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A Large-step Infeasible-interior-point Method for the P_*-matrix LCP

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Abstract

A large-step infeasible-interior-point method is proposed for solving $P_*(\kappa)$-matrix linear complementarity problems. It is new even for monotone LCP. The algorithm generates points in a large neighborhood of an infeasible central path. Each iteration requires only one matrix factorization. If the problem is solvable, then the algorithm converges from an arbitrary positive starting point (x^0, s^0). The computational complexity of the algorithm depends on the quality of the starting point. If a well centered starting point is feasible or close to being feasible, then it has $O((1+\kappa)\sqrt{n}L)$-iteration complexity. With appropriate initialization, a modified version of the algorithm terminates in $O((1+\kappa)^2 nL)$ steps either by finding a solution or by determining that the problem is not solvable. High-order local convergence is proved for problems having a strictly complementary solution. We note that while the properties of the algorithm (e.g. computational complexity) depend on κ the algorithm itself does not.

Key Words: linear complementarity problems, P_*-matrices, path-following, interior-point algorithm, polynomiality, superquadratic convergence.
Abbreviated Title: A large-step method for LCP.

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1 Introduction

The linear complementarity problem (LCP) consists in determining a vector pair $(x, s) \in I\!R^{2n}$ that satisfies the conditions

$$s = Mx + q, \qquad x \geq 0, \qquad s \geq 0, \qquad x^T s = 0, \tag{1.1}$$

where $q \in I\!R^n$ and $M \in I\!R^{n\times n}$. In this paper we consider problem (1.1) with M a P_*-matrix. The class of P_*-matrices was introduced by Kojima, Megiddo, Noma and Yoshise [6] and it contains many types of matrices encountered in practical applications. Let κ be a nonnegative number. A matrix M is called a $P_*(\kappa)-$matrix if

$$(1 + 4\kappa) \sum_{i\in\mathcal{I}_+(x)} x_i[Mx]_i + \sum_{i\in\mathcal{I}_-(x)} x_i[Mx]_i \geq 0 \ , \quad \forall x \in I\!R^n \ , \tag{1.2}$$

where

$$\mathcal{I}_+(x) = \{i \ : \ x_i[Mx]_i > 0\} \ , \quad \mathcal{I}_-(x) = \{i \ : \ x_i[Mx]_i < 0\} \ ,$$

or equivalently if

$$x^T Mx \geq -4\kappa \sum_{i\in\mathcal{I}_+(x)} x_i[Mx]_i \ , \quad \forall x \in I\!R^n \ . \tag{1.3}$$

The class of all $P_*(\kappa)-$matrices is denoted by $P_*(\kappa)$, and the class P_* is defined by

$$P_* = \bigcup_{\kappa\geq 0} P_*(\kappa)$$

i.e., M is a P_*-matrix if $M \in P_*(\kappa)$ for some $\kappa \geq 0$.

Obviously, $P_*(0) = PSD$ (the class of positive semi-definite matrices) and $P_*(\kappa_1) \subset P_*(\kappa_2)$ for $0 \leq \kappa_1 \leq \kappa_2$. Also we have $P_* \supset P$, where P is the the class of all matrices with positive principal minors. This follows from the fact that a P-matrix M is a $P_*(\kappa)$-matrix for $\kappa = \max\left\{-\frac{\lambda_{\min}(M)}{4\gamma(M)}, \ 0\right\}$, where $\lambda_{\min}(M)$ is the least eigenvalue of $(M + M^T)/2$, and $\gamma(M) > 0$ is the so-called P-matrix number of M (see [6, Lemma 3.3]).

Most interior-point methods for linear programming problems have been extended for monotone linear complementarity problems, i.e., for the $P_*(0)$-matrix LCP. Some of them have also been extended to the P_*-matrix LCP (cf. [4], [5], [6], [7]).

In [6] and [7] it is assumed that the starting point (x^0, s^0) is strictly feasible, in the sense that it belongs to the relative interior of the feasible set

$$\mathcal{F} = \{(x, s) \in I\!R^{2n}_+ \ : \ s = Mx + q\}.$$

Such a starting point may be very difficult to find in practical applications. Moreover the existence of a strictly feasible starting point implies that the solution set

$$\mathcal{F}^* = \{(x^*, s^*) \in \mathcal{F} \,:\, {x^*}^T s^* = 0\}$$

is nonempty and bounded, which restricts the class of problems to which the methods apply.

We note that all known infeasible-interior-point algorithms for the P_*-matrix LCP depend on the classification number κ for their implementation. However, sometime it is extremely difficult to estimate κ. In this paper, we propose a large-step path-following algorithm independent of κ. If the problem is solvable, the algorithm is globally convergent when starting from an arbitrary positive starting point (x^0, s^0). The algorithm is defined in a large neighborhood of the central path and requires one matrix factorization per iteration. If a well centered starting point is feasible or close to being feasible, then the algorithm has $O((1+\kappa)\sqrt{n}L)$-iteration complexity. If the starting point is large enough, then the iteration complexity is $O((1+\kappa)^2 nL)$. With appropriate initialization a slightly modified version of the algorithm terminates in $O((1+\kappa)^2 nL)$ steps either by finding a solution or by determining that the problem is not solvable. High-order local convergence is proved for problems having a strictly complementary solution.

We mention that our algorithm is new even for monotone LCP, where it attains $O(nL)$-iteration complexity for infeasible starting points, compared to $O(n^2L)$-iteration complexity of the algorithms, with similar work per iteration, proposed by Wright [16, 14, 15], Wright and Zhang [17], and Zhang [18]. In this paper, we have adapted the algorithm structure of "fast-safe-improve" of Wright and Zhang [17] by using new search directions and new neighborhood.

2 A Generic Large-step Infeasible-interior-point Algorithm

Our convergence results are proved under the assumption that $\mathcal{F}^*$ is not empty. It is easily seen that $(x^*, s^*) \in \mathcal{F}^*$ if and only if (x^*, s^*) is a nonnegative solution of the following nonlinear system

$$F(x,s) := \begin{pmatrix} Xs \\ Mx - s + q \end{pmatrix} = 0. \tag{2.1}$$

For any given $\epsilon > 0$ we define the set of *ϵ-approximate solutions* of (1.1) as

$$\mathcal{F}_\epsilon = \{(x,s) \in I\!R^{2n}_{++}, \quad x^T s \le \epsilon, \quad \|Mx - s + q\| \le \epsilon\}.$$

In what follows we present an algorithm that finds a point in this set in a finite number of steps. The starting point of the algorithm can be any pair of strictly positive vectors $(x^0, s^0) > 0$. First, let us define

$$p = \frac{n}{(x^0)^T s^0} X^0 s^0, \quad \rho_0 = \min\{[p]_i : 1 \le i \le n\}, \quad \tau_0 = (x^0)^T s^0 / n.$$

Obviously, we have

$$\rho_0 \le 1, \quad \sum_{i=1}^n [p]_i = n.$$

The algorithm depends on seven positive parameters:

$$0 < \alpha_{\min} < \alpha_{\max} \le 1, \tag{2.2a}$$

$$0 < \beta_{\min} < \beta_{\max}, \tag{2.2b}$$

$$0 < \rho < \gamma < 1, \qquad \rho < \overline{\rho}, \tag{2.2c}$$

and a nonnegative integer I. The complexity of the algorithm will not depend on $\overline{\rho}$ and I. However the asymptotic order of convergence depends on I.
Note that these parameters are independent of κ and n. We define a neighborhood

$$\begin{aligned}\mathcal{N} = \{(x, s, \tau, \alpha, \beta) \in I\!R^{2n+3}_{++} : \quad & Xs \ge \alpha \tau p, \quad \|Xs - \tau\, p\| \le \beta\tau, \\ & r = (\tau/\tau_0) r^0,\ \tau \le \tau_0,\ \alpha_{\min} \le \alpha \le \alpha_{\max},\ \beta_{\min} \le \beta \le \beta_{\max}\}\end{aligned}$$

and denote

$$\mu = \frac{1}{n} x^T s, \qquad r = s - Mx - q.$$

We note that $\beta \in (0,1)$ is a typical requirement of short-step algorithms. In our algorithm β does no longer have such a restriction. Therefore $\mathcal{N}$ is a large neighborhood of the infeasible central path defined by

$$Xs = \tau p, \tag{2.3a}$$

$$s = Mx + q + (\tau/\tau_0) r^0. \tag{2.3b}$$

Also, α, β may change at each step.
In our algorithm τ is driven to zero in a specified manner. Since all the points $(x, s, \tau, \alpha, \beta)$ are in $\mathcal{N}$, we have

$$\alpha_{\min}\tau \le \mu \le (1 + \beta_{\max})\tau, \tag{2.4}$$

so that μ will be driven to zero at about the same rate as τ. On the other hand $\|r\|$ is driven to zero at exactly the same rate as τ.

At the beginning of the k-th iteration of our algorithm, a point $(x^k, s^k, \tau_k, \alpha_k, \beta_k) \in \mathcal{N}$ is given. The k-th iteration consists of several steps. Each step has input $(x, s, \tau, \alpha, \beta) \in \mathcal{N}$ and output $(x^+, s^+, \tau_+, \alpha_+, \beta_+) \in \mathcal{N}$. Such a step is defined as follows:

- Choose $\lambda \in [0, 1)$ ($\lambda = 0$ for *fast step*, $\lambda \in (0, 1)$ for *safe step*).
- Solve the linear system

$$\begin{aligned} S^k u + X^k v &= \lambda \tau p - Xs, &\quad (2.5a)\\ Mu - v &= (1-\lambda) r. &\quad (2.5b) \end{aligned}$$

- Denote

$$x(\theta) = x + \theta u, \quad s(\theta) = s + \theta v.$$

- Choose α_+, β_+ such that $\alpha_{\min} \le \alpha_+, \quad \beta \le \beta_+ \le \beta_{\max}$.
- Compute $\overline{\theta} \in [0, 1]$ as the largest step length for which

$$X(\theta)s(\theta) \ge \alpha_+(1 - (1-\lambda)\theta)\tau p,$$

and

$$\|X(\theta)s(\theta) - (1 - (1-\lambda)\theta)\tau\, p\| \le \beta_+(1 - (1-\lambda)\theta)\tau,$$

for all $0 \le \theta \le \overline{\theta}$. (2.6)

- Set $x^+ = x + \overline{\theta} u, \;\; s^+ = s + \overline{\theta} v, \;\; \tau_+ = (1 - (1-\lambda)\overline{\theta})\tau$.

Let us note that the computation of $\overline{\theta}$ above involves the solutions of n quadratic equations and a quartic equation. In the following analysis we assume that these equations are solved exactly. The results are true for appropriate approximate solutions of these equations by use of the bisection method (see also [2]).

Algorithm 2.1
Given $\alpha_{\max}, \alpha_{\min}, \beta_{\max}, \beta_{\min}, \gamma, \rho$ *satisfying (2.2a) and* $(x^0, s^0) > 0$;
$t_0 \leftarrow 0, \;\; \alpha_0 \leftarrow \alpha_{\max}, \;\; \beta_0 \leftarrow \beta_{\min}, \;\; \tau_0 \leftarrow \mu_0$;
while $\max\{\mu_k, \|r^k\|\} > \epsilon$

** fast branch **
solve (2.5)-(2.6) with $(x,s)=(x^k,s^k),\ \lambda=0,$
$\alpha_+=\alpha_{\min}+\gamma^{t_k+1}(\alpha_{\max}-\alpha_{\min}),\ \ \beta_+=\beta_{\max}-\gamma^{t_k+1}(\beta_{\max}-\beta_{\min});$
if $\quad 1-\overline{\theta}\le\rho$
then $\quad t_+\leftarrow t_k+1,\ \lambda_k\leftarrow 0;$

else ** safe branch **
solve (2.5)-(2.6) with $(x,s)=(x^k,s^k),\ \lambda\in(0,1),$
$\alpha_+=\alpha_{\min}+\gamma^{t_k}(\alpha_{\max}-\alpha_{\min}),\ \ \beta_+=\beta_{\max}-\gamma^{t_k}(\beta_{\max}-\beta_{\min});$
$t_+\leftarrow t_k,\ \lambda_k\leftarrow\lambda;$
end if
** main points **
$\tau_+\leftarrow\tau_k(1-(1-\lambda_k)\overline{\theta}),$
$(x^+,s^+)\leftarrow(x^k,s^k)+\overline{\theta}(u,v);$
improve $(x^k,y^k,(x^+,s^+,\tau_+,\alpha_+,\beta_+,t_+));$
** final points **
$(x^{k+1},\ s^{k+1},\ \tau_{k+1},\ \alpha_{k+1},\ \beta_{k+1})\leftarrow(x^+,\ s^+,\ \tau_+,\ \alpha_+,\ \beta_+),\quad k\leftarrow k+1;$

end while
The **improve** *procedure is defined as follows:*
Given $\overline{\rho}\in(\rho,1),\ I\ge 0,$
for $i=1,2,\ldots,I$

$(x,s,\tau,\alpha,\beta,t)\leftarrow(x^+,s^+,\tau_+,\alpha_+,\beta_+,t_+)$
if $\quad\max\{\mu_k,\|r^k\|\}\le\epsilon,$ **then** *return;*
solve (2.5)-(2.6) with $\lambda=0,$
$\alpha_+=\alpha_{\min}+\gamma^{t+1}(\alpha_{\max}-\alpha_{\min}),\ \ \beta_+=\beta_{\max}-\gamma^{t+1}(\beta_{\max}-\beta_{\min});$
if $\quad 1-\overline{\theta}\le\rho$
then $\quad t\leftarrow t+1,$

else $t_+\leftarrow t,$ *solve (2.5)-(2.6) with* $\lambda\in(0,1),$
$\alpha_+=\alpha_{\min}+\gamma^{t}(\alpha_{\max}-\alpha_{\min}),\ \ \beta_+=\beta_{\max}-\gamma^{t}(\beta_{\max}-\beta_{\min});$
if $1-(1-\lambda)\overline{\theta}>\overline{\rho}$ **then** *return;*
end if
** intermediary points **
$\tau_+\leftarrow\tau(1-(1-\lambda)\overline{\theta}),$
$(x^+,s^+)\leftarrow(x,s)+\overline{\theta}(u,v);$

end for

The algorithm begins each iteration by trying a fast step, which uses an affine scaling search direction. The fast steps are accepted only if they produce a reduction in τ_k or $\|r^k\|$ of at least a factor of ρ. Otherwise, the algorithm reverts to taking a safe step. Then it goes to **improve** by reusing the coefficient matrix in (2.5) and taking a combination of safe and fast steps, just like the main algorithm. However, the **improve** procedure terminates if τ and $\|r\|$ are not improved by at least a factor of $\overline{\rho} \in (\rho, 1)$. The parameter $\overline{\rho}$ and the nonnegative integer I are supplied by the user, where I is the maximum number of steps that can be taken in **improve**.

In the next lemma we show that if the main points defined by Algorithm 2.1 are generated in the safe branch then the improvement rate τ_+/τ is bounded by a quantity that increases with $\delta = \|Uv\|/\tau$. Later on we will prove global convergence by showing that δ is bounded.

Lemma 2.2 *Assume that $(x^k, s^k, \tau_k, \alpha_k, \beta_k) \in \mathcal{N}$ and suppose that the main points in Algorithm 2.1 are generated by the safe branch. Then $\overline{\theta} \in [0,1]$ defined by (2.6) satisfies*

$$\overline{\theta} \geq \hat{\theta} := \min\{1, \ \frac{\lambda \ \min((1-\alpha)\rho_0 \ \ , \beta)}{\delta}\},$$

where $\delta = \|Uv\|/\tau$. Moreover,

$$\tau^+/\tau \leq 1 - \min\{1, \ \frac{\lambda \ \min((1-\alpha)\rho_0 \ \ , \beta)}{\delta}\}(1-\lambda). \tag{2.7}$$

Proof. Since $(x, s, \tau, \alpha, \beta) \in \mathcal{N}$, we have

$$Xs \geq \alpha\tau p, \quad \|Xs - \tau p\| \leq \beta\tau.$$

By definition we get

$$\begin{aligned} X(\theta)s(\theta) &= Xs + \theta(Su + Xv) + \theta^2 Uv \\ &= (1-\theta)Xs + \theta\lambda\tau p + \theta^2 Uv. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & X(\theta)s(\theta) - \alpha(1-(1-\lambda)\theta)\tau p \\ = \ & (1-\theta)(Xs - \alpha\tau p) + \theta[(1-\alpha)\lambda\tau p + \theta Uv] \\ \geq \ & \theta[(1-\alpha)\lambda\tau p + \theta Uv] \\ \geq \ & \theta\tau[(1-\alpha)\lambda p - \theta\delta e] \\ \geq \ & 0, \ \text{ for all } \ \theta \in [0, \hat{\theta}]. \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
& \|X(\theta)s(\theta) - (1-(1-\lambda)\theta)\tau p\| - \beta(1-(1-\lambda)\theta)\tau \\
= \; & \|(1-\theta)(Xs - \tau p) + \theta^2 Uv\| - \beta(1-(1-\lambda)\theta)\tau \\
\le \; & (1-\theta)\|Xs - \tau p\| + \theta^2\|Uv\| - \beta(1-(1-\lambda)\theta)\tau \\
\le \; & \theta\tau(\theta\delta - \lambda\beta) \\
\le \; & 0, \quad \text{for all} \quad \theta \in [0, \hat{\theta}].
\end{aligned}
$$

Consequently, $\overline{\theta} \ge \hat{\theta}$, and (2.7) follows immediately. $\square$

Theorem 2.3 *For any integer $k \ge 0$, Algorithm 2.1 defines $(x^k, s^k, \tau_k, \alpha_k, \beta_k)$ such that*

$$(x^k, s^k, \tau_k, \alpha_k, \beta_k) \in \mathcal{N}, \tag{2.8}$$

$$r^k = \psi_k r^0, \tag{2.9}$$

where

$$\psi_k = \tau_k/\tau_0 \le \prod_{j=1}^{k}(1-(1-\lambda_j)\overline{\theta}_j). \tag{2.10}$$

Moreover all the intermediary points produced in the **improve** *procedure also satisfy*

$$(x, s, \tau, \alpha, \beta) \in \mathcal{N}. \tag{2.11}$$

Proof. The proof is by induction. First, we note that

$$\|X^0 s^0 - \tau_0 p\| = \|X^0 s^0 - \mu_0 p\| = 0 < \beta_0 \tau_0,$$

and

$$X^0 s^0 \ge \alpha_{\max} X^0 s^0 = \alpha_0 \mu_0 p = \alpha_0 \tau_0 p,$$

which shows that $(x^0, s^0, \tau_0, \alpha_0, \beta_0) \in \mathcal{N}$. Hence (2.8) and (2.9) are satisfied for $k = 0$. Suppose they are satisfied for $k = 0, 1, \ldots, l$ for some $l \ge 0$ and denote $(x, s) = (x^l, s^l)$. Then by Lemma 2.2, $\overline{\theta}$ exists, and we have

$$X(\theta)s(\theta) \ge \alpha(1-(1-\lambda)\theta)\tau p > 0, \quad \text{for all} \quad 0 \le \theta \le \overline{\theta}. \tag{2.12}$$

The positivity of x^+ and s^+ is proved by contradiction. Suppose for example that $[x^+]_i \le 0$ for some i. Then there must exist $\theta', 0 \le \theta' \le \overline{\theta}$, such that $[x(\theta')]_i = 0$. This implies

$$[x(\theta')]_i [s(\theta')]_i = 0,$$

which contradicts (2.12). Hence (2.8) is satisfied for $k = l + 1$. The fact that (2.9) is verified for $k = l + 1$ follows from the definition of Algorithm 2.1. Similarly we can prove (2.11). $\square$

In next section, we will show that with appropriate choice of λ and (x^0, s^0), Algorithm 2.1 achieves $O((\kappa + 1)^2 nL)$-iteration complexity.

3 Global Convergence and Polynomial Complexity

In this section we assume that $\mathcal{F}^*$ is nonempty. Under this assumption we will prove that Algorithm 2.1, with $\epsilon = 0$, is globally convergent in the sense that

$$\lim_{k\to 0} \mu_k = 0, \quad \lim_{k\to 0} r^k = 0,$$

for any starting point $(x^0, s^0) > 0$. We start with the following three technical lemmas.

Lemma 3.1 *Assume that $\mathcal{F}^*$ is nonempty. Then for any $(x^*, s^*) \in \mathcal{F}^*$ and $(x, s, \tau, \alpha, \beta) \in \mathcal{N}$ we have*

$$\psi((x)^T s^0 + (s)^T x^0) \;\leq\; (1 + \beta_{\max})(1 + 4\kappa)(2 + \zeta)n\tau \;, \tag{3.1a}$$
$$(1 - \psi)((x)^T s^* + (s)^T x^*) \;\leq\; (1 + \beta_{\max})(1 + 4\kappa)((1 + \psi) + (1 - \psi)\zeta)n\tau \;, \tag{3.1b}$$

where

$$\psi = \tau/\tau_0, \quad \zeta = ((x^0)^T s^* + (s^0)^T x^*)/((x^0)^T s^0). \tag{3.2}$$

Proof. According to (2.9) we have

$$\begin{aligned}\psi s^0 + (1 - \psi)s^* - s &= \psi(r^0 + M(x^0 - x^*)) - (r + M(x - x^*)) \\ &= M(\psi x^0 + (1 - \psi)x^* - x) \;.\end{aligned}$$

Using the inequalities $(x^*, s^*) \geq 0$, $(x, s) > 0$, and the defining property (1.3) of a $P_*(\kappa)$-matrix we can write

$$\begin{aligned}&[\psi x^0 + (1 - \psi)x^* - x]^T[\psi s^0 + (1 - \psi)s^* - s] \qquad (3.3)\\ &\quad\geq -4\kappa \sum_{i\in\mathcal{I}_+} [\psi x^0 + (1 - \psi)x^* - x]_i[\psi s^0 + (1 - \psi)s^* - s]_i \\ &\quad\geq -4\kappa \sum_{i\in\mathcal{I}_+} \left(\psi^2[x^0]_i[s^0]_i + (1 - \psi)\psi([x^*]_i[s^0]_i + [x^0]_i[s^*]_i) + [x]_i[s]_i\right) \\ &\quad\geq -4\kappa(\psi^2(x^0)^T s^0 + (1 - \psi)\psi((x^*)^T s^0 + (x^0)^T s^*) + x^T s) \;, \qquad (3.4)\end{aligned}$$

where

$$\mathcal{I}_+ = \{i \ : \ [\psi x^0 + (1-\psi)x^* - x]_i[\psi s^0 + (1-\psi)s^* - s]_i > 0\} \ .$$

By expanding (3.3), we obtain

$$\begin{aligned}
&[\psi x^0 + (1-\psi)x^* - x]^T[\psi s^0 + (1-\psi)s^* - s] \\
&\quad = \psi^2 n\mu_0 + (1-\psi)\psi((x^0)^T s^* + (s^0)^T x^*) \\
&\quad\ -\psi((x^0)^T s + (s^0)^T x) + x^T s - (1-\psi)(s^T x^* + x^T s^*) + (1-\psi)(x^*)^T s^*. \qquad (3.5)
\end{aligned}$$

The desired inequalities (3.1) follows from (3.4) and (3.5) by applying Theorem 2.3 and the relations $(x^*)^T s^* = 0$, $s^T x^* + x^T s^* \geq 0$, $s^T x^0 + x^T s^0 > 0$, and $\mu \leq (1+\beta_{\max})\tau$ (see (2.4)) . □

In order to prove global convergence we have to show that $\overline{\theta}$'s are bounded away from zero. An essential tool is the following technical result of [4].

Lemma 3.2 *Let x, s, a, r be four n-dimensional vectors with $x > 0$ and $s > 0$, and let $M \in I\!R^{n\times n}$ be a $P_*(\kappa)$-matrix. Then the solution (u, v) of the linear system*

$$\begin{aligned} Su + Xv &= a \ , \\ Mu - v &= b \end{aligned}$$

satisfies the following relations:

$$\begin{aligned}
\|Du\| &\leq \|\tilde{b}\| + \sqrt{\|\tilde{a}\|^2 + \|\tilde{b}\|^2 + 2\kappa\|\tilde{c}\|^2} \ , \\
\|D^{-1}v\| &\leq \sqrt{\|\tilde{a}\|^2 + \|\tilde{b}\|^2 + 2\kappa\|\tilde{c}\|^2} \ , \\
\|Du\|^2 + \|D^{-1}v\|^2 &\leq \|\tilde{a}\|^2 + 2\kappa\|\tilde{c}\|^2 + 2\|\tilde{b}\|^2 + 2\|\tilde{b}\|\sqrt{\|\tilde{a}\|^2 + \|\tilde{b}\|^2 + 2\kappa\|\tilde{c}\|^2} \\
&\equiv \chi_1^2 \ , \\
\|Uv\|^2 &\leq \frac{1}{8}\|\tilde{a}\|^4 + \frac{1}{4}\chi_1^2(\chi_1^2 - \|\tilde{a}\|^2) \ ,
\end{aligned}$$

where

$$D = X^{-1/2}S^{1/2} \ , \ \tilde{a} = (XS)^{-1/2}a \ , \ \tilde{b} = D^{-1}b \ , \ \tilde{c} = \tilde{a} + \tilde{b} \ .$$

We use the above result to find bounds for the quantity δ arising in the estimate given by Lemma 2.2 when the algorithm chooses the safe step.

Lemma 3.3 *During the safe branch, $\delta = \|Uv\|/\tau$ satisfies*

$$\delta \leq \phi(\kappa)\Big\{(1-\lambda)[\|p\|/(\alpha\rho_0) + \|X^{1/2}S^{-1/2}r\|/\sqrt{\tau}] + \beta/(\alpha\rho_0)\Big\}^2, \tag{3.6a}$$

where

$$\phi(\kappa) := [.25(\sqrt{1+2\kappa}+1)^4 + .125]^{1/2}. \tag{3.6b}$$

Moreover,

$$\|X^{1/2}S^{-1/2}r\|/\sqrt{\tau} \leq \eta\sqrt{n(1+\beta)}, \tag{3.7a}$$

where

$$\eta = \sqrt{n}(1+4\kappa)(2+\zeta)\|(S^0)^{-1}r^0\|_\infty \sqrt{\frac{1+\beta_{\max}}{\rho_0\alpha_{\min}}}, \tag{3.7b}$$

with ζ given by (3.2).

Proof. By applying Lemma 3.2 to the linear system (2.5) with $a = \lambda\tau p - Xs$ and $b = (1-\lambda)r$ we have

$$\begin{aligned}
\|\tilde{a}\| &= \|(XS)^{-1/2}(\lambda\tau p - Xs)\| \\
&\leq (1-\lambda)\tau\|(XS)^{-1/2}p\| + \|(XS)^{-1/2}(Xs-\tau p)\| \\
&\leq (1-\lambda)\sqrt{\tau}\|p\|/(\alpha\rho_0) + \sqrt{\tau}\beta/(\alpha\rho_0), \\
\|\tilde{b}\| &= \|D^{-1}r\| = (1-\lambda)\|X^{1/2}S^{-1/2}r\|, \\
\|\tilde{c}\| &= \|\tilde{a}+\tilde{b}\| \leq \|\tilde{a}\| + \|\tilde{b}\|, \\
\delta &= \|Uv\|/\tau \leq \phi(\kappa)(\|\tilde{a}\| + \|\tilde{b}\|)^2/\tau,
\end{aligned}$$

and (3.6a) follows from the above relations. From Lemma 3.1 we deduce that

$$\begin{aligned}
& \|(XS)^{-1/2}Xr\|/\sqrt{\tau} \leq (\rho_0\alpha)^{-1/2}\tau^{-1}\|Xr\| \\
\leq\ & (\rho_0\alpha)^{-1/2}\tau^{-1}\|Xr\|_1 = \psi(\rho_0\alpha)^{-1/2}\tau^{-1}\|Xr^0\|_1 \\
=\ & \psi(\rho_0\alpha)^{-1/2}\tau^{-1}\sum_{i=1}^n \Big|[x]_i[s^0]_i[r^0]_i/[s^0]_i\Big| \\
\leq\ & (\rho_0\alpha)^{-1/2}\tau^{-1}\|(S^0)^{-1}r^0\|_\infty\psi(s^0)^Tx \\
\leq\ & \eta\sqrt{n(1+\beta)} \qquad \square
\end{aligned}$$

Now let us take

$$\lambda = \overline{\lambda} := \frac{\|p\|/(\alpha\rho_0) + \|X^{1/2}S^{-1/2}r\|/\sqrt{\tau} + \beta/(\alpha\rho_0)}{\|p\|/(\alpha\rho_0) + \|X^{1/2}S^{-1/2}r\|/\sqrt{\tau} + \omega\ \beta/(\alpha\rho_0)}, \tag{3.8}$$

where $\omega > 1$ is a fixed number. Clearly,

$$\overline{\lambda} \in (1/\omega, 1). \tag{3.9}$$

From (3.6a) we have

$$\delta \leq \omega^2 \phi(\kappa)\beta^2/(\alpha\rho_0)^2. \tag{3.10}$$

Then by virtue of (2.7), (3.8), (3.9) and (3.10), we deduce that

$$\begin{aligned}
\tau^+/\tau &\leq 1 - \min\left\{1,\ \lambda\frac{\min((1-\alpha)\rho_0\ \ ,\beta)}{\delta}\right\}(1-\lambda) \\
&\leq 1 - \min\left\{1,\ \frac{\min((1-\alpha)\rho_0\ \ ,\beta)}{\omega^3\phi(\kappa)\beta^2/(\alpha\rho_0)^2}\right\}(1-\overline{\lambda}) \\
&\leq 1 - \min\left\{1,\ \frac{\min\{(1-\alpha)\rho_0\ \ ,\beta\}}{\omega^3\phi(\kappa)\beta^2/(\alpha\rho_0)^2}\right\}\frac{(\omega-1)\beta}{\|p\| + \eta\alpha\rho_0\sqrt{n(1+\beta)} + \omega\beta} \\
&\leq 1 - \theta^*,
\end{aligned} \tag{3.11}$$

where

$$\theta^* := \min\left\{1,\ \frac{\min\{(1-\alpha_{\max})\rho_0\ \ ,\beta_{\min}\}}{\omega^3\phi(\kappa)\beta_{\max}^2/(\alpha_{\min}\rho_0)^2}\right\}\frac{(\omega-1)\beta_{\min}}{\|p\| + \eta\alpha_{\max}\rho_0\sqrt{n(1+\beta_{\max})} + \omega\beta_{\max}} \tag{3.12}$$

Suppose the algorithm takes the fast branch at iteration k. Since the **improve** procedure never increases the value of ψ, we must have $\psi_{k+1}/\psi_k \leq \rho$; if a safe branch is taken we must have $\psi_{k+1}/\psi_k \leq 1 - \theta^*$. Therefore we obtain

$$\psi_{k+1}/\psi_k \leq \max\{1 - \theta^*, \rho\}. \tag{3.13}$$

This observation shows that our algorithm converges at a global linear rate.

Theorem 3.4 *Suppose that the optimal set $\mathcal{F}^*$ is nonempty, and in the safe branch λ_k is defined by (3.8).*

(i) If $\epsilon = 0$ then Algorithm 2.1 either finds an optimal solution $z^* \in \mathcal{F}^*$ in a finite number of steps or produces an infinite sequence $z^k = (x^k, s^k)$ such that

$$\lim_{k\to\infty}(x^k)^T s^k = 0,\ \lim_{k\to\infty}(r^k) = 0.$$

(ii) If $\epsilon > 0$ then Algorithm 2.1 terminates with a $z \in \mathcal{F}_\epsilon$ in at most

$$K_\epsilon = \lceil \frac{|\ln(\frac{\epsilon}{\epsilon_0})|}{|\ln(\max\{1-\theta^*, \rho\})|} \rceil$$

iterations, where $\epsilon_0 = \max\{(1+\beta)(x^0)^T s^0, \|r^0\|\}$, and $\lceil \chi \rceil$ denotes the smallest integer greater than or equal to χ.

For starting points that are feasible or close to being feasible we obtain the following complexity result.

Corollary 3.5 *Under the hypothesis of Theorem 3.4 suppose that $\rho_0 = \Omega(1)$, and the starting point is chosen such that there is a constant C independent of n satisfying the inequality*

$$(2+\zeta)\|(S^0)^{-1}r^0\|_\infty \leq \frac{C}{(1+\kappa)\sqrt{n}}.$$

Then Algorithm 2.1 terminates in at most

$$\tilde{K}_\epsilon = O((1+\kappa)\sqrt{n}\ln(\epsilon_0/\epsilon))$$

iterations.

Most of the complexity results on infeasible interior-point methods are obtained for starting points of the form

$$x^0 = \rho_p e, \quad s^0 = \rho_d e\ , \tag{3.14}$$

where ρ_p and ρ_d are sufficiently large positive constants ("big M initialization"). For such starting points we have clearly $\rho_0 = 1$ and

$$\zeta = \|x^*\|_1/(n\rho_p) + \|s^*\|_1/(n\rho_d), \quad \text{for some } (x^*, s^*) \in \mathcal{F}^*\ ,$$

$$\|(S^0)^{-1}r^0\|_\infty \leq 1 + (\rho_p/\rho_d)\|Me\|_\infty + (1/\rho_d)\|q\|_\infty.$$

Therefore if ρ_p and ρ_d satisfy the inequalities

$$\rho_p \geq n^{-1}\|x^*\|_1\ , \quad \rho_d \geq \max\{\rho_p\|Me\|_\infty\ , \|q\|_\infty, n^{-1}\|s^*\|_1\}\ , \tag{3.15}$$

for some $(x^*, s^*) \in \mathcal{F}^*$ then $\eta \leq O((1+\kappa)\sqrt{n})$. Hence, from (3.12) and Theorem 3.4 we deduce the following computational complexity bound.

Corollary 3.6 *Under the hypothesis of Theorem 3.4 suppose that the starting point satisfies (3.15) for some* $(x^*, s^*) \in \mathcal{F}^*$*. Then Algorithm 2.1 terminates in at most*

$$\tilde{K}_\epsilon = O((1+\kappa)^2 n \ln(\epsilon_0/\epsilon)) \tag{3.16}$$

iterations.

Let us end this section by noting that while our algorithm does not use any information on the classification number κ, its computational complexity depends on κ.

4 Infeasibility Detection

All the above results have been proved under the assumption that $\mathcal{F}^*$ is nonempty. It turns out that Algorithm 2.1 can be modified in such a way that it can detect within polynomial time whether $\mathcal{F}^*$ contains points of norm less than a quantity chosen in advance provided the parameter κ is known. Let $\overline{\rho}_p$, $\overline{\rho}_d$ be such quantities. Also let us define

$$\overline{\zeta} = (\|x^0\|\overline{\rho}_d + \|s^0\|\overline{\rho}_p)/((x^0)^T s^0),$$

$$\overline{\eta} = \sqrt{n}(1+4\kappa)(2+\overline{\zeta})\|(S^0)^{-1}r^0\|_\infty \sqrt{\frac{1+\beta_{\max}}{\alpha_{\min}\rho_0}}, \tag{4.1}$$

$$\overline{\theta}^* := \min\left\{1, \;\; \frac{\min\{(1-\alpha_{\max})\rho_0 \;\;, \beta_{\min}\}}{\omega^3\phi(\kappa)\beta_{\max}^2/(\alpha_{\min}\rho_0)^2}\right\} \frac{(\omega-1)\beta_{\min}}{\|p\| + \overline{\eta}\alpha_{\max}\rho_0\sqrt{n(1+\beta_{\max})} + \omega\beta_{\max}}.$$

In what follows, we always assume that λ is defined by (3.8). Then we can prove the following theorem.

Theorem 4.1 *Suppose that the following instruction*
"If $(x^0)^T s^k + (s^0)^T x^k > (1+4\kappa)[(\tau_k/\tau_0)(x^0)^T s^0 + (\tau_0/\tau_k)(x^k)^T s^k + (1-\tau_k/\tau_0)(\overline{\rho}_d\|x^0\| + \overline{\rho}_p\|s^0\|)]$ **then terminate"** *is inserted just before the* **while** *loop of Algorithm 2.1. Then the new algorithm terminates at most*

$$\overline{K}_\epsilon = \lceil \frac{|\ln(\frac{\epsilon}{\epsilon_0})|}{|\ln(\max\{1-\overline{\theta^*}, \rho\})|} \rceil$$

iterations either by finding a $\epsilon-$ *approximate solution or by determining either there is no* $z^* = (x^*, s^*) \in \mathcal{F}^*$ *with* $\|x^*\| \le \overline{\rho}_p$, $\|s^*\| \le \overline{\rho}_d$, *or* M *is not a* $P_*(\kappa)$*-matrix.*

Proof. Suppose that M is a $P_*(\kappa)$-matrix and that the inequality

$$(x^0)^T s^k + (s^0)^T x^k \le (1+4\kappa)[(\tau_k/\tau_0)(x^0)^T s^0 + (\tau^0/\tau_k)(x^k)^T s^k + (1-\tau_k/\tau_0)(\overline{\rho}_d\|x^0\| + \overline{\rho}_p\|s^0\|)] \tag{4.2}$$

holds for all $0 \le k \le \overline{K}_\epsilon$. Then we have

$$\psi_k((x^0)^T s^k + (s^0)^T x^k) \le (1+4\kappa)(1+\beta_{\max})(2+\overline{\zeta})n\tau_k, \quad 0 \le k \le \overline{K}_\epsilon. \tag{4.3}$$

With the help of (4.3) we can show that $\tau_{k+1}/\tau_k \le \max\{1-\overline{\theta^*},\ \rho\}$ for all $0 \le k \le \overline{K}_\epsilon$, which implies $(x^{\overline{K}_\epsilon}, s^{\overline{K}_\epsilon}) \in \mathcal{F}_\epsilon$.
On the other hand, if there exists $(x^*, s^*) \in \mathcal{F}^*$ such that $\|x^*\| \le \overline{\rho}_p$, $\|s^*\| \le \overline{\rho}_d$, then (3.4) and (3.5) imply that (4.2) must hold for all $k \ge 0$. Therefore, if (4.2) is violated for some $k \le \overline{K}_\epsilon$, then there is no $(x^*, s^*) \in \mathcal{F}^*$ such that $\|x^*\| \le \overline{\rho}_p$, $\|s^*\| \le \overline{\rho}_d$. This completes the proof of our theorem. □
From the above theorem we can obtain polynomial complexity for our new algorithm under certain assumptions on the starting point.

Corollary 4.2 *Under the hypothesis of Theorem 4.1 suppose $\rho_0 = \Omega(1)$ and the starting point satisfies*

$$(2+\overline{\zeta})\|(S^0)^{-1}r^0\|_\infty \le \frac{C}{(1+\kappa)\sqrt{n}},$$

for some constant C independent of n. Then the new algorithm terminates in at most

$$\tilde{K}_\epsilon = O((1+\kappa)\sqrt{n}\ln(\epsilon_0/\epsilon))$$

iterations either by finding an $\epsilon-$approximate solution or by determining either there is no $z^* = (x^*, s^*) \in \mathcal{F}^*$ with $\|x^*\| \le \overline{\rho}_p$, $\|s^*\| \le \overline{\rho}_d$, or M is not a $P_*(\kappa)$-matrix.

Suppose now that the starting point satisfies

$$x^0 = \hat{\rho}_p e, \quad s^0 = \hat{\rho}_d e, \tag{4.4}$$

where

$$\hat{\rho}_p \ge \overline{\rho}_p/\sqrt{n}, \tag{4.5}$$

$$\hat{\rho}_d \ge \max\{\hat{\rho}_p\|Me\|_\infty, \|q\|_\infty, \overline{\rho}_d/\sqrt{n}\}. \tag{4.6}$$

Then we obtain

$$\overline{\zeta} \le 2, \tag{4.7}$$

$$\|(S^0)^{-1}r^0\|_\infty \le 1 + (\hat{\rho}_p/\hat{\rho}_d)\|Me\|_\infty + (1/\hat{\rho}_d)\|q\|_\infty \le 3. \tag{4.8}$$

From (4.1), (4.7) and (4.8), it follows that $\overline{\eta} < O((1+\kappa)\sqrt{n})$. Hence we have the following complexity result.

Corollary 4.3 *Under the assumption of Theorem 4.1 suppose that the starting point satisfies (4.4)-(4.6). Then the new algorithm terminates in at most*

$$\hat{K}_\epsilon = O((1+\kappa)^2 n \ln(\epsilon_0/\epsilon))$$

iterations either by finding an $\epsilon-$ *approximate solution or by determining either there is no* $z^* = (x^*, s^*) \in \mathcal{F}^*$ *with* $\|x^*\| \le \overline{\rho}_p$, $\|s^*\| \le \overline{\rho}_d$, *or* M *is not a* $P_*(\kappa)$*-matrix.*

5 Technical Results

In what follows we study the asymptotic convergence properties of Algorithm 2.1 under the further assumption that (1.1) has a strictly complementary solution. We prove some results about the steps generated in the fast branch of the algorithm. In particular, we analyze the effect of using an approximate Jacobian for solving (2.5) as is done during the **improve** procedure.
Let us denote by $\mathcal{F}^c$ the set of all strictly complementary solutions, i.e.,

$$\mathcal{F}^c = \{(x,s) \in \mathcal{F}^* : [x]_i + [s]_i > 0, i = 1, 2, \cdots, n\} \ .$$

It is well known that there is a unique partition

$$\mathcal{B} \cup \mathcal{N} = \{1, 2, \cdots, n\} \ , \quad \mathcal{B} \cap \mathcal{N} = \emptyset \ ,$$

such that for any $(x,s) \in \mathcal{F}^c$ we have $([x]_i > 0, [s]_i = 0, \forall i \in \mathcal{B})$ and $([x]_i = 0, [s]_i > 0, \forall i \in \mathcal{N})$. Let us denote the corresponding partition of M by

$$M = \begin{pmatrix} M_{BB} & M_{BN} \\ M_{NB} & M_{NN} \end{pmatrix} .$$

Also, for any vector $y \in I\!\!R^n$ we denote by y_B the vector of components $[y]_i, \ i \in \mathcal{B}$, and by y_N the vector of components $[y]_i, \ i \in \mathcal{N}$.

From the next lemma it follows that the points generated by Algorithm 2.1 are bounded.

Lemma 5.1 *There is a constant* $C_1 > 0$ *such that*

$$\|(x,s)\| \le C_1, \tag{5.1}$$

for all $(x, s, \tau, \alpha, \beta) \in \mathcal{N}$.

Proof. The proof is straitforward by observing that (3.1a) implies

$$x^T s^0 + s^T x^0 \leq (1+\beta)(1+4\kappa)(2+\zeta)n\tau_0. \quad \square$$

Lemma 5.2 *There is a constant $C_2 > 0$ such that for any $(x, s, \tau, \alpha, \beta) \in \mathcal{N}$ and for any $z^* := (x^*, s^*) \in \mathcal{F}^*$ we have*

$$\|(x + u^a - x^*,\ s + v^a - s^*)\| \leq C_2 \frac{\|z^k - z^*\|^2}{\tau}, \tag{5.2}$$

where $z := (x, s)$ and (u^a, v^a) satisfy

$$\begin{aligned} Su^a + Xv^a &= -Xs, &(5.3a)\\ Mu^a - v^a &= r. &(5.3b)\end{aligned}$$

Proof. Let $z^* = (x^*, s^*) \in \mathcal{F}^*$. By using (5.3) we have:

$$\begin{pmatrix} S & X \\ M & -I \end{pmatrix} \begin{pmatrix} x + u^a - x^* \\ s + v^a - s^* \end{pmatrix} = \begin{pmatrix} (X - X^*)(s - s^*) \\ 0 \end{pmatrix}.$$

Applying Lemma 3.2 to the above linear system, we obtain

$$\begin{aligned} \|D(x + u^a - x^*)\| &\leq \sqrt{1+2\kappa}\|(XS)^{-\frac{1}{2}}(X - X^*)(s - s^*)\| \\ &\leq \sqrt{1+2\kappa}\frac{\|x - x^*\|\,\|s - s^*\|}{\sqrt{\alpha_{\min}\tau}} \\ &\leq \frac{\sqrt{1+2\kappa}}{2\sqrt{\alpha_{\min}}}\frac{\|z - z^*\|^2}{\sqrt{\tau}}, \end{aligned} \tag{5.4}$$

where $D = X^{-\frac{1}{2}}S^{\frac{1}{2}}$. Thus

$$\begin{aligned} \|x + u^a - x^*\| &= \|D^{-1}D\,(x + u^a - x^*)\| \\ &\leq \|X\|\|(XS)^{-1/2}\|\,\|D(x + u^a - x^*)\| \\ &\leq \frac{\sqrt{1+2\kappa}\,C_1}{2\alpha_{\min}}\frac{\|z - z^*\|^2}{\tau} \\ &= C_2\frac{\|z - z^*\|^2}{\tau}, \quad \text{with} \quad C_2 := \frac{\sqrt{1+2\kappa}C_1}{2\alpha_{\min}}. \end{aligned}$$

The corresponding inequality for $s + v^a - s^*$ can be obtained similarly. $\square$

Lemma 5.3 *If $\mathcal{F}^c$ is nonempty, then there is a constant $C_3 > 0$ such that for any $(x, s, \tau, \alpha, \beta) \in \mathcal{N}$ with $\tau \le \tau_1$,*

$$\|x_N\| \le C_3\, \tau, \quad \|s_B\| \le C_3 \tau,$$

for all $k \ge 1$.

Proof. Let $(x^*, s^*) \in \mathcal{F}^c$. If $\psi = \tau/\tau_0$, then $\psi \le \psi_1 \le 1 - \theta_0$ so that we can write

$$\begin{aligned}(x)^T s^* + (s)^T x^* &\le (1+\beta_{\max})(1+4\kappa)((1+\psi_1)/(1-\psi_1)+\zeta)n\tau_k \\ &\le (1+\beta_{\max})(1+4\kappa)((2-\overline{\theta}_0)/\overline{\theta}_0+\zeta)n\tau_k \,.\end{aligned}$$

Therefore,

$$[x]_i \le \frac{(1+\beta_{\max})(1+4\kappa)((2-\overline{\theta}_0)/\overline{\theta}_0+\zeta)}{[s^*]_i} n\tau_k \,, \quad \forall\, i \in \mathcal{N} \,,$$

and

$$[s]_i \le \frac{(1+\beta_{\max})(1+4\kappa)((2-\overline{\theta}_0)/\overline{\theta}_0+\zeta)}{[x^*]_i} n\tau_k \,, \quad \forall\, i \in \mathcal{B} \,.$$

Hence the desired result holds with

$$C_3 = n\sqrt{n}(1+\beta_{\max})(1+4\kappa)((2-\overline{\theta}_0)/\overline{\theta}_0+\zeta)\max\left\{\max_{i\in\mathcal{B}}\frac{1}{[x^*]_i}\,,\max_{i\in\mathcal{N}}\frac{1}{[s^*]_i}\right\} . \qquad \square$$

Lemma 5.4 *If $\mathcal{F}^c$ is nonempty, then there exists a constant $C_4 > 0$ such that for each $(x, s, \tau, \alpha, \beta) \in \mathcal{N}$ there is a solution $z^* = (x^*, s^*) \in \mathcal{F}^*$ such that*

$$\|z - z^*\| \le C_4 \tau. \tag{5.5}$$

Proof. Consider the following problem:

$$\begin{array}{llllll} M_{BB}x_B & & & & = -q_B \\ M_{NB}x_B & & & -s_N & = -q_N \\ & x_N & & & = 0 \\ & & s_B & & = 0 \\ & & x_B, & s_N & \ge 0 \,. \end{array} \tag{5.6}$$

Under the assumption that $\mathcal{F}^c \ne \emptyset$, (5.6) has a solution and the solution set of (5.6) is $\mathcal{F}^*$. By Hoffman's lemma [3] , there is a constant C_4' such that there is a $z^* \in \mathcal{F}^*$ satisfying

$$\begin{aligned}\|z - z^*\| &\le C_4' \,\|(M_{BB}x_B + q_B, M_{NB}x_B + q_N - s_N, x_N, s_B)\| \\ &\le C_4' \left\|(-M_{BN}x_N + s_B, -M_{NN}x_N^k, x_N, s_B)\right\| + C_4'\|r\| \,.\end{aligned} \tag{5.7}$$

We have

$$\|r\| = \|\psi r^0\| = \frac{\|r^0\|}{\tau_0}(\psi\tau_0) = \frac{\|r^0\|}{\tau_0}\tau. \tag{5.8}$$

Finally, (5.5) follows from Lemma 5.3 and (5.7)-(5.8) . $\square$

Lemma 5.5 *There exists a constant C_5 such that for any $(x, s, \tau, \alpha, \beta) \in \mathcal{N}$ we have*

$$\|(u^a, v^a)\| \le C_5\tau, \tag{5.9}$$

where (u^a, v^a) is the solution of (5.3).

Proof. Let $z^* = (x^*,\ s^*) \in \mathcal{F}^*$ satisfy (5.5). From Lemma 5.2 and Lemma 5.4 we deduce that

$$\begin{aligned} \|u^a\| &\le \|x + u^a - x^*\| + \|x - x^*\| \\ &\le (C_2C_4^2 + C_4)\tau := C_5\tau\ . \end{aligned}$$

The corresponding inequality for $\|v^a\|$ is obtained in a similar manner. $\square$

We now turn to the *approximate* fast steps computed by (2.5)-(2.6), where (x, s) is either the current iterate (x^k, s^k) or some intermediate point generated in the call to **improve** at the k-th iteration. By the definition of the algorithm, we have

$$\tau \le \tau_k. \tag{5.10}$$

Let us assume that the point (x, s) is not too far from (x^k, s^k) in the sense that there is a constant $\chi \ge 1$ independent of k such that

$$\|(x^k - x,\ s^k - s)\| \le \chi\tau_k. \tag{5.11}$$

We will show later on that such a χ exists (see Lemma 6.1).
The following lemma describes some properties of the actual search direction (u, v) calculated from (2.6), in terms of the *exact* search direction (u^a, v^a) that satisfies (5.3).

Lemma 5.6 *Suppose that $\mathcal{F}^c$ is nonempty and consider the notation of Algorithm 2.1. Under the assumption (5.11) suppose that (u, v) is the solution of (2.5) with $\lambda = 0$, then there exists a constant C_6 independent of k and χ such that the following bounds are satisfied:*

$$\|u - u^a\| \le C_6\chi\tau, \quad \|v - v^a\| \le C_6\chi\tau, \tag{5.12a}$$

$$\|(u, v)\| \le C_6\chi\tau, \tag{5.12b}$$

$$\|u_N - u_N^a\| \le C_6\chi\tau\tau_k, \quad \|v_B - v_B^a\| \le C_6\chi\tau\tau_k. \tag{5.12c}$$

Proof. From (2.5), we have that

$$\begin{pmatrix} S^k & X^k \\ M & -I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -Xs \\ r \end{pmatrix}, \tag{5.13}$$

while from (5.3), we get

$$\begin{pmatrix} S & X \\ M & -I \end{pmatrix} \begin{pmatrix} u^a \\ v^a \end{pmatrix} = \begin{pmatrix} -Xs \\ r \end{pmatrix}, \tag{5.14}$$

and therefore

$$\begin{pmatrix} S^k & X^k \\ M & -I \end{pmatrix} \begin{pmatrix} u^a \\ v^a \end{pmatrix} = \begin{pmatrix} -Xs + (S^k - S)u^a + (X^k - X)v^a \\ r \end{pmatrix}. \tag{5.15}$$

By (5.13) and (5.15) we obtain

$$\begin{pmatrix} S^k & X^k \\ M & -I \end{pmatrix} \begin{pmatrix} u^a - u \\ v^a - v \end{pmatrix} = \begin{pmatrix} (S^k - S)u^a + (X^k - X)v^a \\ 0 \end{pmatrix}. \tag{5.16}$$

Then from (5.11) and Lemma 5.5, there is a constant C_6' independent of k and χ such that

$$\|(S^k - S)u^a + (X^k - X)v^a\| \le C_6' \chi \tau \tau_k. \tag{5.17}$$

Applying Lemma 3.2 to the linear system (5.16) with $a = (S^k - S)u^a + (X^k - X)v^a$, $b = 0$, we deduce that

$$\begin{aligned} \|D^k(u^a - u)\| &\le \sqrt{1+2\kappa}\|(X^k S^k)^{-1/2}((S^k - S)u^a + (X^k - X)v^a)\| \\ &\le \frac{\sqrt{1+2\kappa}C_6'}{\sqrt{\alpha_{\min}\rho_0}} \chi\tau\sqrt{\tau_k}, \end{aligned} \tag{5.18}$$

where $D^k = (X^k)^{-1/2}(S^k)^{1/2}$. Therefore

$$\|u^a - u\| \le \|(D^k)^{-1}\| \|D^k(u^a - u)\| \le \frac{\sqrt{1+2\kappa}C_1 C_6'}{\alpha_{\min}\rho_0}\chi\tau. \tag{5.19}$$

Then from (5.18) and Lemma 5.3 we deduce that

$$\begin{aligned} \|u_N^a - u_N\| &\le \|(D_N^k)^{-1}\| \|D^k(u^a - u)\| \\ &= \|(X_N^k)(X_N^k S_N^k)^{-1/2}\| \|D^k(u^a - u)\| \\ &\le \frac{\sqrt{1+2\kappa}C_3 C_6'}{\alpha_{\min}\rho_0}\chi\tau\tau_k. \end{aligned} \tag{5.20}$$

The bounds for $(v^a - v)$ and $(v_B^a - v_B)$ are obtained similarly. Since

$$\|(u,v)\| \leq \|(u^a, v^a)\| + \|(u - u^a, v - v^a)\|,$$

relation (5.12) follows from Lemma 5.5 with an appropriate $C_6 > 0$. $\square$

The next lemma gives an estimate for the step length $\overline{\theta}$ along a (possibly approximate) fast step direction (u, v). The point (x, s) considered in the lemma represents either the main iterate (x^k, y^k) itself or one of the intermediate points generated by the **improve** procedure during the k-th iteration. For the purpose of this result, let us define

$$C_7' = \rho_0(1-\gamma)\min\{\alpha_{\max} - \alpha_{\min},\ \beta_{\max} - \beta_{\min}\},$$
$$C_7'' = \max\{2C_5^2,\ 4C_6(C_1 + C_3)\},$$
$$C_7 = C_7''/C_7'.$$

Lemma 5.7 *During the k-th iteration of Algorithm 2.1 suppose $(x, s, \tau, \alpha, \beta) \in \mathcal{N}$, where*

$$\alpha = \alpha_{\min} + \gamma^t(\alpha_{\max} - \alpha_{\min}), \quad \beta = \beta_{\max} - \gamma^t(\beta_{\max} - \beta_{\min}),$$

and the quantity t satisfies

$$C_7\chi^2 \frac{\tau_k}{\gamma^t} \leq \rho. \tag{5.21}$$

If $\overline{\theta}$ is computed from (2.5)-(2.6) with $\lambda = 0$ and

$$\alpha_+ = \alpha_{\min} + \gamma^{t+1}(\alpha_{\max} - \alpha_{\min}), \quad \beta_+ = \beta_{\max} - \gamma^{t+1}(\beta_{\max} - \beta_{\min}),$$

then

$$1 - \overline{\theta} \leq C_7\chi^2 \frac{\tau_k}{\gamma^t} \leq \rho. \tag{5.22}$$

Proof. By definition, we have

$$\begin{aligned} X(\theta)s(\theta) &= Xs + \theta(Su + Xv) + \theta^2 Uv \\ &= Xs + \theta(Su^a + Xv^a) + \theta[S(u - u^a) + X(v - v^a)] + \theta^2 Uv \\ &= (1-\theta)Xs + \theta[S(u - u^a) + X(v - v^a)] + \theta^2 Uv. \end{aligned} \tag{5.23}$$

Then from Lemmas 5.1, 5.3 and 5.6, we deduce that

$$\begin{aligned} &\|S(u - u^a) + X(v - v^a)\| \\ \leq\ & \|s_B\|\|u_B - u_B^a\| + \|x_B\|\|v_B - v_B^a\| + \|s_N\|\|u_N - u_N^a\| + \|x_N\|\|v_N - v_N^a\| \\ \leq\ & 2C_6(C_1 + C_3)\chi\tau\tau_k \leq \frac{C_7''}{2}\chi^2\tau\tau_k. \end{aligned} \tag{5.24}$$

Let θ'' be the unique positive root of the quadratic equation

$$C_7'\gamma^t(1-\theta) - \frac{C_7''}{2}\chi^2\tau_k\theta - \frac{C_7''}{2}\chi^2\tau_k\theta^2 = 0.$$

Then it is easily seen that $\theta'' \in (0,1)$ and

$$C_7'\gamma^t(1-\theta) + \frac{C_7''}{2}\chi^2\tau_k\theta - \frac{C_7''}{2}\chi^2\tau_k\theta^2 \geq 0, \quad \text{for all} \;\; \theta \in [0,\theta''].$$

Hence, if $\theta \in [0,\theta'']$, then from (5.23), (5.24) and Lemma 5.6, we deduce that

$$\begin{aligned}
& X(\theta)s(\theta) - \alpha_+(1-\theta)\tau p \\
= \;& (1-\theta)(Xs - \alpha_+\tau p) + \theta[S(u-u^a) + X(v-v^a)] + \theta^2 Uv \\
= \;& (1-\theta)(Xs - \alpha\tau p) + (1-\theta)(\alpha-\alpha_+)\tau p + \theta[S(u-u^a) + X(v-v^a)] + \theta^2 Uv \\
\geq \;& (1-\gamma)\gamma^t(\alpha_{\max} - \alpha_{\min})(1-\theta)\tau p - \theta\|S(u-u^k) + X(v-v^k)\|e - \theta^2\|Uv\|e \\
\geq \;& C_7'\gamma^t(1-\theta)\tau e - \frac{C_7''}{2}\chi\tau\tau_k\theta e - \frac{C_7''}{2}\chi^2\tau^2\theta^2 e \\
\geq \;& \tau[C_7'\gamma^t(1-\theta) - \frac{C_7''}{2}\chi^2\tau_k\theta - \frac{C_7''}{2}\chi^2\tau_k\theta^2]e \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
& \|X(\theta)s(\theta) - (1-\theta)\tau p\| - \beta_+(1-\theta)\tau \\
\leq \;& (1-\theta)\|Xs - \tau p\| + \theta\|S(u-u^a) + X(v-v^a)\| + \theta^2\|Uv\| - \beta_+(1-\theta)\tau \\
\leq \;& -(1-\gamma)\gamma^t(\beta_{\max} - \beta_{\min})(1-\theta)\tau + \theta\|S(u-u^a) + X(v-v^a)\| + \theta^2\|Uv\| \\
\leq \;& -C_7'\gamma^t(1-\theta)\tau + \frac{C_7''}{2}\chi^2\tau\tau_k\theta + \frac{C_7''}{2}\chi^2\tau^2 \\
\leq \;& \tau[-C_7'\gamma^t(1-\theta) + \frac{C_7''}{2}\chi^2\tau_k + \frac{C_7''}{2}\chi^2\tau_k] \leq 0.
\end{aligned}$$

Therefore, $\overline{\theta} \geq \theta''$ which implies

$$\begin{aligned}
1 - \overline{\theta} \;&\leq\; 1 - \theta'' = \frac{1}{C_7'\gamma^t}(\frac{C_7''}{2}\chi^2\tau_k\theta'' + \frac{C_7''}{2}\chi^2\tau_k(\theta'')^2) \\
&\leq\; C_7(\chi^2\tau_k/\gamma^t) \leq \rho. \quad \square
\end{aligned}$$

Lemma 5.8 *There exists a constant $C_8 \in (0,1)$ such that*

$$\frac{\tau_{k+1}}{\gamma^{t_{k+1}}} \leq C_8\frac{\tau_k}{\gamma^{t_k}}, \quad \textit{for all} \;\; k \geq 0. \tag{5.25}$$

Proof. When the fast step is taken with a certain t, then $t_+ = t+1$ and

$$\tau_+ \le \rho\tau.$$

Hence

$$\frac{\tau_+}{\gamma^{t_+}} \le \rho\frac{\tau}{\gamma^{t+1}} \le \frac{\rho}{\gamma}\frac{\tau}{\gamma^t}.$$

On the other hand, if a safe step is taken, we have from (3.13) and the definition of the algorithm that

$$\tau_+ \le \max\{1-\theta^*, \overline{\rho}\}\tau,$$

while the value of t keeps unchanged. So it follows that

$$\frac{\tau_+}{\gamma^{t_+}} = \frac{\tau_+}{\gamma^t} \le \max\{1-\theta^*, \overline{\rho}\}\frac{\tau}{\gamma^t}.$$

Therefore, (5.25) holds with $C_8 = \max\{\rho/\gamma,\ 1-\theta^*,\ \overline{\rho}\}$. $\square$

6 Local Convergence

We are ready to show that there exists a threshold value of τ_k/γ^{t_k} below which both the main algorithm and the procedure **improve** take only fast steps.

Lemma 6.1 *Define*

$$\chi = \max\left\{1, \quad C_5 \exp\left(\frac{C_6\rho}{1-\rho}\right)\right\}, \tag{6.1}$$

and let K be the smallest index such that

$$C_7\chi^2\frac{\tau_K}{\gamma^{t_K+I}} \le \rho. \tag{6.2}$$

Then at the k-th iteration with $k \ge K$, the fast branch is taken in the main algorithm and, moreover, I fast steps are taken in the call to **improve**.

Proof. The existence of K is guaranteed by Lemma 5.8 because

$$\frac{\tau_k}{\gamma^{t_k+I}} = \frac{1}{\gamma^I}\frac{\tau_k}{\gamma^{t_k}} \le \frac{\tau_0}{\gamma^{t_0+I}}C_8^k \ \to\ 0, \ \text{ as }\ k\to\infty.$$

Now let us choose any $k \ge K$. Our proof proceeds by showing first that the step taken from (x^k, s^k) in the main algorithm is a fast step. We then prove by induction that I fast steps

are taken inside the procedure **improve**. Our main tool is Lemma 5.7.
For the first part of the proof, we apply Lemma 5.7 with

$$(x, s) = (x^k, s^k), \quad t = t_k. \tag{6.3}$$

Note that the point (x, s) trivially satisfies the assumption (5.11). Since

$$C_7\chi^2 \frac{\tau_k}{\gamma^t} = C_7\chi^2 \frac{\tau_k}{\gamma^{t_k}} \leq C_7\chi^2 \frac{\tau_K}{\gamma^{t_K}} \leq C_7\chi^2 \frac{\tau_K}{\gamma^{t_K+I}} \leq \rho,$$

the condition (5.21) of Lemma 5.7 is satisfied by the choices (6.3), and therefore a fast step is taken by the main algorithm.
We turn now to the procedure **improve**. Our aim is to show inductively that if (x, s) is the current vector pair at the commencement of the i-th iteration of this procedure, then

$$\|(x^k, s^k) - (x, s)\| \leq \left[C_5 \prod_{l=0}^{i-1} (1 + C_6\rho^l) \right] \tau_k, \tag{6.4}$$

and that a fast step is taken from this vector pair (x, s) during the i-th iteration of **improve**.
Note for future reference that

$$\ln \prod_{l=1}^{i-1} (1 + C_6\rho^l) = \sum_{l=1}^{i-1} \ln(1 + C_6\rho^l) \leq \sum_{l=1}^{i-1} C_6\rho^l \leq \frac{C_6\rho}{1-\rho},$$

and thus

$$C_5 \prod_{l=1}^{i-1} (1 + C_6\rho^l) \leq \chi, \quad i = 1, \ldots, I.$$

Consider the case $i = 1$, that is the first iteration of **improve**. Then we have

$$(x, s) = (x^k, s^k) + \overline{\theta}(u^k, v^k),$$

where (u^k, v^k) is the solution of (5.3) with $(x, s) = (x^k, s^k)$ since a fast step was just taken at iteration k in the main algorithm. Hence by Lemma 5.5, we deduce that

$$\|(x^k, s^k) - (x, s)\| \leq \|(u^k, v^k)\| \leq C_5\tau_k.$$

Thus the bound (6.4), and therefore also (5.11), holds for the point (6). Because

$$C_7\chi^2 \frac{\tau_k}{\gamma^t} = C_7\chi^2 \frac{\tau_k}{\gamma^{t_k}} \leq C_7\chi^2 \frac{\tau_K}{\gamma^{t_K}} \leq C_7\chi^2 \frac{\tau_K}{\gamma^{t_K+I}} \leq \rho,$$

Lemma 5.7 applies, which implies that a fast step is taken at the first iteration of **improve**. We now consider the i-th iteration of the internal loop of **improve**. Let us assume that our assertions hold for iterations 1 through $i-1$. Let (x^-, s^-) denote the value of (x, s) at the start of iteration $i-1$, and let (u^-, v^-) be the search direction calculated during this iteration. We reserve the notation (x, s) for the current point at the start of iteration i. To obtain an estimate of $\|(x^k, s^k) - (x, s)\|$, we note that our inductive hypothesis (6.4) implies that

$$\|(x^k, s^k) - (x^-, s^-)\| \leq \left[C_5 \prod_{l=0}^{i-2}(1 + C_6\rho^l)\right] \tau_k.$$

Applying Lemma 5.6 to the step (u^-, v^-) taken during iteration $i-1$, with χ replaced by $C_5 \prod_{l=0}^{i-2}(1 + C_6\rho^l)$, we deduce that

$$\begin{aligned}
& \|(x^k, s^k) - (x, s)\| \\
\leq\ & \|(x^k, s^k) - (x^-, s^-)\| + \|(x^-, s^-) - (x, s)\| \\
\leq\ & \left[C_5 \prod_{l=0}^{i-2}(1 + C_6\rho^l)\right] \tau_k + \|(u^-, v^-)\| \\
\leq\ & \left[C_5 \prod_{l=0}^{i-2}(1 + C_6\rho^l)\right] \tau_k + C_6 \left[C_5 \prod_{l=0}^{i-2}(1 + C_6\rho^l)\right] \tau^- \\
\leq\ & \left[C_5 \prod_{l=0}^{i-2}(1 + C_6\rho^l)\right] (1 + C_6\rho^{i-1})\tau_k.
\end{aligned}$$

The final inequality follows from the fact that $\tau^- \leq \rho^{i-1}\tau_k$, since (x^-, s^-) is obtained by taking $i-1$ fast steps (one step in the main algorithm, followed by $i-2$ iterations of the **improve** loop). We have shown that the bound (6.4) holds at iteration i as well. Since

$$C_7\chi^2\frac{\tau_k}{\gamma^t} = C_7\chi^2\frac{\tau_k}{\gamma^{t_k+i}} \leq C_7\chi^2\frac{\tau_K}{\gamma^{t_k+I}} \leq C_7\chi^2\frac{\tau_K}{\gamma^{t_K+I}} \leq \rho,$$

Lemma 5.7 applies again, and therefore a fast step is accepted at iteration i. This completes our induction argument and proves our lemma. $\square$

We end this section by showing that the sequence $\{\mu_k\}$ converges to zero with Q-order at least $I+2$. Since $\gamma_{\min}\rho_0\tau \leq \mu \leq (1+\beta_{\max})\tau$, this is equivalent to show that for any $\epsilon > 0$,

$$\limsup_{k\to\infty}\frac{\tau_{k+1}}{{\tau_k}^{I+2-\epsilon}} = 0.$$

An equivalent characterization of the Q-order $I+2$ convergence is

$$\liminf_{k\to\infty}\frac{\ln \tau_{k+1}}{\ln \tau_k} \geq I+2 \tag{6.5}$$

(see Ortega and Rheinboldt [9] or Potra [10]).

Theorem 6.2 *Suppose $\mathcal{F}^c$ is not empty, and λ during the safe branch is defined by (3.8), then the main sequence $\{\mu_k\}, k = 0, 1, \ldots,$ converges to zero with Q-order at least $I+2$.*

Proof. We will show (6.5). Consider $k \geq K$. By Lemma 6.1, a fast step is taken by the main algorithm and all I iterations of **improve**, so we can apply inequality (5.22) $I+1$ times to bound τ_{k+1}:

$$\tau_{k+1} \leq \frac{C_7^{I+1} \chi^{2(I+1)} \tau_k^{I+1}}{\gamma^{(I+1)t_k + I(I+1)/2}}. \tag{6.6}$$

From Lemma 5.7 and (6.6) we deduce that

$$\frac{\tau_{k+1}}{\tau_k} \leq C_9 \left(\frac{\tau_k}{\gamma^{t_k}} \right)^{I+1} \rightarrow 0, \tag{6.7}$$

where

$$C_9 := \frac{C_7^{I+1} \chi^{2(I+1)}}{\gamma^{I(I+1)/2}}.$$

This means that $\{\tau_k\}$ (and therefore $\{\mu_k\}$) converges to zero at least Q-superlinearly. By taking logarithms in (6.6), we get

$$\ln \tau_{k+1} \leq \ln C_9 + (I+2) \ln \tau_k - (I+1) t_k \ln \gamma. \tag{6.8}$$

We may assume that k is sufficiently large so that $\tau_k < 1$. Then from (6.8) we have

$$\frac{\ln \tau_{k+1}}{\ln \tau_k} \geq I + 2 + \frac{C_9}{\ln \tau_k} - ((I+1)\ln \gamma) \frac{t_k}{\ln \tau_k}. \tag{6.9}$$

Obviously, as $k \rightarrow \infty$, the second term in the right-hand side vanishes.
Since $t_k \leq (I+1)k + 1$, it suffices to prove

$$\lim_{k \rightarrow \infty} \frac{k}{\ln \tau_k} = 0. \tag{6.10}$$

Suppose, by contradiction, that (6.10) does not hold. Then there exists a subsequence, also denoted by τ_k, such that there exists $\nu > 0$,

$$\frac{k}{-\ln \tau_k} \geq \nu, \quad \text{for all} \quad k > 0,$$

which implies $(1/e)^k \le \tau_k^\nu$. From (6.7), there exists a positive integer J such that for all $k \ge J$, $\tau_{k+1} \le (.5/e)^{1/\nu}\tau_k$. Hence, for all $k > J$,

$$(1/e)^k \le \tau_k^\nu \le \left((.5/e)^{(k-J)/\nu}\tau_J\right)^\nu = (.5/e)^{(k-J)}\tau_J^\nu.$$

It follows that for all $k > J$ we must have $2^k \le (.5/e)^{-J}\tau_J^\nu$, which is a contradiction. This completes the proof. $\square$

7 Concluding Remarks

- Our algorithm can solve all P_*-matrix linear complementarity problems , without knowing the classification number κ in advance. The computational complexity of the algorithm depends on κ.
- The algorithm is new even for monotone LCP where it attains $O(nL)$-iteration complexity, which improves the $O(n^2L)$-iteration complexity of the algorithm of [17].
- The algorithm requires only one matrix factorization at each iteration, compared to two matrix factorizations of predictor-corrector algorithms (cf. [8], [11], [12], [13]).
- Asymptotically, the algorithm requires $I+1$ backsolves at each iteration to get fast local convergence of Q-order at least $I+2$ for problems having a strictly complementary solution.
- The algorithm can be easily extended to horizontal LCP and mixed LCP (see [1]).

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