Solution of the Robbins Problem

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Abstract. In this article we show that the three equations known as commutativity, associativity, and the Robbins equation are a basis for the variety of Boolean algebras. The problem was posed by Herbert Robbins in the 1930s. The proof was found automatically by a theorem-proving program for equational logic. We present the proof and the search strategies that enabled the program to find the proof.

Key words: Associative-commutative unification, Boolean algebra, EQP, paramodulation, Robbins algebra, Robbins problem.

1. Introduction

In 1933, E. V. Huntington presented the following basis for Boolean algebra [5, 4]:

$$x + y = y + x, \qquad \text{(commutativity)}$$
$$(x + y) + z = x + (y + z), \qquad \text{(associativity)}$$
$$n(n(x) + y) + n(n(x) + n(y)) = x. \qquad \text{(Huntington equation)}$$

The unary operation n can be read as "complement". (Boolean algebra is ordinarily presented in terms of addition, multiplication, complement, 0, and 1. From Huntington's basis, one can show that a 0 and a 1 with the appropriate properties exist, and if multiplication is defined in the obvious way, it also has the appropriate properties.)

Shortly thereafter, Herbert Robbins posed the question of whether the Huntington equation can be replaced with the following equation, which is shorter by one occurrence of n:

$$n(n(x + y) + n(x + n(y))) = x. \qquad \text{(Robbins equation)}$$

The Robbins equation is clearly valid in all Boolean algebras, so the question can be rephrased as "Does the Huntington equation follow from commutativity, associativity, and the Robbins equation?"

Robbins and Huntington could not find a proof or counterexample, and the problem was later studied by Tarski, his students, and others [2, p. 245]. Algebras satisfying commutativity, associativity, and the Robbins equation became known

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as Robbins algebras, and the question was sometimes phrased as "Are all Robbins algebras Boolean?"[1]

As far as we know, automated deduction was first attempted on the problem in 1979, when Steve Winker, a student visiting Argonne, learned of the problem from Joel Berman. Larry Wos, one of Winker's advisors at Argonne, suggested attacking the problem by looking for properties, which we call *sufficient conditions*, that force Robbins algebras to be Boolean. For example, it is nearly trivial to show that Robbins algebras satisfying $n(n(x)) = x$ are Boolean.[2]

Examples of conditions that were shown to be sufficient automatically by Argonne's theorem provers are (1) $\forall x, x + x = x$, (2) $\exists c \forall x, c + x = x$, and (3) $\exists c \forall x, c + x = c$.

Winker then proved (by hand, with insight from theorem prover searches) several weaker conditions sufficient. The two such conditions that play a role in the present work are contained in the following two lemmas.

Lemma 1 (S. Winker [13, 14]). *A Robbins algebra satisfying* $\exists c \exists d, c + d = c$ *is a Boolean algebra.*

Lemma 2 (S. Winker [13, 14]). *A Robbins algebra satisfying* $\exists c \exists d, n(c+d) = n(c)$ *is a Boolean algebra.*

Appendix B contains a computer proof of Lemma 1. Note that Lemma 2 is a strengthening of Lemma 1.

2. The Solution

This section contains the proof of the key result (Lemma 3). The theorem that Robbins algebras and Boolean algebras coincide then follows directly from Lemma 1 (also from Lemma 2).

Lemma 3. *All Robbins algebras satisfy* $\exists c \exists d, c + d = c$.

Proof. The proof (found automatically by computer) starts with the Robbins equation and uses paramodulation (Section 3.3) with built-in associative-commutative (AC) unification (Section 3.1) and simplification with built-in AC matching. We abbreviate $x+x$ as $2x$, $x+x+x$ as $3x$, and so on. The justification "$m \rightarrow n$" indicates paramodulation from m into n, and "simp:n" indicates simplification with n.

[1] In [5], Huntington included $x + x = x$ in his basis (along with commutativity, associativity, and the Huntington equation) and incorrectly stated that the four equations are independent; the correction appeared in [4], where he showed that $x + x = x$ can be derived from the other three (which are independent). Folklore incorrectly has it that the Robbins problem arose from Huntington's mistake, in particular, that the error in Huntington's paper is that the Robbins equation appears in place of the Huntington equation [5]. According to Robbins [11], the Robbins problem is not directly related to Huntington's error.

[2] *Proof.* With the Robbins equation, let x be $n(x)$, complement both sides of the equation, and simplify with $n(n(x)) = x$ to obtain the Huntington equation.

7	$n(n(n(x)+y)+n(x+y)) = y$	[Robbins Equation[3]]
10	$n(n(n(x+y)+n(x)+y)+y) = n(x+y)$	[7 → 7]
11	$n(n(n(n(x)+y)+x+y)+y) = n(n(x)+y)$	[7 → 7]
29	$n(n(n(n(x)+y)+x+2y)+n(n(x)+y)) = y$	[11 → 7]
54	$n(n(n(n(n(x)+y)+x+2y)+n(n(x)+y)+z)+n(y+z)) = z$	[29 → 7]
217	$n(n(n(n(n(n(x)+y)+x+2y)+n(n(x)+y)+n(y+z)+z)+z) = n(y+z)$	[54 → 7]
674	$n(n(n(n(n(n(x)+y)+x+2y)+n(n(x)+y)+n(y+z)+z)+z+u)+$ $n(n(y+z)+u)) = u$	[217 → 7]
6736	$n(n(n(n(3x)+x)+n(3x))+n(n(n(3x)+x)+5x)) = n(n(3x)+x)$	[10 → 674]
8855	$n(n(n(3x)+x)+5x) = n(3x)$	[6736 → 7,simp:54,flip]
8865	$n(n(n(n(3x)+x)+n(3x)+2x)+n(3x)) = n(n(3x)+x)+2x$	[8855 → 7]
8866	$n(n(n(3x)+x)+n(3x)) = x$	[8855 → 7,simp:11]
8870	$n(n(n(n(3x)+x)+n(3x)+y)+n(x+y)) = y$	[8866 → 7]
8871	$n(n(3x)+x)+2x = 2x$	[8865,simp:8870,flip]

Equation 8871 asserts the existence of an object c, namely $2x$, and an object d, namely $n(n(3x)+x)$, such that $c + d = c$. Q.E.D.

(The preceding proof is what the program presents to the user. The AC instantiations and substitutions performed by the program are difficult to reconstruct, so we include a detailed version of the proof in Appendix A. See Section 5 for statistics on the proof search.)

Note that Equation 8855 satisfies the Lemma 2 condition: $\exists c$, namely $3x$, and $\exists d$, namely $n(n(3x)+x)+2x$, such that $n(c+d) = n(c)$, so we could have stopped there and used Lemma 2. See Section 5.

Theorem. *The three equations {commutativity, associativity, Robbins} are a basis for the variety of Boolean algebras.*

Proof. This follows directly from Lemma 1, Lemma 3, and the observation that the Robbins equation is valid in all Boolean algebras.

3. The Theorem Prover EQP

The theorem prover that found the proof of Lemma 3 is EQP, presented in [9].[4] EQP is restricted to equational logic and can perform associative-commutative (AC) unification and matching. The search algorithm is similar to Knuth-Bendix completion [7], with the option of using the "basic" restriction [3].

We describe here some of EQP's features that were used in the proof of Lemma 3. See [9] for details on these and the other features of EQP.

[3] EQP automatically rearranges AC subterms and renames variables.

[4] EQP is similar in many ways to our more well known theorem prover Otter [8]; the most important differences are that Otter (1) does not have associative-commutative unification or matching, (2) applies to full first-order statements, (3) has fewer paramodulation options, and (4) always uses the given clause algorithm to derive the search.

3.1. AC Unification and Matching

Associative-commutative (AC) unification [12] builds the properties of associativity and commutativity of a binary operation into the inference process so that the corresponding equations need not be present as explicit axioms. Two terms are *AC identical* if they can be made identical by reassociating and commuting subterms. An *AC unifier* of two terms is a substitution (of terms for variables) that makes the two terms AC identical. *AC unification* is the process of finding the set of most general AC unifiers (which is always finite) for a pair of terms. *AC matching* is a special case of AC unification in which only one of the two given terms is instantiated; AC matching is used to simplify derived equations and to determine whether one equation subsumes another. EQP uses Stickel's AC unification algorithm [12],

A pair of terms can have a great number of most general AC unifiers, and we have an optional heuristic, the *super-0 strategy*, that eliminates the more complicated unifiers. Stickel's algorithm constructs a linear homogeneous Diophantine equation that represents identity of the two terms to be unified then computes the basis of solutions (the basis has the property that every solution is a linear combination of the basis). To find all most general AC unifiers, the algorithm ordinarily considers each subset of the basis. The effect of the super-0 strategy is that if a subset S produces a potential unifier, then no supersets of S are considered. Since a unique variable is associated with each member of the subset, the corresponding AC unifier, if it exists, is more complicated for larger subsets.

The super-0 strategy causes incompleteness of the proof procedure, because it eliminates some of the most general unifiers, but we have not seen any cases in practical work where it blocks all proofs. (We have seen cases, however, in which it block short proofs, increasing the time required to find a proof.) The heuristic was used to find the proof of Lemma 3, and its role seems to have been important.

The AC matching algorithm is of our own design, with ideas from Kapur's and Zhang's RRL [6]. See [9] for details on EQP's AC unification and matching.

3.2. Paramodulation and Demodulation

The search uses paramodulation (an inference rule for equality) and demodulation (simplification of inferred equations) [15]. A simplification ordering $\succ$ on terms is used to orient equations and to guarantee termination of demodulation. The term ordering is $t_1 \succ t_2$ if $length(t_1) > length(t_2)$ and no variable has more occurrences in t_2 than in t_1.[5] Every input and derived equation is oriented, if possible, so that the left hand side is greater, and each oriented equation is added to the set of demodulators.

Paramodulation is not permitted from or into right hand sides of oriented equations, and paramodulation is not permitted from or into variables. Each

[5] This ordering is primitive but very fast.

equation inferred by paramodulation is simplified with the set of demodulators. If the simplified equation passes the retention tests (typically subsumption and a length limit) and is orientable, it is then used to simplify all other equations in memory.

We have the option of using the "basic" restriction on paramodulation. In summary, the "basic" restriction says that terms that arise by instantiation alone are ineligible as "into" terms. In more detail, consider each equation as a pair, ⟨skeleton,substitution⟩. Input equations have an empty (or identity) substitution, and derived equations are constructed as follows. The "into" term must exist as a nonvariable term in the skeleton of the "into" parent. The skeleton of the paramodulant is constructed from the skeletons of the parents by simple equality replacement, and the substitution of the paramodulant is constructed from the substitutions of the parents and the unifier for the inference.[6] Without the "basic" restriction, terms that exist only in the substitution part would be admissible "into" terms as well. The "basic" restriction, which is complete for several variants of paramodulation [1, 10], imposes an order on derivations, but, like most other strategies, its use can interfere with searches as well as help them. It appears to have been a key strategy in finding the proof of Lemma 3 and similar proofs.

3.3. The Search Algorithm

A paring algorithm is used to select the next equations for application of paramodulation. Let the weight of a pair of equations be the sum of the lengths of its members, and let the age of a pair be the sum of the ages of its members. (The age of an equation is determined by the time of its derivation.) In each iteration of the search loop, either the lightest pair not yet selected or the oldest pair not yet selected is chosen. The *pair selection ratio*, one of EQP's important search parameters, is used to specify the ratio, *lightest:oldest*. The default value is 1:0, that is, to always select the lightest pair.

4. Use of EQP

EQP is not an interactive program. The user states the conjecture, sets a few search parameters, and starts the search. As EQP searches, it sends derived equations and some statistics to the output file. If the search fails or does not look promising, the user can adjust the parameters and try again. By iterating in this way, we try to achieve a *well-behaved search* [9]. We had up to three workstations available when working on the this project, so we ran (independent) searches in parallel as well.

[6] The "basic" restriction is not implemented by storing equations as ⟨skeleton,substitution⟩. Instead, "basic" positions in equations are marked, and the marks are inherited during paramodulation.

The attack that led to the solution of the Robbins problem took place over the course of five weeks (September 6 through October 11, 1996). The search parameters were varied in the following ways in various combinations. (1) The limit on the length of retained equations started at 36 and was raised to 40, 50, 60, 70, and 80. (2) Searches were run with and without the super-0 restriction on AC unifiers. (3) Searches were run with and without the "basic" restriction on paramodulation. (4) Several searches were run with pair selection ratio 1:0, and several with 1:1.

After about 14 multiday searches, using a total of about 5 CPU-weeks of computer time, a proof of the Lemma 2 sufficient condition, $\exists c \exists d, n(c+d) = n(c)$, was found. The successful search took almost 8 days and used about 30 megabytes of memory. The successful search parameters were a length limit of 70, the super-0 restriction, the "basic" restriction, and a pair selection ratio of 1:1.

5. More Computer Proofs

We ran several more searches to try to (1) determine whether a proof could have been found earlier or quicker if we had used different search parameters, (2) find a simpler proof, and (3) find proofs of other sufficient conditions. Table I lists a summary of the experiments. All of the listed searches started with the Robbins

Table I. Statistics For Various Proofs

Search	Max-weight	Ratio	Proof	Days	Length
m5-70	70	1:1	Cond. 2	7.85	15
m5-60a	60	1:1	Cond. 2	5.71	15
			Cond. 1	5.72	17
			Hunt.	8.83	194
m5-60b	60	4:1	Cond. 2	3.08	15
			Cond. 1	3.09	17
			Hunt.	5.68	194
m5-50	50	1:1	Cond. 2	10.03	8
			Cond. 1	10.03	12
			Hunt.	10.75	86
m5-50b	50	4:1	Cond. 2	4.89	8
			Cond. 1	4.90	12
			Hunt.	5.56	86

equation and used AC unification with the super-0 strategy and the "basic" restriction on paramodulation. All searches except m5-70 had multiple goals, including the Lemma 1 condition (Cond. 1), the Lemma 2 condition (Cond. 2),

and the Huntington equation (Hunt.). For the searches with multiple goals, EQP was told to keep searching and prove as many goals as it could.

The first proof found and the shortest proofs found are of the Lemma 2 condition. The Lemma 1 condition was always found shortly thereafter in just two or four more steps. We chose to present the proof of the Lemma 1 condition because it leads to a simpler overall proof—Winker's proof (and the computer proof in Appendix B) that Cond. 1 is sufficient is much less complicated than his proof (or our computer proof [9]) that Cond. 2 is sufficient. Note that since EQP proved the Huntington equation, we have an automatic proof that does not rely on either of the Winker lemmas; we have not presented it here because of its length.

As a final note, the relationships between the various proofs say something about the searches. Although different selection ratios and different weight limits usually lead to very different proofs, the three proofs of length 15 are identical, the two proofs of length 17 are identical, the two proofs of length 8 are identical, the two proofs of length 12 are identical, the two proofs of length 86 are identical, and the two proofs of length 194 have the same steps, but in a different order. The Cond. 2, proofs of length 8 and 15 have 6 steps in common, and the Cond. 1, proofs of length 12 and 17 have same 6 steps in common. The Huntington proofs (lengths 86 and 194) have 22 steps in common. All of the commonality indicates that the searches and proofs are closer to being canonical than those we ordinarily see (for example, using our theorem prover Otter [8]). The main causes appear to be AC unification and the "basic" restriction on paramodulation.

Appendix A

This appendix contains a detailed proof of Lemma 3. We start with the Robbins equation (7) and derive (8871) with AC unification and matching. The numbering of the steps is the same as in the less detailed proof in Section 2.

$$n(n(n(x)+y)+n(x+y)) = y \tag{7}$$
$$n(n(3x)+x)+2x = 2x \tag{8871}$$

STEP 10

With 7, let x be $n(x)+y$ and y be $n(x+y)$:

$$n(\underline{n(n(n(x)+y)+n(x+y))}+n(n(x)+y+n(x+y))) = n(x+y).$$

Use (7) to replace the underlined term; then rearrange:

$$n(n(n(x+y)+n(x)+y)+y) = n(x+y). \qquad [7\to 7]\ (10)$$

STEP 11

With 7, let y be $n(n(x)+y)$ and x be $x+y$:

$$n(\underline{n(n(x+y)+n(n(x)+y))}+n(x+y+n(n(x)+y))) = n(n(x)+y).$$

Use (7) to replace the underlined term; then rearrange:

$$n(n(n(n(x)+y)+x+y)+y) = n(n(x)+y). \qquad [7\to7]\ (11)$$

STEP 29

With 7, let x be $n(n(x)+y)+x+y$ and y be y:

$$n(\underline{n(n(n(n(x)+y)+x+y)+y)}+n(n(n(x)+y)+x+2y)) = y.$$

Use (11) to replace the underlined term; then rearrange:

$$n(n(n(n(x)+y)+x+2y)+n(n(x)+y)) = y. \qquad [11\to7]\ (29)$$

STEP 54

With 7, let x be $n(n(n(x)+y)+x+2y)+n(n(x)+y)$ and y be z:

$$n(n(\underline{n(n(n(n(x)+y)+x+2y)+n(n(x)+y))}+z)+n(n(n(n(x)+y)+x+2y)+ n(n(x)+y)+z)) = z.$$

Use (29) to replace the underlined term; then rearrange:

$$n(n(n(n(n(x)+y)+x+2y)+n(n(x)+y)+z)+n(y+z)) = z. \qquad [29\to7]\ (54)$$

STEP 217

With 7, let x be $n(n(n(x)+y)+x+2y)+n(n(x)+y)+z$ and y be $n(y+z)$:

$$n(\underline{n(n(n(n(n(x)+y)+x+2y)+n(n(x)+y)+z)+n(y+z))}+n(n(n(n(x)+y)+x+2y)+ n(n(x)+y)+z+n(y+z))) = n(y+z).$$

Use (54) to replace the underlined term; then rearrange:

$$n(n(n(n(n(x)+y)+x+2y)+n(n(x)+y)+n(y+z)+z)+z) = n(y+z). \qquad [54\to7]\ (217)$$

STEP 674

With 7, let y be u and x be $n(n(n(n(x)+y)+x+2y)+n(n(x)+y)+n(y+z)+z)+z$:

$$n(n(\underline{n(n(n(n(x)+y)+x+2y)+n(n(x)+y)+n(y+z)+z)+z)}+u)+n(n(n(n(x)+y)+x+2y)+n(n(x)+y)+n(y+z)+z)+z+u)) = u.$$

Use (217) to replace the underlined term; then rearrange:

$$n(n(n(n(n(x)+y)+x+2y)+n(n(x)+y)+n(y+z)+z)+z+u)+n(n(y+z)+u)) = u. \quad [217\to 7]\ (674)$$

STEP 6736

With 10, let x be $3v$ and y be $n(n(3v)+v)+2v$:

$$n(n(n(3v+n(n(3v)+v)+2v)+n(3v)+n(n(3v)+v)+2v)+n(n(3v)+v)+2v) = n(3v+n(n(3v)+v)+2v). \quad (10')$$

With 674, let x be $3v$, y be v, z be $2v$ and u be $n(n(3v)+v)$:

$$n(\underline{n(n(n(n(n(3v)+v)+5v)+n(n(3v)+v)+n(3v)+2v)+2v+n(n(3v)+v))}+n(n(3v)+n(n(3v)+v))) = n(n(3v)+v). \quad (674')$$

Replace the underlined term of (674′), which is AC-identical to the left-hand side of (10′), with the right-hand side of (10′):

$$n(n(3v+n(n(3v)+v)+2v)+n(n(3v)+n(n(3v)+v))) = n(n(3v)+v).$$

Rename the variable and rearrange:

$$n(n(n(n(3x)+x)+n(3x))+n(n(n(3x)+x)+5x)) = n(n(3x)+x). \quad [10\to 674]\ (6736)$$

STEP 8855

With 7, let x be $n(n(3x)+x)+n(3x)$ and y be $n(n(n(3x)+x)+5x)$:

$$n(\underline{n(n(n(n(3x)+x)+n(3x))+n(n(n(3x)+x)+5x))}+n(n(n(3x)+x)+n(3x)+n(n(n(3x)+x)+5x))) = n(n(n(3x)+x)+5x).$$

Use (6736) to replace the underlined term:

$$n(n(n(3x)+x)+n(n(n(3x)+x)+n(3x)+n(n(n(3x)+x)+5x))) = n(n(n(3x)+x)+5x).$$

With 54, let x be $3x$, z be $n(3x)$ and y be x:

$$n(n(n(n(n(3x)+x)+5x)+n(n(3x)+x)+n(3x))+n(x+n(3x))) = n(3x)$$

The left-hand sides of the preceding two equations are AC-identical; hence

$$n(n(n(3x)+x)+5x) = n(3x). \quad [6736\to 7{:}54]\ (8855)$$

STEP 8865

With 7, let y be $n(n(3x)+x)+2x$ and x be $3x$:

$$n(n(n(3x)+n(n(3x)+x)+2x)+\underline{n(3x+n(n(3x)+x)+2x)}) = n(n(3x)+x)+2x.$$

Uset (8855) to replace the underlined term; then rearrange:

$$n(n(n(n(3x)+x)+n(3x)+2x)+n(3x)) = n(n(3x)+x)+2x. \qquad [8855\to 7]\ (8865)$$

STEP 8866

With 7, let x be $n(n(3x)+x)+4x$ and y be x:

$$n(n(n(n(n(3x)+x)+4x)+x)+\underline{n(n(n(3x)+x)+5x)}) = x.$$

Replace the underlined term with the right-hand side of (8855):

$$n(\underline{n(n(n(n(3x)+x)+4x)+x)}+n(3x)) = x. \qquad (A2)$$

With 11, let x be $3x$ and y be x:

$$n(n(n(n(3x)+x)+4x)+x) = n(n(3x)+x). \qquad (A3)$$

Use (A3) to replace the underlined term of (A2):

$$n(n(n(3x)+x)+n(3x)) = x. \qquad [8855\to 7{:}11]\ (8866)$$

STEP 8870

With 7, let x be $n(n(3x)+x)+n(3x)$:

$$n(n(\underline{n(n(n(3x)+x)+n(3x))}+y)+n(n(n(3x)+x)+n(3x)+y)) = y.$$

Use (8866) to replace the underlined term; then rearrange:

$$n(n(n(n(3x)+x)+n(3x)+y)+n(x+y)) = y. \qquad [8866\to 7]\ (8870)$$

STEP 8871

With 8870, let y be $2x$:

$$n(n(n(n(3x)+x)+n(3x)+2x)+n(3x)) = 2x.$$

Use the preceding equation to simplify (8865):

$n(n(3x)+x)+2x = 2x.$ [8865:8870] (8871)

Q.E.D.

Appendix B

This appendix contains a proof of Lemma 1, conjectured and first proved by Winker [13, 14], then later proved automatically by EQP [9].

To simplify the presentation, we first prove a stronger condition sufficient. Both computer proofs were found by EQP, the first in about 5 seconds, and the second in about 2319 seconds.

Lemma 0. *A Robbins algebra satisfying* $\exists c, c + c = c$ *is a Boolean algebra.*

Proof. We assert that the Huntington equation fails to hold, and we derive a contradiction. The terms A, B, and C are constants. (In the justification, n' indicates the extension of equation n; that is, if n is $t_1 = t_2$, n' is $t_1 + x = t_2 + x$.)

1	$C + C = C$	[hypothesis]
2	$n(n(n(x) + y) + n(x + y)) = y$	[Robbins Equation]
3	$n(B + n(A)) + n(n(B) + n(A)) \neq A$	[denial of Huntington Equation]
4	$n(n(C) + n(C + n(C))) = C$	$[1 \to 2]$
5	$n(n(C + n(C) + x) + n(C + x)) = C + x$	$[1' \to 2]$
8	$n(n(C + x) + n(n(C) + n(C + n(C)) + x)) = x$	$[4 \to 2]$
9	$n(C + n(C + n(C) + n(C))) = n(C)$	$[4 \to 2]$
13	$n(n(C) + n(C + n(C) + n(C + n(C)))) = C$	$[1 \to 8]$
20	$n(n(C) + n(C + n(C) + n(C))) = C$	$[9 \to 2$,simp:1]
22	$n(C + n(C) + n(C)) = n(C + n(C))$	$[9 \to 2$,simp:20,flip]
24	$n(C + n(C + n(C))) = n(C)$	[9,simp:22]
32	$C + n(C + n(C)) = C$	$[24 \to 5$,simp:13,flip]
35	$n(C + n(C)) + x = x$	$[32' \to 2$,simp:8,flip]
42	$n(n(n(x)) + n(x)) = n(C + n(C))$	$[35 \to 2$,simp:35]
50	$n(n(n(n(x)) + x)) = n(n(x))$	$[42 \to 2$,simp:35]
52	$n(n(n(x))) = n(x)$	$[42 \to 2$,simp:35,50]
58	$n(n(x)) = x$	$[2 \to 52$,simp:2]
87	$n(n(x) + y) + n(x + y) = n(y)$	$[2 \to 58$,flip]
88	$A \neq A$	[3,simp:87,58]

Lemma 1 (S. Winker [13, 14]). *A Robbins algebra satisfying* $\exists c \exists d, c + d = c$ *is a Boolean algebra.*

Proof.

2	$D + C = C$	[hypothesis]
3	$n(n(n(x) + y) + n(x + y)) = y$	[Robbins Equation]
4	$n(n(C) + n(D + n(C))) = D$	$[2 \to 3]$

8	$n(n(D+n(C+x)+y)+n(C+x+y))=D+y$	$[2' \to 3]$
20	$n(D+n(C+n(D+n(C))))=n(D+n(C))$	$[4 \to 3]$
34	$n(n(n(n(x)+y)+n(x+y)+z)+n(y+z))=z$	$[3 \to 3]$
35	$n(n(n(n(x)+y)+x+y)+y)=n(n(x)+y)$	$[3 \to 3]$
56	$n(n(C)+n(D+n(C+n(x))+n(C+x)))=D$	$[2 \to 34]$
151	$n(n(D+n(C+n(D+n(C)))+x)+n(n(D+n(C))+x))=x$	$[20 \to 3]$
152	$n(n(D+n(C))+n(C+n(D+n(C))))=D$	$[20 \to 3,\text{simp:}2]$
173	$n(D+n(D+n(C)+n(C+n(D+n(C)))))=n(C+n(D+n(C)))$	$[152 \to 3]$
197	$n(n(C+n(D+n(C)))+n(C+n(C+n(D+n(C)))))=C$	$[2' \to 151]$
280	$n(n(n(n(n(x)+y)+x+y)+y+z)+n(n(n(x)+y)+z))=z$	$[35 \to 3]$
837	$n(C+n(D+n(C)))=n(C)$	$[4 \to 151,\text{simp:}173]$
839	$n(n(C)+n(C+n(C)))=C$	$[197,\text{simp:}837,837]$
842	$n(n(C+x)+n(n(C)+n(C+n(C))+x))=x$	$[839 \to 3]$
844	$n(C+n(C+n(C+n(C))))=n(C+n(C))$	$[839 \to 3]$
883	$n(n(C+n(C))+n(C+C+n(C+n(C))))=C$	$[844 \to 3]$
946	$n(C+n(C+n(C)+n(C+C+n(C+n(C)))))=n(C+C+n(C+n(C)))$	$[883 \to 3]$
1706	$n(C+C+n(C+n(C)))=n(C)$	$[839 \to 280,\text{simp:}946]$
1734	$D+n(C+n(C))=D$	$[1706 \to 8,\text{simp:}56,\text{flip}]$
1745	$C+n(C+n(C))=C$	$[2' \to 1734',\text{simp:}2]$
1802	$n(C+n(C))+x=x$	$[1745' \to 3,\text{simp:}842,\text{flip}]$

From Equation 1802, we have a term e, namely $n(C+n(C))$, such that $e+e=e$. Hence, the result follows by Lemma 0. Q.E.D.

Appendix C

The program EQP (including the source code), the input files, and the EQP proofs referred to in this article are available on the World Wide Web through the page

`http://www.mcs.anl.gov/home/mccune/ar/robbins/`

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