THE EXISTENCE OF MULTIPLE SOLUTIONS FOR A GINZBURG-LANDAU TYPE MODEL OF SUPERCONDUCTIVITY

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Abstract. We study the bifurcation curve for a system of two second order differential equations with cubic nonlinearity which model a thin film of superconductor subjected to a tangential magnetic field. We verify some recent conjectures of one of the authors about this curve, showing in particular that for a range of parameter values the relevant boundary value problem has three symmetric solutions.

1. Introduction

In this paper we study the following system of ordinary differential equations, in unknown functions (ϕ, a):

$$\phi'' = K(\phi(x)^2 + a(x)^2 - 1)\phi \tag{1.1}$$

$$a'' = L\phi(x)^2 a(x) \tag{1.2}$$

together with boundary conditions

$$\phi'(0) = \phi'(1) = 0 \tag{1.3}$$

$$a(0) = 0 \tag{1.4}$$

where we further impose the conditions

$$\phi(x) > 0, \;\; \phi'(x) \leq 0 \tag{1.5}$$

on the relevant interval $0 \leq x \leq 1$. The goal is to determine how the quantity $h = a'(1)$ varies with the undetermined initial condition $\beta = \phi(0)$.

This problem was recently studied in [2], and some of its physical significance is described there, together with many further references. Of these, we cite [1], [3] , and [4] as being accessible and containing the needed background. Here we shall just observe that the variable ϕ is called the "order parameter", and the variable a is the potential, a

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scalar quantity in the one dimensional case represented by the above system. In higher dimensions, ϕ is complex valued and the potential is a vector quantity.

K and L are parameters, with L being the appropriate length scale to reduce the problem to the non-dimensional form of (1.1)-(1.2) and K related to curvature. Also, h is a non-dimensionalized quantity proportional to the external magnetic field in the 1-d case.

We now summarize the main results in [2] on the problem (1.1)-(1.5). First the existence of a solution for fixed positive K and L is discussed. To give the results, let $(\phi(x,\alpha,\beta), a(x,\alpha,\beta))$ denote the unique solution to (1.1)-(1.2) such that $\phi(0) = \beta, \phi'(0) = 0, a(0) = 0, a'(0) = \alpha$. It is shown that for each $\beta \in (0,1)$, there is a unique $\alpha > 0$ such that (ϕ, a) satisfy (1.1)-(1.5). In addition, it is proved that α is continuous and monotonically decreasing over the interval $0 \leq \beta \leq 1$ and that $\alpha(1) = 0$. We can also define $h(0)$ by letting $\phi = \beta u$ and considering the differential equations and boundary conditions for (u, a). If we let $h(\beta) = a'(1, \alpha(\beta), \beta)$, then $h(0) > 0, h'(0) = 0, h(1) = 0$. It is further shown in [2] that h is a continuous function on $[0,1]$ and that $h'(0) = 0$.

The reference [2] also contains an extensive numerical study of the problem. It is found that for some values of K and L, the function h is monotone decreasing in $[0,1]$, while for others, h first increases and then decreases to 0. Most interesting for us was that for a small range of K and L, the numerical results indicated that $h(\beta)$ first decreases, then increases, and then decreases to zero. This, if true, would mean that for some fixed values of h, there are three solutions to (1.1)-(1.5) such that $a'(1) = h$. It is this which we set out to prove in Theorem 1.

We did further numerical experiments, and extended the range for K and L given in [2] for which the function h has both a local minimum and a local maximum; i.e., there can be three solutions for a given value of h. We found that the phenomenon became more pronounced as K and L were increased, and this led us to consider what happens as K and L tend to infinity. It turns out that the ratio of L to K is significant. The main results in this paper are the following:

Theorem 1. *Suppose that r is a number in the interval $(0,2)$. Let $L = rK$. Then for sufficiently large K, the function $h(\beta)$ first decreases, then increases, and finally decreases to 0 at $\beta = 1$.*

Theorem 2. *If $r \geq 2$ and $L = rK$, then for sufficiently large K, $h(\beta)$ increases above $h(0)$ and then decreases to 0 at $\beta = 1$.*

Corollary to Theorems 1 and 2. *If $0 < r < 2$ then for sufficiently large K, the problem (1.1)-(1.5) has at least three solutions with $a'(1) = h$. If $r \geq 2$, then there are values of h such that there are at least two such solutions.*

We do not show that the function $h()$ has only one maximum or one minimum.

In this section we shall just introduce one key tool in the proof, which is the use of an "energy" function. If we set

$$E(x) = (\phi')^2 + \frac{(a')^2}{r} - K(\frac{\phi^4}{2} - \phi^2 + \beta^2 - \frac{\beta^4}{2} + a^2\phi^2),$$

then use of (1.1)-(1.2) shows that $E'(x) = 0$; i.e., E is constant. ¿From (1.3)-(1.5) we further see that

$$E(0) = E(x) = \alpha^2. \tag{1.6}$$

2. Preliminaries.

Throughout the rest of the paper we let $L = rK$ in (1.2) and assume that for a given $\beta \in (0,1)$, the pair (ϕ, a) refers to a solution of (1.1)-(1.5), so that $a'(0) = \alpha(\phi(0))$. At several points in our analysis we refer to the function

$$H(x) = \phi(x)^2 + a(x)^2 - 1,$$

which is seen to satisfy the initial value problem

$$H'' = 2K\phi^2 H + 2((\phi')^2 + (a')^2 + rK\phi^2 a^2) \tag{2.1}$$

$$H(0) = \beta^2 - 1, \;\; H'(0) = 0. \tag{2.2}$$

Lemma 2.1. *For each $\beta \in (0,1)$ there is a unique $\hat{x} = \hat{x}(\beta) \in (0,1)$ such that $H(\hat{x}) = 0$ and $H > 0, \;\; H' > 0$ on $(\hat{x}, 1)$.*

Proof: Since $\phi'(0) = \phi'(1) = 0$, we see that

$$0 = \int_0^1 \phi''(x)dx = \int_0^1 \phi(x)H(x)dx,$$

and since $\phi > 0$, there must be some $\hat{x}$ such that $H(\hat{x}) = 0$. That $\hat{x}$ is unique, and $H, H' > 0$ in $(\hat{x}, 1)$ follows from (2.1)-(2.2).

Clearly, then, $\phi'' < 0$ on $[0, \hat{x})$ and $\phi'' > 0$ on $(\hat{x}, 1]$.

In addition to H we also need to know basic properties of the ratio $\frac{a'}{a}$.

Lemma 2.3. *Let $\beta \in (0,1)$. Then*

$$(\frac{a'}{a})' < 0$$

and

$$\frac{a'}{a} \geq \sqrt{rK}\phi > 0$$

for $0 < x < 1$.

Proof: Let $S = \frac{a'}{a}$. Then S satisfies

$$S' = rK\phi^2 - S^2. \tag{2.3}$$

¿From (2.1) and (2.2) we conclude that $a' > 0$ and $a > 0$ on $(0,1)$, and that

$$\lim_{x \to 0+} S = \infty. \tag{2.4}$$

Differentiating this gives $S'' + 2SS' = 2Kr\phi\phi'$. From this we see that $S'' < 0$ at any point where $S = 0$, which, with (2.8), imply the result.

3. Behavior of $h(\beta)$ near $\beta = 0$.

In this section we begin our analysis of the the function

$$h(\beta) = a'(1, \alpha(\beta), \beta).$$

For β near 0 it is convenient to rescale (1.1)-(1.2) by setting $\phi = \beta u$. Then u satisfies

$$u'' = Ku(\beta^2 u^2 + a^2 - 1) \tag{3.1}$$

$$a'' = K\beta^2 u^2 a \tag{3.2}$$

$$u(0) = 1, \quad u'(0) = u'(1) = 0, \quad a(0) = 0.$$

Recall that $a'(0) = \alpha(\beta) > 0$. Kwong proves that as $\beta \to 0$, $h(\beta) \to h_0 = \alpha(0) > 0$ where h_0 is the number such that the linear problem

$$u'' = Ku(h_0^2 x^2 - 1) \tag{3.4}$$

$$u(0) = 1, \quad u'(0) = u'(1) = 0 \tag{3.5}$$

has a solution u with $u > 0$ on $[0,1]$. A simple Sturm-Liouville argument shows that h_0 is unique.

Lemma3.1. *For all positve K, $\sqrt{K} < h_0 < \sqrt{K+3}$.*

Proof: To obtain the first inequaltiy, we compare the solutions of (3.4) with those of the related problem

$$w'' = Kw(Kx^2 - 1)$$

$$w(0) = 1, \quad w'(0) = 0.$$

The solution of this problem is $w(x) = e^{-Kx^2/2}$. Setting $q = \frac{w'}{w}$,we see that $q = -Kx$ satsifies

$$q' + q^2 = K(Kx^2 - 1). \tag{3.6}$$

Let $\rho = \frac{u'}{u}$. Then ρ satisfies

$$\rho' + \rho^2 = K(h_0^2 x^2 - 1) \tag{3.7}$$

$$\rho(0) = \rho(1) = 0. \tag{3.8}$$

Let $f = \rho - q$. Then f satisfies

$$f' + (f + 2q)f = Kx^2(h_0^2 - K), \tag{3.9}$$

$$f(0) = 0, \;\; f(1) = K.$$

¿From (3.9) we see that if $h_0^2 \leq K$,then $f \leq 0$ on (0,1] so that $f(1) \leq 0$, a contradiction which proves the first inequality of Lemma 3.1.

Turning to the proof of the second part, note that in addition to (3.7) we have $\rho'(0) = -K$, $\rho''(0)$ =0, and $\rho'' + 2\rho\rho' = 2Kh_0^2 x$. From what we have already shown, it follows that $\rho'''(0) > 0$. Therefore $\rho'' > 0$, $\rho' > -K$ on some interval $(0, \bar{x})$. In addition,

$$(\rho'' e^{\int_0^x 2\rho\, dt})' = 2(Kh_0^2 - \rho'^2)e^{\int_0^x 2\rho\, dt} > 0$$

for $0 < x < 1$ as long as $(\rho')^2 < K$.

Thus, $\rho'' > 0$ as long as $\rho' > -K$. Hence, $\rho' > -K$, $\rho'' > 0$ on (0,1). It is then easily found that $\rho''' \geq 2(Kh_0^2 - K^2)$ on (0,1). Integrating this twice show that $\rho \geq -Kx + \frac{x^3}{3}(Kh_0^2 - K^2)$, and then the relation $\rho(1) = 0$ easily implies that $h_0^2 \leq K + 3$. This completes the proof of Lemma 3.1.

4 The Dip in $h(\beta)$.

In this section and the next we restrict our attention to values of β in the interval $(1 - \frac{1}{\sqrt{K}}, 1)$. Our goal now is to prove that for each large K, there is a $\beta = \beta_1(K) \in (1 - \frac{1}{\sqrt{K}}, 1)$ such that

$$h(\beta_1(K)) = a'(1) \sim (\frac{rK}{2})^{1/2} \tag{4.1}$$

as $K \to \infty$. The consequence of this result is that for a fixed $r \in (0, 2)$, $h(\beta_1) < h(0)$ for large K. Thus, as β increases from 0 to β_1, the function h decreases to $h(\beta_1)$. We do not show that this function is monotone in this interval.

Choose $K \geq e$, so that

$$\frac{1}{\sqrt{K}} \leq \frac{\ln K}{\sqrt{K}} < 1. \tag{4.2}$$

Lemma4.1. *Suppose that $0 < r < 2$. Then there is a $K_1 > e$ such that if $K \geq K_1$ and $\beta = 1 - \frac{1}{\sqrt{K}}$,then $\phi(\frac{1}{2}) \leq \frac{100}{\sqrt{K}}$.*

Proof: We assume, on the contrary, that $\phi(\frac{1}{2}) > \frac{100}{\sqrt{K}}$ for large K (or, if necessary, along an unbounded sequence of $K's$). We claim that

$$\phi(\frac{1}{8}) > \frac{1}{8}$$

for large K. If not, then there is a unique $x_0 \in (0, \frac{1}{8})$ such that $\phi(x_0) = \frac{1}{8}$. By the mean value theorem there is a $\gamma \in (0, \frac{1}{8})$ such that

$$\phi'(\gamma) = \frac{\phi(x_0) - \phi(1)}{x_0} \leq 8(x_0 - 1) \leq -6$$

for large K. By Lemma 2.1, there is a unique $\hat{x} \in (0,1)$ such that $\phi\,''(x) = 0$. If $\hat{x} > \frac{1}{4}$, then $\phi' < -6$ over $[\frac{1}{8}, \frac{1}{4}]$ and an integration gives $\phi(\frac{1}{4}) < 0$, a contradiction. If $\hat{x} \leq \frac{1}{4}$, then $\phi''(\frac{1}{4}) \geq 0$ so that $a^2(\frac{1}{4}) \geq 1 - \phi(\frac{1}{4})^2$. Therefore $a(\frac{1}{4}) \geq \sqrt{\frac{63}{64}}$ since $\phi(\frac{1}{4}) \leq \phi(\frac{1}{8}) \leq \frac{1}{8}$. This and the mean value theorem imply that there is a z in $(0, \frac{1}{4})$ such that $a'(z) = 4a(\frac{1}{4}) \geq 3$.

We now use the function H. Since $\hat{x} \leq \frac{1}{4}$, Lemma 2.1 shows that $H(\frac{1}{4}) \geq 0$ and $H' \geq 0$ on $[\frac{1}{4}, \frac{1}{2}]$. Furthermore, Equation (1.2) implies that $a'' > 0$ on $(\frac{1}{4}, \frac{1}{2}]$ and therefore, using (2.1), we conclude that on this interval, $H'' \geq 18$. Two integrations give $H \geq 9(x - \frac{1}{4})^2$. Substituting this in (1.1) and integrating shows that $\phi'(\frac{1}{2}) \geq \phi'(\frac{1}{4}) + \frac{300}{64}\sqrt{K}$.

If $\phi'(\frac{1}{4}) \geq -\sqrt{K}$, then $\phi'(\frac{1}{2}) > 0$, a contradiction. But note that by (1.1), $\phi'' \geq -K\phi$. Multiplying this by ϕ' and integrating shows that $\phi' \geq -\sqrt{K}$ in [0,1]. Thus we conclude that $\phi(\frac{1}{8}) \geq \frac{1}{8}$ for large K.

Now we consider the interval $[0, \frac{1}{32}]$. If $a(\frac{1}{32})^2 < \frac{1}{2\sqrt{K}}$,then from (1.1) we conclude that $\phi'' \leq -\frac{\sqrt{K}}{16}$over $[0, \frac{1}{32}]$, because $\frac{1}{8} \leq \phi \leq 1 - \frac{1}{\sqrt{K}}$ and $a' > 0$. Two integrations give $\phi(\frac{1}{32}) < 0$ for large K, a contradiction. Hence, there is an x_1 in $(0, \frac{1}{32})$ such that $a(x_1) = \frac{K^{-1/4}}{\sqrt{2}}$. From Lemma 2.3 we conclude that $a' \geq \frac{\sqrt{rK}}{8}a$ over $[x_1, \frac{1}{8}]$. Integrating this from $\frac{1}{32}$to $\frac{1}{8}$ gives $a(\frac{1}{8})^2 \geq \frac{e^{\frac{\sqrt{rK}}{128}}}{2\sqrt{K}}$. Since $a' > 0$, we find from (1.1) that $\phi'' \geq 25e^{\sqrt{rK}/128}$ for $\frac{1}{8} \leq x \leq \frac{1}{4}$ and K large. Integrating this for K large shows that $\phi'(\frac{1}{2}) > 0$, a final contradiction which proves Lemma 4.1.

To prove the asymptotic formula (4.1), we use the energy identity (1.6), which reduces, at $x = 1$, to

$$\frac{a'(1)^2}{rK} = \frac{a'(0)^2}{rK} + \beta^2 - \frac{\beta^4}{2} - \phi(1)^2 + \frac{\phi(1)^4}{2} + a(1)^2\phi(1)^2$$

In Lemma 4.1 we set $\beta = 1 - \frac{1}{\sqrt{K}}$ and found that $\phi(\frac{1}{2}) \leq \frac{100}{\sqrt{K}}$. To obtain the desired result, we would like to show that $a(1)\phi(1) \to 0$ and $a'(0) \to 0$, as $K \to \infty$. However we were unable to prove this for $\beta = 1 - \frac{1}{\sqrt{K}}$, so we must consider higher values of β.

Lemma4.2. *Let $0 < r < 2$. For each sufficiently large K, there is a $\beta_1 = \beta_1(K)$ such that $\phi(\frac{1}{2}) = \frac{\ln K}{\sqrt{K}}$. In addition, a and ϕ satisfiy the following:*

$$\begin{array}{ll} (i) & \phi(1) \to 0 \text{ as } K \to \infty \\ (ii) & a'(0) \to 0 \text{ as } K \to \infty \\ (iii) & a(1)\phi(1) \to 0 \text{ as } K \to \infty \\ (iv) & \dfrac{a'(1)^2}{rK} \to \dfrac{1}{2} \text{ as } K \to \infty \end{array}$$

(Property (iv) is equivalent to (4.1)). Proof: For $\beta = 1 - \frac{1}{\sqrt{K}}$, Lemma 4.1 implies that $\phi(\frac{1}{2}) \leq \frac{\ln K}{\sqrt{K}}$ for large K. Thus, since $\phi(\frac{1}{2}) \to 1$ as $\beta \to 1$, there is a first $\beta_1(K) \in (1-\frac{1}{\sqrt{K}}, 1)$ such that

$$\phi(\frac{1}{2}) = \frac{\ln K}{\sqrt{K}}.$$

This, in particular, proves (i). Next, if (ii) is false, then there exists a $\mu > 0$ such that $a'(0) \geq \mu$ for all large K (or, if necessary, along a sequence of $K's$ tending to ∞). Since $\phi' < 0$ on (0,1), (1.2) implies that

$$a'' \geq r \ln(K)^2 a \tag{4.3}$$

for $0 \leq x \leq \frac{1}{2}$. A standard Sturm-Liouville argument leads to

$$a \geq \frac{\mu}{\sqrt{r}\ln K} \sinh(\sqrt{r}\ln K\, x), \quad 0 \leq x \leq \frac{1}{2}.$$

For large K,and $\frac{1}{4} \leq x \leq \frac{1}{2}$,this further implies that

$$a \geq 2\mu e^{\sqrt{r}\ln K/8}. \tag{4.4}$$

Substituting (4.3) and (4.4) into (1.1), we obtain

$$\phi'' \geq \sqrt{K}\ln K\ \mu^2 e^{\sqrt{r}\ln K/4} \tag{4.5}$$

for $\frac{1}{4} \leq x \leq \frac{1}{2}$ and $K >> 1$.

Recalling from the proof of Lemma 4.1 that $\phi' \geq -\sqrt{K}$, we integrate (4.5) and obtain $\phi'(\frac{1}{2}) > 0$ for large K, a contradiction. This proves (ii), and we turn to (iii).

Once again turning to the mean value theorem, we find that there must be a $z = z(K) \in (0, \frac{1}{2})$ such that

$$\phi'(z) = 2(\frac{\ln K}{\sqrt{K}} - \beta_1).$$

Therefore, for large K, $\phi'(z) \leq -1$. If $\phi'' < 0$ over $[0, \frac{9}{16}]$,then $\phi' \leq -1$ on $[\frac{1}{2}, \frac{9}{16}]$,and an integration over this interval gives $\phi(\frac{9}{16}) \leq \frac{\ln K}{\sqrt{K}} - \frac{1}{16} < 0$ for large K, again a contradiction.

It follows that for large K, $\phi''(\hat{x}) = 0$ at some $\hat{x} \in (0, \frac{9}{16})$, and therefore $\phi'' > 0$ for $\hat{x} < x < \frac{9}{16}$.

Next, we consider the interval $[\frac{5}{8}, \frac{3}{4}]$. If $\lim_{K\to\infty} \phi'(\frac{5}{8}) \neq 0$, then there exists an $\eta > 0$ such that $\phi'(\frac{5}{8}) \leq -\eta$ for arbitrarily large values of K. Then, $\phi' \leq -\eta$ over $[\frac{9}{16}, \frac{5}{8}]$, since $\phi'' > 0$ over this interval. Again, an integration gives $\phi(\frac{5}{8}) < 0$ for large K, a contradiction. Hence we see that $\phi'(\frac{5}{8}) \to 0$ as $K \to \infty$. Also, since $\phi''(\frac{5}{8}) > 0$ and $\phi(\frac{5}{8}) \leq \frac{\ln K}{\sqrt{K}}$, we can conclude that $a(\frac{5}{8})^2 \geq 1 - \frac{\ln(K)2}{K}$. Again using the mean value theorem, there is a point in $(0, \frac{5}{8})$ where $a' = \frac{8}{5}a(\frac{5}{8}) \geq \frac{7}{5}$. Thus, $a'(\frac{5}{8}) \geq \frac{7}{5}$ because $a'' > 0$ on (0,1]. Therefore, an integration of $a' \geq \frac{7}{5}$ over $[\frac{5}{8}\ , \frac{3}{4}]$ shows that $a(\frac{3}{4}) \geq 1 + \delta a(\frac{3}{4})^2$ for large K and some $\delta > 0$ independent of K. Thus, $\phi'' \geq K\phi\delta a^2$on $[\frac{3}{4}, 1]$. If, for K large, $K\phi\delta a^2 > 1$ everywhere in $[\frac{3}{4}, \frac{7}{8}]$, then $\phi'' \geq 1$ and an integration over this last interval gives the contradiction that $\phi'(\frac{7}{8}) > 0$.

Hence we arrive at

$$K\phi(\check{x})\delta a(\check{x})^2 \leq 1 \tag{4.6}$$

for some $\check{x} \in [\frac{3}{4}, \frac{7}{8}]$. Since $\phi' < 0$ it follows that

$$\phi(1) \leq \phi(\check{x}) \leq \frac{1}{K\delta a(\check{x})^2}. \tag{4.7}$$

To obtain an upper bound for $a(1)$ we recall that (4.2) holds, and so $\rho = \frac{a'}{a}$ satsifies $\rho' \leq r\ln(K)^2 - \rho^2$ for $\frac{1}{2} \leq x \leq 1$. An integration from $\frac{1}{2}$ to $\frac{5}{8}$ gives $\rho(\frac{5}{8}) \leq 2\sqrt{r}\ln(K)$ for large K. Hence $\rho \leq 2\sqrt{r}\ln K$ over $[\frac{5}{8}, 1]$. Setting $\rho = \frac{a'}{a}$ and integrating from x̌ to 1 gives $a(1) \leq a(\check{x})e^{2\sqrt{r}\ln K\ (1-\check{x})}$ which implies that

$$a(1) \leq a(\check{x})e^{\ln K/2}. \tag{4.8}$$

Combining (4.6) and (4.8), we obtain that

$$\phi(1)a(1) \leq \frac{-\frac{K}{2}}{\delta a(\check{x})}.$$

Therefore, $\phi(1)a(1) \to 0$ as $K \to \infty$. Part (iv) of Lemma 4.2, and hence relation (4.1), follow from (i), (ii), (iii), and (4.2).

5. The region where h exceeds $h(\beta_1)$.

We have now shown that for large K, $h(0) \geq \sqrt{K}$, while

$$\lim_{K\to\infty} \frac{h(\beta_1(K))}{\sqrt{(rK)}} = \frac{1}{2}.$$

To prove Theorem 1, we have left to show that for large K, we can find a β such that $h(\beta) > h(\beta_1(K))$.

The strategy here is to pick a number $\lambda \in (0, 1)$, and then choose $\beta_2(K) > \beta_1(K)$ so that

$$\phi(\beta_2(K)) = \lambda.$$

This can be done because Lemma 4.1 implies that $\phi(1, \alpha(\beta_1(K)), \beta_1(K)) \to 0$ as $K \to \infty$, while for fixed K, $\phi(1, \alpha(\beta), \beta) \to 1$. We will then show that λ can be chosen so that

$$\liminf_{K\to\infty} \frac{h(\beta_2(K))}{\sqrt{rK}} > \frac{1}{\sqrt{2}}. \tag{5.1}$$

To do this, fix r in $(0, 2)$. For large K, let $\tilde{x} = 1 - \frac{1}{\sqrt{K}}$. Also, set $\delta_1 = \frac{1}{2}(\sqrt{\frac{r}{2}} + 1)$ and $\delta_2 = \frac{1}{2}(\sqrt{\frac{r}{2}} + \delta_1)$. Finally, let $\delta = \delta_1 - \delta_2$. From now on, we restrict λ to satisfy, at least,

$$0 < \lambda < \sqrt{1 - \delta_1^2}.$$

Lemma 5.1. *For large K, if $0 < a(\tilde{x}) < \delta$, then $a'(1) > \delta_2\sqrt{K} > \sqrt{\frac{r}{2}}\sqrt{K}$.*

Remark: This shows that under the hypotheses of the Lemma, h has increased from its minimum, for large K.

Proof: Using the mean value theorem as before,

$$a(1) - a(\tilde{x}) \leq \frac{a'(1)}{\sqrt{K}}.$$

Hence, using the inequalities defining δ, we find that

$$a'(1) \geq \sqrt{K}(a(1) - a(\tilde{x})) \geq \sqrt{K}(\sqrt{1-\lambda^2} - \delta) \geq \delta_2\sqrt{K}.$$

This proves Lemma 5.1.

We have left to consider the case $a(\tilde{x}) \geq \delta$ for large K. Note that δ depends on r, but not on K or λ. We now consider the energy identity (1.6). Recall that $\beta_2(K) > 1 - \frac{1}{\sqrt{K}}$, so that $\beta_2(K) \to 1$ as $K \to \infty$. Hence, if $\beta = \beta_2(K)$, then

$$\frac{a'(1)^2}{K} - (\frac{\lambda^4}{2} + (a(1)^2 - 1)\lambda^2 + a'(0)^2) \to \frac{1}{2}.$$

If $a(1) \geq 1$, we see that

$$\liminf_{K\to\infty} \frac{a'(1)^2}{rK} \geq \frac{1+\lambda^4}{2},$$

which proves (5.1). Hence we have to consider only those large $K's$, if there are any, where $a(1) < 1$.

The following Lemma plays a crucial role. As above, we denote by $\hat{x}$ the unique point in (0,1) where $\phi'' = 0$. Also, recall that (ϕ, a) satisfies (1.1)-(1.5) with $L = rK$.

Lemma 5.2. *If $\phi(\hat{x}\ ) \geq 2\phi(1)$, then $a(1) \geq 1$.*

Proof: Using the mean value theorem and the fact that $\phi'' > 0$ on $(\hat{x}, 1)$, we find that

$$\phi(1) - \phi(\hat{x}) \geq \phi'(\hat{x})(1 - \hat{x}). \tag{5.2}$$

Also, $H'(\hat{x}) \geq 0$, so that

$$\phi'(\hat{x}) \geq -\frac{a(\hat{x})a'(\hat{x})}{\phi(\hat{x})}. \tag{5.3}$$

Further, again using the mean value theorem,

$$a(1) - a(\hat{x}) \geq a'(\hat{x})(1 - \hat{x}). \tag{5.4}$$

Substituting (5.3) in (5.2), and then substituting for $-a'(\hat{x})(1-\hat{x})$ using (5.4), we find that

$$\phi(1) - \phi(\hat{x}) \geq \frac{a(\hat{x})(a(\hat{x}) - a(1))}{\phi(\hat{x})}.$$

Now use $\phi(1) \leq \frac{\phi(\hat{x})}{2}$ and $H(x) = \phi(\hat{x})^2 + a(\hat{x})^2 - 1 = 0$ to obtain that $a(1) \geq 1$.

So now we can assume that $a(1) < 1$ and that $\phi(\hat{x}) < 2\phi(1) = \lambda$. As before, we let $\tilde{x} = 1 - \frac{1}{\sqrt{K}}$.

We now consider two cases: (i) $\phi(\tilde{x}) \geq 3\lambda$, and (ii) $\phi(\tilde{x}) < 3\lambda$. In case (i), Lemma 5.2 implies that $\tilde{x} < \hat{x}$. Since $\phi(\tilde{x}) - \phi(\hat{x}) \geq \lambda$, further use of the mean value theorem, and the fact that $\phi'' < 0$ on $(\tilde{x}, \hat{x})$, show that

$$\phi'(\hat{x}) \leq -\sqrt{K}\lambda.$$

The assumption that $a(1) < 1$ implies that

$$\phi'' < K\phi^3 < 8\lambda^3 K \tag{5.5}$$

on $(\hat{x}, 1)$. Since $1 - \hat{x} < \frac{1}{\sqrt{K}}$, it follows from two integrations of (5.5) that for small λ, $\phi'(1) < 0$, a contradiction.

Remark: The use of the upper bound on ϕ' which is negative and linear in λ, and the upper bound on ϕ'' which is cubic in λ, is crucial here. However it is interesting to note that the numerical results, as in Figure —, seem to indicate that the maximum of h occurs for fairly large λ.

Turning to case (ii), let $\bar{x} = 1 - \frac{1}{2\sqrt{K}}$. Lemma 2.3 implies that

$$a'(\tilde{x}) \geq \sqrt{rK}a(\tilde{x})\phi(\tilde{x}) \geq \sqrt{rK}\delta\phi,$$

by our earlier assumption on $a(\tilde{x})$, just after Lemma 5.2. Therefore $a' \geq \sqrt{rK}\delta\phi$ on $[\tilde{x}, 1]$. Hence

$$H' \geq 2(\phi\phi' + \delta^2\sqrt{rK}\phi)$$

on $[\tilde{x}, 1]$. Suppose that $\phi' \leq -\frac{\delta^2\sqrt{rK}}{2}$ somewhere in $[\tilde{x}, 1]$. Then because we are in case (ii), so that $\phi'' < 8K\lambda^3$, we again obtain that $\phi'(1) < 0$ for small λ, a contradiction. Therefore we can assume that $\phi' \geq -\frac{\delta^2\sqrt{rK}}{2}$ in $[\tilde{x}, 1]$ and hence that $H' \geq \delta^2\sqrt{rK}\lambda$ on this interval.

We claim that $H(\bar{x}) \geq -\frac{\delta^2\lambda\sqrt{r}}{8}$ for large K. If not, then $H < -\frac{\delta^2\lambda\sqrt{r}}{8}$ on $[\tilde{x}, \bar{x}]$, so $\phi'' < -\frac{k\delta^2\lambda^2}{8}$ on this interval, which, on integrating, gives an upper bound of the form $\phi'(\bar{x}) \leq -c\lambda^2\sqrt{K}$ for some positive c independednt of λ. For small λ this leads to the same contraction as before, namely, $\phi'(1) < 0$.

Thus it must be the case that $H(\bar{x}) \geq -\frac{\delta^2\lambda\sqrt{r}}{8}$ and $H' \geq \frac{\delta^2\sqrt{rK}\lambda}{2}$ over $[\bar{x}, 1]$. An integration gives $H(1) \geq \frac{\delta^2\sqrt{r}\lambda}{8}$ so that $a(1)^2 \geq 1 - \lambda^2 + \frac{\delta^2\sqrt{r}\lambda}{8} > 1$ for small λ, a contradiction. This completes the proof of Lemma 5.1, and Theorem 1 follows.

The proof of Theorem 2 for $r > 2$is included, since for $r > 2$, the proof of Lemma 4.2 is still valid. Together with Lemma 3.1, this shows that $h(\beta) > h(0)$ for some β, which gives the result. For the case $r = 2$ we need one final Lemma.

Lemma 5.3. *Let $r = 2$. Then, for small $\lambda > 0$,*

$$\liminf_{K \to \infty} \frac{a'(1)^2}{K} \geq 1 + \lambda^4.$$

Proof: Following the proof of Lemma 5.1, we immediately conclude that $\phi(\tilde{x}) \leq 3\lambda$. Next, we claim that $\liminf_{\lambda \to 0}(\liminf_{K \to \infty} a(\bar{x})) > 0$. Otherwise, we obtain $H \leq (3\lambda)^2 + a^2 - 1 \leq -\frac{1}{2}$ for small λ, large K, and $\tilde{x} \leq x \leq \bar{x}$. This and (1.2) give $\phi'' \leq -\frac{K\lambda}{2}$ over $[\tilde{x}, \bar{x}]$ and therefore, $\phi'(\bar{x}) \leq -\frac{\sqrt{K}\lambda}{4}$. Since $\phi \leq 3\lambda$, (1.2) gives $\phi'' \leq 27K\lambda^3$, and an integration leads to $\phi'(1) < 0$ for small λ, a contradiction. Therefore it must be the case that $a(\bar{x}) \geq \delta > 0$ for some $\delta >$) independent of K and λ. The remainder of the proof follows by repeating the arguments of Lemma 5.1, except that $\tilde{x}$ and $\bar{x}$ are replaced by $\bar{x}$ and $x^* = 1 - \frac{1}{4\sqrt{K}}$,, respectively.

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This completes the proof of Theorem 2.

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Figure Legend

Figure 1: $K = 15, r = \frac{4}{3}, L = 20, h(0) = 3.8732$

Figure 2: $K = 15, r = \frac{8}{3}, L = 40, h(0) = 3.8732$

Figure 3: $K = .15, r = \frac{8}{3}, L = .4, h(0) = 1.74$