro at 4.44 seconds).

3,2	$x-(y-x)=x$	
4	$x-(x-y)=y-(y-x)$	
5	$(x-y)-z=(x-z)-(y-z)$	
7,6	$x\cdot y=x-(x-y)$	
8	$(A\cdot B)\cdot C=A\cdot(B\cdot C),\ A\cdot B=B\cdot A \;\rightarrow\; \Box$	
9	$(A-(A-B))-((A-(A-B))-C)=A-(A-(B-(B-C))) \;\rightarrow\; \Box$	[copy,8 :7,7,7,7,7,7 :4]
10	$(x-y)-(z-y)=(x-z)-y$	[flip 5]
11	$(x-y)-y=x-y$	[2 $\rightarrow$ 2]
14,13	$(x-y)-(x-y)=y-y$	[11 $\rightarrow$ 4 :3]
15	$(x-y)-((x-y)-x)=x-(y-(y-x))$	[4 $\rightarrow$ 4, flip]
22,21	$(x-(x-y))-(y-(y-x))=x-x$	[4 $\rightarrow$ 13 :14]
23	$x-x=y-y$	[4 $\rightarrow$ 13 :22,14]
24	$((x-y)-z)-(y-z)=(x-y)-z$	[11 $\rightarrow$ 5, flip]
28	$(x-y)-((z-x)-y)=x-y$	[2 $\rightarrow$ 5, flip]
44	$(x-(x-y))-y=z-z$	[5 $\rightarrow$ 23 :3]
48,47	$x-(y-y)=x$	[23 $\rightarrow$ 4 :3]
56,55	$(x-x)-y=x-x$	[47 $\rightarrow$ 2]
76	$(A-(A-B))-((B-(B-A))-C)=A-(A-(B-(B-C))) \;\rightarrow\; \Box$	[4 $\rightarrow$ 9]
103	$x-x=((y-(y-z))-u)-z$	[44 $\rightarrow$ 10 :56]

107,106	$((x-y)-z)-y=(x-y)-(z-y)$	[11 → 10, flip]
112,111	$(x-y)-(z-x)=x-(y-(z-x))$	[2 → 10, flip]
126,125	$(x-y)-x=x-x$	[55 → 10, flip]
128	$((x-(x-y))-z)-y=u-u$	[flip 103]
132	$x-(y-(y-x))=x-y$	[15 :126,48, flip]
148	$(x-(y-z))-y=x-y$	[125 → 10 :48, flip]
153,152	$x-(x-(x-y))=x-y$	[125 → 4 :48, flip]
178	$(x-y)-(x-(y-z))=z-z$	[148 → 125 :14,14]
227,226	$((x-(x-y))-z)-((y-x)-z)=(x-(x-y))-z$	[132 → 28]
294	$(x-(x-y))-z=(y-(y-x))-z$	[4 → 24 :227]
313	$x-(x-((y-(y-x))-z))=(y-(y-x))-z$	[128 → 4 :48, flip]
403	$x-(x-((y-x)-z))=z-z$	[2 → 178]
521	$x-(x-((y-x)-z))=x-x$	[132 → 403 :14,14,14]
615,614	$(x-y)-(z-y)=(x-y)-z$	[521 → 132 :48,107, flip]
770	$x-(y-(z-x))=x-y$	[28 :615,112]
774	$(x-y)-z=(x-z)-y$	[10 :615]
842,841	$x-((y-(z-x))-u)=x-(y-u)$	[774 → 770]
847,846	$(x-(x-y))-z=y-(y-(x-z))$	[313 :842, flip]
861	$x-(x-(y-z))=y-(y-(x-z))$	[294 :847,847]
862	$B-(B-(A-(B-C)))=A-(A-(B-(B-C)))\ \rightarrow\ \square$	[76 :847,847,153]
863	$\square$	[862,861]

Problem 104.

$$\left\{\begin{array}{l} x-(y-x)=x \\ x-(x-y)=y-(y-x) \end{array}\right\} \Rightarrow \{(x-y)-z=(x-z)-(y-z) \Leftrightarrow (x-y)-z=(x-z)-y\}.$$

Proof (⇒) found by Otter 3.0.3+ on gyro at 2.40 seconds.

3,2	$x-(y-x)=x$	
4	$x-(x-y)=y-(y-x)$	
5	$(x-y)-z=(x-z)-(y-z)$	
6	$(A-C)-B\neq(A-B)-C$	
7	$(x-y)-(z-y)=(x-z)-y$	[flip 5]
8	$(x-y)-y=x-y$	[2 → 2]
11,10	$(x-y)-(x-y)=y-y$	[8 → 4 :3]
12	$(x-y)-((x-y)-x)=x-(y-(y-x))$	[4 → 4, flip]
19,18	$(x-(x-y))-(y-(y-x))=x-x$	[4 → 10 :11]
20	$x-x=y-y$	[4 → 10 :19,11]
35	$(A-B)-(C-B)\neq(A-B)-C$	[5 → 6]
46,45	$x-(y-y)=x$	[20 → 4 :3]
53	$(x-x)-y=x-x$	[45 → 2]
77,76	$((x-y)-z)-y=(x-y)-(z-y)$	[8 → 7, flip]
92,91	$(x-y)-x=x-x$	[53 → 7, flip]
98	$x-(y-(y-x))=x-y$	[12 :92,46, flip]

114	$(x-(y-z))-y=x-y$	[91 → 7 :46, flip]
146	$(x-y)-(x-(y-z))=z-z$	[114 → 91 :11,11]
339	$x-(x-((y-x)-z))=z-z$	[2 → 146]
455	$x-(x-((y-x)-z))=x-x$	[98 → 339 :11,11,11]
501	$(x-y)-(z-y)=(x-y)-z$	[455 → 98 :46,77, flip]
503	$\square$	[501,35]

Proof ($\Leftarrow$) found by Otter 3.0.3+ on gyro at 19.01 seconds.

3,2	$x-(y-x)=x$	
4	$x-(x-y)=y-(y-x)$	
5	$(x-y)-z=(x-z)-y$	
6	$(A-B)-C\neq(A-C)-(B-C)$	
7	$(A-C)-(B-C)\neq(A-B)-C$	[flip 6]
8	$(x-y)-y=x-y$	[2 → 2]
11,10	$(x-y)-(x-y)=y-y$	[8 → 4 :3]
19,18	$((x-y)-z)-y=(x-y)-z$	[8 → 5, flip]
21	$(x-(x-y))-z=(y-z)-(y-x)$	[4 → 5]
22	$(x-y)-(z-x)=x-y$	[2 → 5, flip]
29	$(x-y)-(x-z)=(z-(z-x))-y$	[flip 21]
33,32	$(x-y)-((x-z)-y)=z-(z-(x-y))$	[5 → 4]
34	$x-((y-x)-z)=x$	[5 → 2]
50	$x-x=y-y$	[34 → 4 :19,11]
55,54	$x-(y-y)=x$	[50 → 34]
57	$(x-x)-y=(z-y)-z$	[50 → 5]
61	$(x-y)-x=(z-z)-y$	[flip 57]
67,66	$(x-x)-y=x-x$	[54 → 2]
68	$(x-y)-x=z-z$	[61 :67]
75	$x-(x-(x-y))=x-y$	[68 → 4 :55, flip]
103,102	$((x-y)-z)-(u-(x-z))=(x-z)-y$	[5 → 22]
767	$(x-y)-((z-(z-x))-y)=(x-z)-((x-z)-(x-y))$	[29 → 4]
841	$(x-(x-y))-((x-(x-y))-(x-z))=y-(y-(x-z))$	[75 → 32 :33, flip]
2094,2093	$(x-y)-(x-(x-z))=(x-y)-z$	[75 → 102 :103, flip]
2125,2124	$(x-y)-((x-y)-(x-z))=(x-y)-z$	[29 → 102 :19]
2130,2129	$(x-(x-y))-z=y-(y-(x-z))$	[841 :2125]
2132	$(x-y)-(z-y)=(x-z)-y$	[767 :2130,2094,2125]
2133	$\square$	[2132,7]

Problem 105.

$$\left\{\begin{array}{l} x-(y-x)=x \\ x-(x-y)=y-(y-x) \end{array}\right\} \overset{(gL)}{\Longrightarrow} \{x=y\}.$$

Proof (found by Otter 3.0.3+ on gyro at 2.45 seconds).

1	$A \neq B$	
4,3	$x-(y-x)=x$	
5	$x-(x-y)=y-(y-x)$	
6	$x-(y-z)=x-(u-z)$	[3 → 3 :(gL) :4]
7	$(x-y)-z=x-z$	[3 → 3 :(gL)]
18,17	$(x-y)-(z-x)=x$	[3 → 7]
60	$x-(y-(y-x))=x$	[5 → 5 :18]
63	$x-(x-(x-y))=x$	[17 → 5, flip]
72	$x-(x-y)=y-(z-x)$	[6 → 5]
104	$x-x=x$	[17 → 60]
141,140	$(x-y)-x=x$	[7 → 104]
185,184	$x-y=x$	[140 → 63 :141,141, flip]
186	$x=y$	[72 :185,185,185,185]
187	$\square$	[186,1]

Problem 106.

$$\left\{\begin{array}{l}\mathrm{RC}\\ x-(y-x)=x\\ x-(x-y)=y-(y-x)\end{array}\right\} \Rightarrow \{x=y\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.79 seconds).

1	$A=B \ \rightarrow\ \square$	
2	$x=x$	
3	$x-y=z,\ u-v=z,\ w-y=v_6,\ v_7-v=v_6,\ x-v_8=v_9,\ u-v_{10}=v_9 \ \rightarrow\ w-v_8=v_7-v_{10}$	
5,4	$x-(y-x)=x$	
6	$x-(x-y)=y-(y-x)$	
9,8	$(x-y)-y=x-y$	[4 → 4]
15,14	$x-(y-(z-u))=x-u$	[3,8,8,2,8,4,8 :9]
17,16	$(x-y)-z=x-z$	[3,2,2,8,2,8,8]
23,22	$x-y=x$	[3,16,6,6,6,2,6 :17,5,15, flip]
24	$x=y$	[3,16,6,6,2,16,6 :23,23,23,23]
25	$\square$	[24,1]

Problem 107.

$$\left\{\begin{array}{l}\mathrm{TC}\\ x-(y-x)=x\\ x-(x-y)=y-(y-x)\end{array}\right\} \Rightarrow \{x=y\}.$$

Proof (found by Otter 3.0.3+ on gyro at 0.64 seconds).

1	$A=B \ \rightarrow\ \square$	
2	$x=x$	

3	$x-y=z,\ u-v=z,\ x-w=v_6,\ v_7-v=v_6\ \rightarrow\ u-w=v_7-y$	
5,4	$x-(y-x)=x$	
6	$x-(x-y)=y-(y-x)$	
9,8	$(x-y)-y=x-y$	[4 → 4]
11,10	$x-(y-(x-z))=x-z$	[3,8,2,4,8 :9]
14	$(x-y)-((x-y)-z)=z-y$	[3,8,4,2,6]
41,40	$(x-y)-(z-((u-y)-v))=(u-y)-v$	[3,10,10,14,14 :11, flip]
44,43	$(x-y)-((z-y)-x)=x-y$	[3,10,2,8,14 :9,41, flip]
49,48	$x-y=x$	[3,14,10,14,4 :5,44, flip]
50	$x=y$	[3,14,8,14,8 :49,49,49,49,49,49,49,49]
51	$\square$	[50,1]

Problem 36A. Find a single identity for the following equational theory T: the type is (3,0) with a ternary $f(x,y,z)$ and a nullary 'e'. The identities satisfied by T are

$$\{f(e,e,e)=e, f(x,y,f(z,x,y))=z\}.$$

Status: Open Problem.

Proof (found by Otter 3.0.3+ on lutra.mcs.anl.gov at 3.27 seconds).

1	$x=x$	
2	$f(f(x,f(f(e,e,e),y,x),e),z,f(u,y,z))=u$	
4	$f(e,e,e)=e,\ f(a,b,f(c,a,b))=c\ \rightarrow\ \square$	
5	$f(f(x,f(f(e,e,e),y,x),e),f(z,u,y),z)=f(v,f(f(e,e,e),u,v),e)$	[2 → 2]
8	$f(x,f(f(e,e,e),y,x),e)=f(z,f(f(e,e,e),y,z),e)$	[5 → 5]
12	$f(f(x,f(f(e,e,e),e,x),e),f(y,f(f(e,e,e),z,y),e),u)=f(v,f(f(e,e,e),f(f(e,e,e),z,u),v),e)$	[8 → 5]
16	$f(f(e,f(x,f(f(e,e,e),y,x),e),e),z,f(u,f(f(e,e,e),y,f(e,e,e)),z))=u$	[8 → 2]
71,70	$f(x,f(f(e,e,e),f(f(e,e,e),y,f(z,e,f(u,f(f(e,e,e),y,u),e))),x),e)=z$	[2 → 12, flip]
80	$f(f(e,x,e),e,x)=f(e,e,e)$	[70 → 16 :71]
93	$f(x,f(y,z,f(f(e,e,e),u,f(x,e,f(v,f(f(e,e,e),u,v),e)))),y)=f(w,f(f(e,e,e),z,w),e)$	[70 → 5]
94	$f(x,y,f(z,f(f(e,e,e),u,f(x,e,f(v,f(f(e,e,e),u,v),e))),y))=z$	[70 → 2]
98	$f(x,f(f(e,e,e),y,x),e)=f(z,f(u,y,f(f(e,e,e),v,f(z,e,f(w,f(f(e,e,e),v,w),e)))),u)$	[flip 93]
105	$f(x,f(f(e,e,e),f(f(e,e,e),e,f(y,e,f(e,f(e,e,e),e))),x),e)=y$	[80 → 70]
139	$f(f(e,f(e,e,e),e),x,f(y,e,x))=y$	[80 → 2]
162,161	$f(e,e,e)=e$	[80 → 139]
183,182	$f(e,x,f(y,e,x))=y$	[139 :162,162]
200,199	$f(x,f(e,y,x),e)=y$	[105 :162,162,162,162,183]
206,205	$f(x,f(y,z,x),y)=z$	[98 :162,200,162,162,200,183, flip]
208,207	$f(x,y,f(z,x,y))=z$	[94 :162,162,206,183]
217	$\square$	[4 :162,208 :1,1]

Problem 36B. Find a single identity for the following equational theory T: the type is (3,1,0) with a ternary f, unary g, and a nullary 'e'. The identities satisfied by T are

$$\{f(e,e,e) = e, f(x,y,f(z,x,y)) = z, g(x) = x\}.$$

Status: Open Problem.

We ran Otter searches on several thousand identities, but failed to find any single axioms.

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