A REMARK ON A WEIGHTED LANDAU INEQUALITY OF KWONG AND ZETTL

R. C. Brown
D. B. Hinton
M. K. Kwong

Abstract. In this note we extend a Theorem of Kwong and Zettl concerning the inequality

$$\int_0^\infty t^\beta |u'|^p \le K \left(\int_0^\infty t^\gamma |u|^p \right)^{1/2} \left(\int_0^\infty t^\alpha |u''|^p \right)^{1/2}$$

to all α, β, γ such that $\beta = (\alpha+\gamma)/2$ except for the triple: $\alpha = p-1$, $\beta = -1$, $\gamma = -1-p$. In this case the inequality is false; however u satisfies the inequality

$$\int_0^\infty t^\beta |u'|^p \le K_1 \left\{ \left(\int_0^\infty t^\gamma |u|^p \right)^{1/2} \left(\int_0^\infty t^\alpha |u''|^p \right)^{1/2} + \int_0^\infty t^\gamma |u|^p \right\} .$$

1. Notation. Let $I = (a,b)$, $-\infty \le a < b \le \infty$, and "$AC_{loc}(I)$" denote the class of locally absolutely continuous functions on I. If α, γ are real numbers define

$$\begin{aligned}
\mathcal{D}_{\alpha\gamma}(I) &:= \left\{ u : u' \in AC_{loc}(I) : \int_I t^\gamma |u|^p,\ \int_I t^\alpha |u''|^p < \infty \right\} , \\
\mathcal{D}^0_L(I) &:= \{ u \in \mathcal{D}_{\alpha\gamma}(I)) : \lim_{t \to a^+} u(t) = 0 \} , \\
\mathcal{D}^1_L(I) &:= \{ u \in \mathcal{D}_{\alpha\gamma}(I)) : \lim_{t \to a^+} u'(t) = 0 \} , \\
\mathcal{D}^0_R(I) &:= \{ u \in \mathcal{D}_{\alpha\gamma}(I) : \lim_{t \to b^-} u(t) = 0 \} , \\
\mathcal{D}^1_L(I) &:= \{ u \in \mathcal{D}_{\alpha\gamma}(I)) : \lim_{t \to b^-} u'(t) = 0 \} , \\
\mathcal{D}_L(I) &:= \mathcal{D}^0_L(I) \cap \mathcal{D}^1_L(I) , \\
\mathcal{D}_R(I) &:= \mathcal{D}^0_R(I) \cap \mathcal{D}^1_R(I) .
\end{aligned}$$

Additionally let K denote a constant of interest whose value may change from line to line; if required different constants will be denoted by K_1, K_2, etc.

2. A weighted multiplicative inequality. In [5, Theorem 9] Kwong and Zettl proved the following result:

Theorem 1. *Suppose $1 \le p < \infty$, β, γ, and α are real numbers such that*

$$\beta = \frac{\alpha + \gamma}{2}\,. \tag{1}$$

Then there is a constant K independent of u such that the inequality

$$\int_0^\infty t^\beta |u'|^p \le K \left(\int_0^\infty t^\gamma |u|^p \right)^{1/2} \left(\int_0^\infty t^\alpha |u''|^p \right)^{1/2} \tag{2}$$

holds for $u \in \mathcal{D}_{\alpha\gamma}((0,\infty))$ if $\beta > -1$ and $\gamma > -1 - p$.

We are going to refine and extend this Theorem by proving:

Theorem 2. *Let $u \in \mathcal{D}_{\alpha\gamma}((0,\infty))$, $1 \le p < \infty$, then the inequality (2) holds if and only if the following conditions are satisfied:*

(i) $\{\alpha, \beta, \gamma\} \ne \{p-1, -1, -1-p\}$.
(ii) *β satisfies (1).*
(iii) $\lim_{t\to 0^+} u'(t) = 0$ *when* $\beta \le -1$ *and* $\beta > \alpha - p$.
(iv) $\lim_{t\to\infty} u'(t) = 0$ *when* $\beta \ge -1$ *and* $\beta < \alpha - p$.

Also in the exceptional case $\{\alpha, \beta, \gamma\} = \{p-1, -1, -1-p\}$ the inequality

$$\int_0^\infty t^{-1} |u'|^p \le K_1 \left\{ \left(\int_0^\infty t^{-1-p} |u|^p \right)^{1/2} \left(\int_0^\infty t^{p-1} |u''| \right)^{1/2} + \int_0^\infty t^{-1-p} |u|^p \right\} \tag{3}$$

is valid.

Proof. That (2) implies (1) is a statement of "dimensional balance" and follows if we introduce the change of variables $t = \lambda s$ in (2). Next suppose that $\beta \ne \alpha - p$. If $\beta > \alpha - p$, we get that $(\alpha - \gamma)/2p < 1$, and if $\beta < \alpha - p$ then $(\alpha - \gamma)/2p > 1$; thus in either case $(\alpha - \gamma)/2p \ne 1$. *Case (i):* Let $\beta \le -1$ and $\beta > \alpha - p$. Assume that (2) holds. Now

$$u'(t) = u'(s) - \int_t^s u''\,.$$

Since $\alpha < p - 1$, $\alpha p'/p < 1$, so that $t^{-\alpha p'/p}$ is integrable if $p > 1$ on right neighborhoods of 0; also $t^{|\alpha|}$ is bounded near 0 if $p = 1$. These facts and Hölder's inequality imply that $\lim_{t\to 0^+} \int_t^s u''$ is finite; consequently $u'(0^+)$ exists. Since $t^\beta |u'|^p$ is integrable (by (2)) while t^β is not integrable on right neighborhoods of 0, $u'(0^+) = 0$. On the other hand, suppose $u'(0^+) = 0$. Because $\beta > \alpha - p$ a form of Hardy's inequality for $\mathcal{D}^1_L((0,1])$ (see [6, Example 6.8(i)) gives

$$\int_0^1 t^\beta |u'|^p \le K \int_0^1 t^{\beta+p} |u''|^p < K \int_0^1 t^\alpha |u''|^p$$

when $\beta < -1$, and

$$\int_0^1 t^\beta |u'|^p < K \int_0^1 t^{\alpha-p} |u''|^p \leq K \int_0^1 t^\alpha |u''|^p$$

when $\beta = -1$. The sum inequality on $\mathcal{D}_L^1((0,1])$

$$\int_0^1 t^\beta |u'|^p < K \left\{ \int_0^1 t^\gamma |u|^p + \int_0^1 t^\alpha |u''|^p \right\} \tag{4}$$

follows trivially. By existing theory (take $I = (1,\infty)$, $\delta := (\alpha-\gamma)/2p < 1$, and $\epsilon = 1$ in [1, Example 1]) we obtain the sum inequality

$$\int_1^\infty t^\beta |u'|^p \leq K \left\{ \int_1^\infty t^\gamma |u|^p + \int_1^\infty t^\alpha |u''|^p \right\} \tag{5}$$

on $\mathcal{D}_{\gamma\alpha}([1,\infty))$. Addition of (4) and (5) gives the sum inequality on $\mathcal{D}_L^1((0,\infty)$. Next set $t = \lambda s$. Then $u_\lambda := u(\lambda s)$ is in $\mathcal{D}_{\alpha\gamma}((0,\infty))$ so that

$$\int_0^\infty s^\beta |u_\lambda'(s)|^p \, ds < K \left\{ \int_0^\infty s^\gamma |u_\lambda(s)|^p \, ds + \int_0^\infty s^\alpha |u_\lambda''(s)|^p \, ds \right\},$$

which is equivalent to the inequality

$$\int_0^\infty t^\beta |u'(t)|^p \, dt < K \left\{ \lambda^\phi \int_0^\infty t^\gamma |u(t)|^p \, dt + \lambda^{-\phi} \int_0^\infty t^\alpha |u''(t)|^p \right\} dt \tag{6}$$

where $\phi = (\alpha-\gamma)/2 - p$. (2) follows by minimizing the right side of (6) with respect to λ (the minimization is possible since $(\alpha-\gamma)/2p \neq -1$).

The other possibilities concerning β follow a similar logic. *Case (ii):* Assume $\beta > \{-1, \alpha - p\}$. Then Hardy's inequality for $\mathcal{D}_R^1((0,1]$ (see [6, Example 6.8(ii)]), Minkowski's inequality, and the integrability of t^β on $(0,1]$ gives

$$\int_0^1 t^\beta |u'|^p \leq K \left\{ \int_0^1 t^\alpha |u''|^p + |u'(1)|^p \right\}. \tag{7}$$

Since [1, Lemma 2.1]

$$|u'(1)| \leq K \left\{ \int_1^2 |u| + \int_1^2 |u''| \right\}, \tag{8}$$

a standard Hölder's inequality argument applied to (8) in conjunction with (7) yields that

$$\int_0^1 t^\beta |u'|^p < K \left\{ \int_0^\infty t^\gamma |u|^p + \int_0^\infty t^\alpha |u''|^p \right\}. \tag{9}$$

Since (5) remains valid, addition of (5) and (9) gives the sum inequality on $(0,\infty)$ and the the same scaling argument as in the previous case may be applied. *Case (iii):* If $\beta < \alpha - p$ and $\beta < -1$, Hardy's inequality for $\mathcal{D}^1_L([1,\infty))$ (see [6, Example 6.9(i)]), Minkowski's inequality, Lemma 2.1 of [1], etc., give as in *Case (ii)* the sum inequality (5). On the other hand since $(\alpha-\gamma)/2p \geq 1$, existing theory (see [1, Example 2]) gives

$$\int_0^1 t^\beta |u'|^p \leq K \left\{ \int_0^1 t^\gamma |u|^p + \int_0^1 t^\alpha |u''|^p \right\} ; \tag{10}$$

we then add and scale as before. *Case (iv):* If $\beta < \alpha - p$ and $\beta \geq -1$, we can show by an argument similar to *Case (i)* that $u'(\infty) := \lim_{t\to\infty} u'(t) = 0$ is a necessary condition for (2). Furthermore if $u'(\infty) = 0$ Hardy's inequality for $\mathcal{D}^1_R([1,\infty))$ (see [6, Example 6.9(ii)]) then leads trivially to (5). Adding this to (10) (the argument of *Case (iii)* continues to apply) and finishing the argument as before completes the proof.

Now suppose that $\beta = \alpha - p$. Let $N = t^\beta$, $W = t^\gamma$, $P = t^\alpha$, $f(t) = t^\delta$ where $\delta := (\alpha-\gamma)/2p = 1$, and $J_{t,\epsilon} = [t, t(1+\epsilon)]$. Then a calculation (see [1, (2.13)]) shows that

$$\int_{J_{t,\epsilon}} N|u'|^p \leq K \left\{ \epsilon^{-p} S_2 \int_{J_{t,\epsilon}} W|u|^p + \epsilon^p S_1 \int_{J_{t,\epsilon}} P|u''|^p \right\} \tag{11}$$

where

$$\begin{aligned} S_1 &:= t^p \left((\epsilon t)^{-1} \int_t^{t(1+\epsilon)} s^{\alpha-p}\, ds \right) \left((\epsilon t)^{-1} \int_t^{t(1+\epsilon)} s^{-\alpha p'/p}\, ds \right)^{p-1} \\ &\leq \max\{1, (1+\epsilon)^\beta\} \max\{1, (1+\epsilon)^{-\alpha}\} \\ S_2 &:= t^{-p} \left((\epsilon t)^{-1} \int_t^{t(1+\epsilon)} s^{\alpha-p}\, ds \right) \left((\epsilon t)^{-1} \int_t^{t(1+\epsilon)} s^{\gamma p'/p}\, ds \right)^{p-1} \\ &\leq \max\{1, (1+\epsilon)^\beta\} \max\{1, (1+\epsilon)^{-\gamma}\} . \end{aligned} \tag{12}$$

Let $\mathcal{J}_\epsilon$ denote the collection of all $J_{t,\epsilon}, t \in (0,\infty)$. Let n be a positive integer. Since every $s \in [1/n, n]$ is the center of some $J_{t,\epsilon} \in \mathcal{J}_\epsilon$ (take $t = 2s/(2+\epsilon)$) we may appeal to the Besicovitch covering theorem (cf. [4, p.2]) to extract finitely many families $\Gamma_1, \ldots, \Gamma_s$ of disjoint intervals in $\mathcal{J}_\epsilon$ where s is independent of n to cover $[1/n, n]$. From (11) and (12) it follows that

$$\begin{aligned} \int_{\Gamma_i} N|u'|^p &\leq K(\epsilon) \left\{ \epsilon^{-p} \int_{\Gamma_i} W|u|^p + \epsilon^p \int_{\Gamma_i} P|u''|^p \right\} \\ &\leq K(\epsilon) \left\{ \epsilon^{-p} \int_0^\infty W|u|^p + \epsilon^p \int_0^\infty P|u''|^p \right\} \end{aligned}$$

where $\Gamma_i := \cup \{J_{t,\epsilon} : J_{t,\epsilon} \in \mathit{\Gamma}_i\}$. Hence

$$\int_{1/n}^{n} N|u'|^p \le sK(\epsilon)\left\{\epsilon^{-p}\int_0^\infty W|u|^p + \epsilon^p \int_0^\infty P|u''|^p\right\}.$$

Since n is arbitrary, taking $\epsilon = 1$ gives the sum inequality

$$\int_0^\infty t^\beta |u|^p \le K\left\{\int_0^\infty t^\gamma |u|^p + \int_0^\infty t^\alpha |u''|^p\right\}, \tag{13}$$

so that in particular $t^\beta|u|^p$ is integrable on $(0,\infty)$.

To obtain (2) we modify an argument previously given in the proof of [2, Theorem 2.1]: Define a bi-infinite partition $\{t_i\}_{-\infty}^{\infty}$ by letting $t_0 = 1$ and $t_i = 2^i$. Let ϕ be a C_0^∞ function with support on $[-3/4, 1]$ such that $0 \le \phi \le 1$ and $\phi = 1$ on $[1/2, 1]$. For $m \in \mathbb{Z}$ and $u \in \mathcal{D}_{\alpha\gamma}((0,\infty))$ set

$$y_m(t) = u(t)\phi((t - t_m)/t_m), \tag{14}$$

where t_m is a point of the partition defined above. It follows that y_m has support on $[t_{m-2}, t_{m+1}]$ and $y_m = u$ on $[t_{m-1}, t_m]$. It is not difficult to show applying Leibniz's rule of differentiation that there is a constant C independent of u and m such that

$$|y_m''(t)| \le C\sum_{i=0}^{2} |u^{(i)}|/t_m^{2-i}, \tag{15}$$

a.e. Next we recall that if $\alpha = \beta = \gamma = 0$ (2) is a special case of a far more general and well known Gabushin) inequality (cf. [3]). (Also note that if $p > 1$ the unweighted inequality follows from *Case (ii)* above.) Substituting (14) into this inequality and using (15) gives

$$\begin{aligned}\left(\int_{t_{m-1}}^{t_m} |u'|^p\right) &\le \left(\int_{t_{m-2}}^{t_{m+1}} |y_m'|^p\right)\\ &\le K\left(\int_{t_{m-2}}^{t_{m+1}} |y_m|^p\right)^{1/2}\left(\int_{t_{m-2}}^{t_{m+1}} |y_m''|^p\right)^{1/2}\\ &\le KC^{1/2}\left(\int_{t_{m-2}}^{t_{m+1}} |u|^p\right)^{1/2}\left(\int_{t_{m-2}}^{t_{m+1}} \sum_{i=0}^{2} |u^{(i)}|^p/(t_m^{2-i})^p\right)^{1/2}. \end{aligned} \tag{16}$$

We multiply the last line of (16) by t^β, noting both that β satisfies (1) and that if $t \in [t_{m-2}, t_m]$, then $1/4 \le t/t_m \le 2$ because of the nature of the partition. This gives

$$\int_{t_{m-1}}^{t_m} t^\beta |u'|^p \le K_1\left(\int_{t_{m-2}}^{t_{m+1}} t^\gamma |u|^p\right)^{1/2}\left(\int_{t_{m-2}}^{t_{m+1}} \sum_{i=0}^{2} t_m^{\alpha-(2-i)p}|u^{(i)}|^p\right)^{1/2} \tag{17}$$

for a constant K_1 independent of u. Summing (17) over m and using the discrete sum form of the Cauchy-Schwartz inequality yields that

$$\int_0^\infty t^\beta |u'|^p \le K_1 \left(\sum_{m=1}^\infty \int_{t_{m-2}}^{t_{m+1}} t^\gamma |u|^p \right)^{1/2} \left(\sum_{m=0}^\infty \int_{t_{m-2}}^{t_{m+1}} \left(\sum_{i=0}^2 t_m^{\alpha-(2-i)p} |u^{(i)}|^p \right) \right)^{1/2}$$

Because each t belongs in at most three intervals $[t_{m-2}, t_{m+1}]$ and by Minkowski's inequality applied to the last integral in (18), it follows that

$$\int_0^\infty t^\beta |u'|^p \le K_2 \left(\int_0^\infty t^\gamma |u|^p \right)^{1/2} \left[\sum_{i=0}^2 \left(\int_0^\infty t_m^{\alpha-(2-i)p} |u^{(i)}|^p \right) \right]^{1/2} . \tag{19}$$

Assume now that $\beta < -1$ so that $\gamma < -1$. Because $\alpha < p-1$, $\beta < -1$, and $t^\beta |u'|^p$ is integrable an argument given in *Case (i)* shows that $\lim_{t \to 0^+} u'(t) = 0$. Similarly the fact that

$$\frac{-\beta p'}{p} > \frac{1}{p-1} > 0$$

together with Hölder's inequality applied to the integral in the identity

$$u(t) = u(s) - \int_t^s u'$$

demonstrates that $\lim_{t \to 0^+} u(t)$ exists. Since $\gamma < -1$ the limit is 0. This shows that $\mathcal{D}_{\alpha\gamma}((0,\infty)) = \mathcal{D}_L((0,\infty))$. Since $\alpha - p < -1$, iteration of Hardy's inequality for $\mathcal{D}_L((0,\infty))$ (cf. [6, Example 6.7]) gives

$$\int_0^\infty t^{\alpha-(2-i)p} |u^{(i)}|^p \le K_3 \int_0^\infty t^\alpha |u''|^p . \tag{20}$$

Substitution of (20) into (19) yields (2). The case $\beta > -1$, $\gamma > -1$ is covered by Theorem 1. However an argument similar to the previous case shows that this alternative implies that $\lim_{t \to \infty} u^{(i)}(t) = 0$ for $i = 0, 1$; so that $\mathcal{D}_{\alpha\gamma}((0,\infty)) = \mathcal{D}_R((0,\infty))$. Since (20) holds on $\mathcal{D}_R((0,\infty))$ if $\alpha - p > -1$ we can substitute it into (19) as before to complete the proof. Summarizing, (2) holds for all choices of α, β, and γ satisfying (1) except possibly for

$$\begin{aligned} \alpha &= p-1 , \\ \beta &= -1 , \\ \gamma &= -1-p , \end{aligned} \tag{21}$$

which was to be proved.

We next show by a counterexample that (2) cannot hold in the exceptional case (21), Let

$$u_\delta(t) := \begin{cases} u_{1,\delta}(t) \equiv t^{1+\delta} & \text{for } t \in [0,1] \\ u_{2,\delta} \equiv ((1+\delta)t^{1-\delta} - 2\delta)/(1-\delta) & \text{for } t \in (1,\infty) \end{cases}$$

where $\delta > 0$ is a parameter. Since u_δ and u'_δ are continuous at 1, $u_\delta \in \mathcal{D}_{\alpha\gamma}((0,\infty))$. To prove that (2) cannot hold for this family of functions it is sufficient to show that if

$$Q(u_\delta) := \frac{\left(\int_0^\infty t^{-1}|u'_\delta|^p\right)^2}{\left(\int_0^\infty t^{-1-p}|u_\delta|^p\right)\left(\int_0^\infty t^{p-1}|u''_\delta|\right)},$$

then

$$\lim_{\delta\to 0} Q(u_\delta) = \infty\,. \tag{22}$$

A calculation yields that

$$\int_0^\infty t^{-1}|u'_\delta|^p = \frac{2(1+\delta)^p}{p\delta} \tag{23}$$

$$\int_0^\infty t^{p-1}|u''_\delta|^p = \frac{2\delta^p(\delta+1)^p}{p\delta}\,. \tag{24}$$

Moreover

$$\begin{aligned} \int_1^\infty t^{-1-p}|u_{2,\delta}|^p &< \frac{\int_1^\infty (1+\delta)t^{1-\delta}}{1-\delta} \\ &< \frac{(1+\delta)^p}{p\delta(1-\delta)^p}\,, \end{aligned} \tag{25}$$

so that

$$\begin{aligned} \int_0^\infty t^{-1-p}|u_\delta|^p &= \frac{1}{p\delta} + \int_1^\infty t^{-1-p}|u_2|^p \\ &< \frac{1}{p\delta} + \frac{(1+\delta)^p}{p\delta(1-\delta)^p}\,. \end{aligned} \tag{26}$$

Combining (25) and (26) and substituting them together with the estimates (23) and (24) into (22) gives

$$\lim_{\delta\to 0} Q(u_\delta) \geq \lim_{\delta\to 0} \frac{2(1+\delta)^p}{\delta^p\left(1+\frac{(1+\delta)^p}{(1-\delta)^p}\right)}$$
$$= \infty .$$

It remains to prove (3) in the exceptional case: Let $f(t) = t$, $N = t^{-1}$, $W = t^{-1-p}$, and $P = t^{p-1}$ in condition (C_3) of [1]. Then a calculation similar to (12) above shows that

$$S_1 \leq \epsilon$$
$$S_2 \leq \epsilon^{-p}(1+\epsilon)^{p+1} .$$

This yields as in the Besicovitch covering argument leading to (13) the sum inequality

$$\int_0^\infty t^{-1}|u|^p \leq K\left\{\epsilon^{-p}(1+\epsilon)^{p+1}\int_0^\infty t^{-1-p}|u|^p + \epsilon^p\int_0^\infty t^{p-1}|u''|^p\right\} . \tag{27}$$

We substitute the elementary inequality

$$(1+\epsilon)^{p+1} \leq 2^p\left(1+\epsilon^{p+1}\right)$$

into (27) and then minimize

$$2^p\epsilon^{-p}\int_0^\infty t^{-1-p}|u|^p + \epsilon^p\int_0^\infty t^{p-1}|u''|^p$$

on the right-hand side. The result is (3) with $K_1 = \max\{2K, \epsilon_0^p\}$ where ϵ_0 is minimizing value of ϵ . The proof is complete.

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Department of Mathematics University of Alabama Tuscaloosa, AL 35487-0350

Department of Mathematics University of Tennessee Knoxville, TN 37996-1300

Mathematics and Computer Science Division Bldg 221 Argonne National Laboratory Argonne, IL 60439