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Model Updating by Adding Known Masses

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SUMMARY

New approaches are developed that use measured data to adjust the analytical mass and stiffness matrices of a system so that the agreement between the analytical modes of vibration and the modal survey is improved. By adding known masses to the structure of interest, measuring the modes of vibration of this mass-modified system, and finally using this set of new data in conjunction with the initial modal survey, the analytical mass matrix of the structure can be corrected, after which the analytical stiffness matrix can be readily updated. Manipulating the correction matrices into vector forms, the connectivity information can be enforced, thereby preserving the physical configuration of the system and reducing the sizes of the least squares problems that need to be solved. Solution techniques for updating the system matrices are introduced, and the numerical issues associated with solving overdetermined and underdetermined least squares problems are investigated. The effects of round-off errors are also studied, and heuristic criteria are given for determining the minimum number of modes that need to be measured in order to ensure sufficiently accurate updated mass and stiffness

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matrices. Numerical experiments are presented to validate the proposed model updating techniques, to illustrate the effects of the number of measured modes on the quality of the updated model, to show how the magnitudes and locations of the added masses influence the updated matrices, and to highlight the numerical issues discussed in this paper.

KEY WORDS: Model updating, Added masses, Least squares

1. Introduction

Highly accurate and detailed analytical models are required to analyze and predict the dynamical behavior of complex structures during analysis and design. With the proliferation of digital computers, new methods of analysis have been developed, in particular through the method of finite elements. Once the finite element model of a physical system is constructed, its accuracy is often tested by comparing its analytical modes of vibration (or natural frequencies and mode shapes) with those obtained from the physical system during a modal survey. If the agreement between the two is good, then more credence is given to the analytical model, and it can be used with confidence for future analysis. If the correlation between the two is poor, then assuming that the experimental measurements are correct, the analytical model must be adjusted so that the agreement between predictions and test results is improved. The updated model may then be considered to be a better dynamical representation of the structure than the initial analytical model. The updated model can subsequently be used with reasonable accuracy to assess the stability and control characteristics and to predict the dynamical responses of the structure. The above process of correcting the system matrices is known as *model updating*.

In recent years various methods have been developed to improve the quality of the analytical finite element models using test data. A detailed discussion of every approach is beyond the scope of this paper, and interested readers are referred to the recent survey paper by Mottershead and Friswell [1]. In the following paragraphs, some commonly used model updating techniques will be briefly reviewed.

Berman [2] proposed an updating scheme based on the Lagrange multipliers formalism that uses the measured mode shapes to correct the mass matrix of a structure. This updating algorithm identifies a set of minimum changes in the analytical mass matrix so that the measured modes are orthogonal to the updated mass matrix of the system. Using essentially the method first introduced by Baruch and Bar Itzhack [3], Wei [4] developed an optimal method to update the stiffness matrix of a structure. He also employed the Lagrange multipliers approach to adjust the stiffness matrix, subjected to the constraints of satisfying the generalized eigenvalue problem, the orthogonality condition of the measured mode shapes and the symmetry property of the stiffness matrix.

The Lagrange multipliers approaches to update the system matrices return fully populated (or *dense*) mass and stiffness matrices that may not bear any resemblance to the physical system being analyzed. To preserve the physical load paths of the original analytical model, Kabe [5] assumed the analytical mass matrix to be correct and incorporated the readily available structural connectivity information in addition to the test data to optimally adjust the stiffness matrix. The adjustments he performed ensure that zero and nonzero elements of the analytical model are preserved, and the adjusted model exactly reproduces the modes used in the identification. He also utilized a Lagrange multipliers technique, so that the percentage change to each stiffness element is minimized. While Kabe's approach to updating the stiffness matrix is straightforward, the assumption that the actual mass matrix is identical to the analytical mass matrix remains debatable [6].

Using an approach based on matrix perturbation theory, Chen, Kuo and Garba [7] found the correction mass and stiffness matrices by enforcing the orthogonality conditions of the measured mode shapes with respect to the system matrices. Like the schemes proposed by Berman and Wei, however, the updating algorithms also return fully populated mass and stiffness matrices, thus failing to preserve the physical connectivity of the system. In addition, because the approach outlined in [7] is based on perturbation theory, the updating algorithm can only be applied when the deviations of the actual parameters from the analytical values are small. Finally, the derivation carried out by Chen *et al.* requires that the measured modal matrix, $[X]$, be properly normalized with respect with the actual mass and stiffness matrices, $[M]$ and $[K]$, of the system, such that it satisfies $[X]^T[M][X] = [I]$ and $[X]^T[K][X] = [\Lambda]$. Because the objective of model updating is to correct the system matrices, $[M]$ and $[K]$ are not known *a priori*. Thus, the orthogonality constraints cannot be enforced, and their updating algorithm cannot be applied in practice.

In this paper, new model updating schemes are introduced that adjust the system mass and stiffness matrices from an incomplete set of measured modes. The underlying principle of the proposed scheme is to add known masses to the physical structure, measure the modes of vibration of the new system, and then use the new set of measurements in conjunction with the original set of experimental data to correct the mass and stiffness matrices of the actual structure.

2. Proposed Model Updating Algorithm

Consider the analytical model of a given structure, with N degrees-of-freedom, whose modes of vibration are given by the solutions of the following generalized eigenvalue problem:

$$[K_o][X_o] = [M_o][X_o][\Lambda_o] \tag{1}$$

where $[M_o]$ and $[K_o]$ are the symmetric analytical mass and stiffness matrices of the system, $[X_o]$ is the $N \times N$ modal matrix (whose columns correspond to the eigenvectors or mode shapes) of the analytical model, and $[\Lambda_o]$ is an $N \times N$ diagonal matrix whose elements correspond to the eigenvalues (or the square of the natural frequencies) of the analytical model. A mode of vibration constitutes a given natural frequency and its corresponding mode shape (or the square root of an eigenvalue and its corresponding eigenvector).

Experimentally, it is often difficult if not impossible to measure the same number of modes as the number of degrees-of-freedom of the analytical model. Thus, the measured data are said to be incomplete. A problem unrelated to that previously described, but also commonly referred to as "incomplete" occurs when the measured eigenvector contains fewer coordinates than are available from the analytical model. In common with other model updating techniques, the measured eigenvectors must first be expanded before the proposed algorithms can be applied. Various mode expansion techniques can be found in references [8] and [9]. In this paper, we will assume that all the coordinates of the eigenvectors can be measured. This allows us to focus our attention on the quality of the proposed updating algorithm, and not confound the resulting updates with errors introduced by mode shape expansion. Therefore, we will reserve the word "incomplete" to mean when the test measurements contain fewer modes than those of the analytical model.

Regardless of whether the measured modes are complete or incomplete, the modes of vibration of the actual system must satisfy the following generalized eigenvalue problem:

$$[K][X] = [M][X][\Lambda] \tag{2}$$

where $[M]$ and $[K]$ are the actual $N \times N$ symmetric mass and stiffness matrices of the physical system, $[X]$ is the measured $N \times N_e$ modal matrix (N_e denotes the number of measured modes; $N_e \leq N$), and $[\Lambda]$ is an $N_e \times N_e$ diagonal matrix whose elements correspond to the measured eigenvalues of the system. Knowing $[M_o]$, $[K_o]$, $[X]$ and $[\Lambda]$, the objective of model updating is to correct $[M_o]$ and $[K_o]$ so that the new analytical system matrices yield modes of vibration that are closer to the measured data than they were initially.

2.1. Mass Updating

To update the mass matrix of the analytical model, we add a known mass matrix, $[M_a]$, to the physical structure, at locations coincident with the nodes of the finite element model in order to preserve the size of the initial analytical model. The resulting system satisfies

$$[K][X_a] = ([M] + [M_a])\,[X_a][\Lambda_a] \tag{3}$$

where $[X_a]$ corresponds to the $N \times N_e$ measured modal matrix of the new system, and $[\Lambda_a]$ is an $N_e \times N_e$ diagonal matrix, whose elements are the measured eigenvalues of this mass-modified system. It is assumed that the added masses do not alter the stiffnesses of the system significantly. Taking the transpose of Eq. (2) and postmultiplying the resulting matrix equation by $[X_a]$, we get

$$[X]^T[K][X_a] = [\Lambda][X]^T[M][X_a] \tag{4}$$

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Premultiplying Eq. (3) by $[X]^T$, we have

$$[X]^T[K][X_a] = [X]^T\left([M]+[M_a]\right)[X_a][\Lambda_a] \tag{5}$$

Subtracting Eq. (5) from (4), we obtain

$$[\Lambda][X]^T[M][X_a] - [X]^T[M][X_a][\Lambda_a] = [Q] \tag{6}$$

where

$$[Q] = [X]^T[M_a][X_a][\Lambda_a] \tag{7}$$

Let

$$[P] = [X]^T[M][X_a] \tag{8}$$

then Eq. (6) simplifies to

$$[\Lambda][P] - [P][\Lambda_a] = [Q] \tag{9}$$

Because both $[\Lambda]$ and $[\Lambda_a]$ are diagonal matrices, Eq. (9) can be easily expanded so that its (i, j)th element yields

$$(\lambda_i - \lambda_{aj})P_{ij} = Q_{ij} \tag{10}$$

where λ_{aj} is the jth measured eigenvalue of the mass-modified system and $i, j = 1, \ldots, N_e$. Assuming the N_e measured eigenvalues of the original and the mass-modified systems do not coincide, we can solve for all of the unknowns P_{ij} and then construct matrix $[P]$. If any two measured eigenvalues of the original and the mass-modified systems coincide, we simply change the added masses or their locations to make the eigenvalues distinct. See Section 4.4 for further discussion on the selection of the number, placement and magnitude of the added masses.

The actual and the analytical mass matrices of the system are related as follows

$$[M] = [M_o] + [\delta M] \tag{11}$$

where $[\delta M]$ represents the correction to the analytical mass matrix. Rewriting Eq. (8), we obtain

$$[X]^T[\delta M][X_a] = [P] - [X]^T[M_o][X_a] \tag{12}$$

Because $[X]$ and $[X_a]$ are both rectangular matrices (assuming $N_e < N$), they have no inverses. However, Eq. (12) can be rewritten so that $[\delta M]$ appears as an unknown column vector $\boldsymbol{\delta}\mathbf{m}$ as follows:

$$[\,A\,]\,\boldsymbol{\delta}\mathbf{m} = \mathbf{r} \tag{13}$$

where

$$\boldsymbol{\delta}\mathbf{m} = [\delta m_{11} \cdots \delta m_{1N} \mid \delta m_{21} \cdots \delta m_{2N} \mid \cdots \mid \delta m_{N1} \cdots \delta m_{NN}]^T \tag{14}$$

and

$$\mathbf{r} = [r_{11} \cdots r_{1N_e} \mid r_{21} \cdots r_{2N_e} \mid \cdots \mid r_{N_e 1} \cdots r_{N_e N_e}]^T \tag{15}$$

In Eq. (14), δm_{ij} corresponds to the (i, j)th element of $[\delta M]$. Matrix $[A]$ is of size $N_e^2 \times N^2$, whose elements can be determined by expanding the left hand side of Eq. (12); vector $\mathbf{r}$ is of length N_e^2, whose components can be obtained by expanding the right hand side of Eq. (12). The technique to solve Eq. (13) will be discussed in Section 2.3. We now turn our attention to updating the stiffness matrix of the system.

2.2. Stiffness Updating

From Eqs. (4) and (5), we get

$$[\Lambda]^{-1}[X]^T[K][X_a] = [X]^T[M][X_a] \tag{16}$$

and

$$[X]^T[K][X_a][\Lambda_a]^{-1} = [X]^T\left([M] + [M_a]\right)[X_a] \tag{17}$$

Subtracting Eq. (16) from Eq. (17), we have

$$[X]^T[K][X_a][\Lambda_a]^{-1} - [\Lambda]^{-1}[X]^T[K][X_a] = [S] \tag{18}$$

where

$$[S] = [X]^T[M_a][X_a] \tag{19}$$

Defining

$$[U] = [X]^T[K][X_a] \tag{20}$$

then Eq. (18) simplifies to

$$[U][\Lambda_a]^{-1} - [\Lambda]^{-1}[U] = [S] \tag{21}$$

Because both $[\Lambda]$ and $[\Lambda_a]$ are diagonal, Eq. (21) can be easily expanded so that its (i, j)th element yields

$$\left(\frac{1}{\lambda_{aj}} - \frac{1}{\lambda_i}\right) U_{ij} = S_{ij} \tag{22}$$

where $i, j = 1, \ldots, N_e$. Assuming that the N_e measured eigenvalues of the original and the mass-modified structures are distinct, the unknowns U_{ij} can be computed and matrix $[U]$ can be assembled.

As in the case of the mass matrix, the actual and the analytical stiffness matrices are related by

$$[K] = [K_o] + [\delta K] \tag{23}$$

where $[\delta K]$ represents the matrix of stiffness offsets. In order to find $[\delta K]$, we first rewrite Eq. (20) as

$$[X]^T[\delta K][X_a] = [U] - [X]^T[K_o][X_a] \tag{24}$$

As before, Eq. (24) can be manipulated into the following form

$$[\,B\,]\,\delta\mathbf{k} = \mathbf{h} \tag{25}$$

where

$$\delta\mathbf{k} = [\delta k_{11}\cdots\delta k_{1N} \mid \delta k_{21}\cdots\delta k_{2N} \mid \;\cdots\; \mid \delta k_{N1}\cdots\delta k_{NN}]^T \tag{26}$$

and

$$\mathbf{h} = [h_{11}\cdots h_{1N_e} \mid h_{21}\cdots h_{2N_e} \mid \;\cdots\; \mid h_{N_e1}\cdots h_{N_eN_e}]^T \tag{27}$$

In Eq. (26), δk_{ij} represents the (i,j)th element of $[\delta K]$. Matrix $[B]$ is of size $N_e^2 \times N^2$ and vector $\mathbf{h}$ is of length N_e^2. The components of $[B]$ and $\mathbf{h}$ can be obtained by expanding the left and right hand sides of Eq. (24), respectively.

2.3. Solution Technique

Equations (13) and (25) are of the general form

$$[\,G\,]\,\mathbf{y} = \mathbf{z} \tag{28}$$

where matrix $[\,G\,]$ and vector $\mathbf{z}$ are both known and of size $N_e^2 \times N^2$ and length N_e^2, respectively. Because the modal matrices $[X]$ and $[X_a]$ are always of full rank, matrix $[G]$ will also be of full rank (see Appendix A for a detailed proof). When $N_e = N$, Eq. (28) can be solved exactly by using simple Gaussian elimination. When $N_e < N$, Eq. (28) yields an underdetermined problem (that is, the number of equations is less than the number of unknowns), which, because the system has full rank, will have an infinite number of solutions. To render the solution unique, we may choose a solution vector **y** such that the Euclidean norm of vector **y** is minimized. The resulting solution is referred to as the minimum norm least squares solution to Eq. (28). Because the analytical and the actual system matrices are presumed to be close, it is reasonable to use the unique minimum norm solutions to update the analytical mass and stiffness matrices, respectively.

Initially it may appear that one must solve two underdetermined, least squares problems of size $N_e^2 \times N^2$ (assuming $N_e < N$) in order to update the system matrices [see Eqs. (13) and (25)]. However, the optimal matrix storage scheme commonly used in finite elements [10] can be applied to pass along the available sparsity information of the analytical system, and to impose the condition that all zero elements in the analytical system matrices remain zeros in the adjusted system matrices. Mathematically, this can be achieved by eliminating all of the known zero elements from **y** and by deleting all the corresponding columns in $[\,G\,]$. This drastically reduces the size of the problem to be solved.

To illustrate how the connectivity information can be used to reduce the size of the least squares problem, consider a system whose analytical mass matrix is diagonal. Then $\delta m_{ij} = 0$ for $i \neq j$, and Eq. (13) reduces to

$$[\,A'\,]\,\delta\mathbf{m}' = \mathbf{r} \tag{29}$$

where $[\,A'\,]$ is obtained from $[\,A\,]$ by deleting all the columns that multiply by δm_{ij} for $i \neq j$, and

$$\delta\mathbf{m}' = [\delta m_{11} \quad \delta m_{22} \quad \cdots \quad \delta m_{NN}]^T \tag{30}$$

Thus, the initial problem of size $N_e^2 \times N^2$ is reduced to one of size $N_e^2 \times N$. The resulting least squares problem will be either overdetermined (that is, the number of equations is greater than

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or equal to the number of unknowns) or underdetermined, depending on whether $N_e^2 \geq N$ or $N_e^2 < N$, respectively.

Similarly, the connectivity information of the analytical stiffness matrix can also be enforced to reduce the size of the least squares problem to be solved. For instance, if the analytical stiffness matrix is tri-diagonal, then $\delta k_{ij} = 0$ for $|i-j| > 1$, and Eq. (25) reduces to

$$[\,B'\,]\,\delta\mathbf{k}' = \mathbf{h} \tag{31}$$

where $[\,B'\,]$ is obtained from $[\,B\,]$ by deleting all the columns that multiply by δk_{ij} for $|i-j| > 1$, and

$$\delta\mathbf{k}' = [\delta k_{11} \quad \delta k_{12} \mid \delta k_{21} \quad \delta k_{22} \quad \delta k_{23} \mid \cdots \mid \delta k_{NN-1} \quad \delta k_{NN}]^T \tag{32}$$

Thus, the initial problem of size $N_e^2 \times N^2$ is reduced to one of size $N_e^2 \times (3N-2)$.

The proposed updating algorithms rely on the correctness of the connectivity information. Because the basis of model updating is the analytical model, the analytical model must capture certain physical attributes of the actual system. Here, we assume that the connectivity information in the analytical model is correct, and the proposed updating algorithms maintain that physical sparsity pattern in the mass and stiffness matrices. It will be shown that the proposed updating schemes are very forgiving when the mass and stiffness parameters vary substantially between the analytical and the actual systems.

3. Numerical Issues

The numerical issues encountered when solving a least squares problem differ in nature depending on whether the problem is overdetermined or underdetermined. Because matrix $[G]$ of Eq. (28) is of full rank, a least squares solution to Eq. (28) always exists. However, since the reduced systems of Eqs. (29) and (31) were obtained by deleting certain columns of $[G]$ to enforce the sparsity information, matrix $[G]$ may suffer rank-deficiency. While there exist techniques for determining the numerical rank of such systems and subsequently finding the unique minimum norm solution, numerical perturbations in the data, either in the form of measurement or round-off errors, can still give rise to computationally induced inaccuracies. It is therefore necessary to understand how $\mathbf{y}$ is affected by perturbations in $[G]$ and $\mathbf{z}$.

To this end, let us examine the least squares solution of a general $m \times n$ linear system

$$[A]\mathbf{x} = \mathbf{b} \tag{33}$$

Let $\hat{\mathbf{x}}$ represent the least squares solution of the $m \times n$ perturbed system

$$[\tilde{A}]\hat{\mathbf{x}} = \tilde{\mathbf{b}} \tag{34}$$

where

$$[\tilde{A}] = [A] + [\delta A] \tag{35}$$

and

$$\tilde{\mathbf{b}} = \mathbf{b} + \delta\mathbf{b} \tag{36}$$

The perturbations $[\delta A]$ and $\delta\mathbf{b}$ can reflect either numerical inaccuracies in the floating point representations of $[A]$ and $\mathbf{b}$, or measurement errors during testing.

A problem is said to be numerically *ill-conditioned* when small errors in data lead to a relatively large error in the solution, regardless of how stable the algorithm is for solving the problem. The *condition number* of a linear system reveals the accuracy or inaccuracy of the computed result due to small perturbations. For the linear system of Eq. (33), its condition number is given by

$$\text{cond}\,([A]) = ||[A]|| \cdot ||[A]^{\dagger}|| \tag{37}$$

where $||[A]||$ denotes the Euclidean norm of $[A]$ and $||[A]^{\dagger}||$ is the pseudoinverse of $[A]$, defined as

$$[A]^{\dagger} = \begin{cases} \left([A]^T[A]\right)^{-1}[A]^T \ , & \text{for } m \geq n \\ [A]^T\left([A][A]^T\right)^{-1} \ , & \text{for } m < n \end{cases}$$

3.1. Overdetermined Systems

If a general $m \times n$ linear system of Eq. (33) is overdetermined ($m \geq n$) and full rank, the least squares solution to the system is *unique*. Premultiplying Eq. (33) by $[A]^T$ yields

$$[A]^T[A]\mathbf{x} = [A]^T\mathbf{b} \tag{38}$$

Because $[A]$ is full rank and $m \geq n$, $([A]^T[A])^{-1}$ exists. Thus, the unique least squares solution of Eq. (33) is given by

$$\mathbf{x} = \left([A]^T[A]\right)^{-1}[A]^T\mathbf{b} \tag{39}$$

where $\mathbf{x}$ is the best solution in the sense of minimizing $||[A]\mathbf{x} - \mathbf{b}||$.

The effects of perturbing $[A]$ and $\mathbf{b}$ separately on the solution of Eq. (33) have been well studied and are detailed in [11]. The results become more complicated when $[A]$ and $\mathbf{b}$ are perturbed simultaneously. Golub and van Loan in [12] provide the following theorem that can be used to analyze the condition of the system when both $[A]$ and $\mathbf{b}$ undergo perturbations. Defining

$$\epsilon = \max\left\{\frac{||[\delta A]||}{||[A]||}, \frac{||\delta\mathbf{b}||}{||\mathbf{b}||}\right\} \tag{40}$$

and letting θ be such that (for $||\mathbf{b}|| \neq 0$)

$$\sin(\theta) = \frac{||[A]\mathbf{x} - \mathbf{b}||}{||\mathbf{b}||} \neq 1 \tag{41}$$

then the upper bound on the relative solution error is given by

$$\frac{||\hat{\mathbf{x}} - \mathbf{x}||}{||\mathbf{x}||} \leq \epsilon \left(\frac{2\ \mathrm{cond}([A])}{\cos{(\theta)}} + \tan{(\theta)} \cdot (\mathrm{cond}([A]))^2 \right) + \mathcal{O}(\epsilon^2) \tag{42}$$

where

$$\tan{(\theta)} = \frac{||[A]\mathbf{x} - \mathbf{b}||}{\sqrt{||\mathbf{b}||^2 - ||[A]\mathbf{x} - \mathbf{b}||^2}} \tag{43}$$

Thus, if the residual norm $||[A]\mathbf{x} - \mathbf{b}||$ is nonzero, it is the *square* of the $\mathrm{cond}([A])$ which measures the sensitivity of the least squares solution to numerical perturbations.

The Normal Equations Method and the QR factorization algorithms are commonly used to solve an overdetermined least squares problem (for example, see [11]). Of the QR methods, Householder transformations, Givens rotations, or modified Gram-Schmidt orthogonalizations can be implemented. QR factorizations tend to be more numerically reliable, but are more expensive, in terms of total FLOP counts, by at least a factor of two. According to Golub and van Loan [12], the Normal Equations solution method produces an $\hat{\mathbf{x}}$ whose relative error always depends on the *square* of $\mathrm{cond}([A])$, whereas the QR approaches produce solutions which depend on both $\mathrm{cond}([A])$ and the product $||[A]\mathbf{x} - \mathbf{b}|| \cdot (\mathrm{cond}[A])^2$. Thus, if the norm of the residual $||[A]\mathbf{x} - \mathbf{b}||$ is small, QR has a distinct advantage over Normal Equations. However, if $||[A]\mathbf{x} - \mathbf{b}||$ is large and the problem is ill-conditioned, both methods will produce inaccurate results. Finally, because the extra computational expense necessary for implementing the QR solution method is a relatively small price to pay for increased numerical stability, a QR approach is used in this paper to solve overdetermined systems.

3.2. Underdetermined Systems

The fundamental difference between the solution of an overdetermined and an underdetermined least squares problem of the form of Eq. (33) of full rank is that while there is a *unique* least squares solution to the overdetermined system ($m \geq n$), there are an *infinite* number of solutions to the underdetermined system ($m < n$).

This can be understood when we consider that if $[A]\mathbf{x} = \mathbf{b}$ is an underdetermined system of equations, then every least squares solution x can be represented as

$$\mathbf{x} = \mathbf{x}_p + \mathbf{x}_N \tag{44}$$

where $\mathbf{x}_p$ is any solution satisfying $[A]\mathbf{x}_p = \mathbf{b}$, and $\mathbf{x}_N$ is in the null space of $[A]$. That is,

$$[A]\mathbf{x}_N = \mathbf{0} \tag{45}$$

The complete solution set is given by

$$\mathbf{x} = [A]^\dagger \mathbf{b} + \left([I] - [A]^\dagger [A]\right) \mathbf{y} \tag{46}$$

where $\mathbf{y}$ is an arbitrary n-vector. Recall that since $m < n$, $[A]^\dagger$ is defined by $[A]^\dagger = [A]^T([A][A]^T)^{-1}$. Comparing Eqs. (39) and (46), we note the absence of the arbitrary vector, $\mathbf{y}$, in Eq. (39).

When there are an infinite number of solutions, it is commonly the minimum norm solution that is sought and then used to correct the system matrices. Additionally, perturbation analysis has been done mainly for the minimum norm case. Golub and van Loan [12] show that for an underdetermined system of full rank, an upper bound on the relative error in the minimum norm solution is given by

$$\frac{||\hat{\mathbf{x}} - \mathbf{x}||}{||\mathbf{x}||} \leq \mathrm{cond}([A]) \left(\frac{||[\delta A]||}{||[A]||} \min(2, n - m + 1) + \frac{||\delta \mathbf{b}||}{||\mathbf{b}||} \right) + \mathcal{O}(\epsilon^2) \tag{47}$$

Equation (47) shows that the sensitivity of an underdetermined system now depends on $\mathrm{cond}([A])$, as opposed to the overdetermined case in which the sensitivity depends on $(\mathrm{cond}[A])^2$ (see Eq. (42)). Because $\mathrm{cond}([A]) \geq 1$ for any matrix $[A]$, the error bound for the overdetermined case can become fairly large unless the norm of the residual is sufficiently small. However, in the underdetermined case, the error bound does not depend explicitly on the norm of the residual, but only on the perturbations in $[A]$ and $\mathbf{b}$.

Algorithms for finding the least squares solution of an underdetermined system include the Normal Equations method and the QR factorization algorithms. A solution approach may either find some non-unique least squares solution to $[A]\mathbf{x} = \mathbf{b}$, or find the unique solution $\mathbf{x}$ such that the Euclidean norm of $\mathbf{x}$, $||\mathbf{x}||$, is minimized. The latter, which is the minimum norm solution, is given explicitly by

$$\mathbf{x} = [A]^{\dagger}\mathbf{b} \tag{48}$$

Of course, it is entirely possible that another solution besides the one with minimum norm may yield a more accurate updated model. However, as discussed previously, because $[\delta M]$ and $[\delta K]$ are assumed to be relatively small, it is reasonable to use the unique minimum norm solution in this case as corrections to the initial analytical matrices.

3.3. Relating Accuracy to N_e and N

For systems in which there is no special sparsity pattern to be maintained, the least squares system will always have dimension $N_e^2 \times N^2$ [see Eqs. (13) and (25)]. If the measured modes are incomplete (for $N_e < N$), the problem will always be underdetermined.

As stated previously, an overdetermined system of full rank has a *unique* least squares solution, whereas an underdetermined system of full rank admits a unique solution only when the additional minimum norm constraint is imposed. Often the physical structure will possess a special sparsity pattern in the analytical system that we will have to maintain. Our numerical experiments indicate that in certain cases, when the sparsity pattern of a physical structure is imposed, more accurate solutions are achieved when the least squares system is overdetermined as opposed to underdetermined, even if the systems are rank-deficient. This may be surprising in light of the fact that our error bounds (42) and (47) seem to indicate that there is more room for error in the overdetermined case. However, these are upper bounds and not exact error measurements. Additionally, if the norm of the residual vector is small compared to the perturbations in $[A]$ and $\mathbf{b}$, the error bound for the overdetermined system could in fact be smaller than that of the underdetermined system. Also, these experimental results make some intuitive sense: the more information we can gather about the physical system, the better our updated model becomes. Thus, to improve the accuracy of the updated model, solving an overdetermined system for this particular application is preferred.

The above discussion has direct implications about the minimum number of experimentally determined modes, N_e, which should be measured before implementing the proposed model updating algorithm. For example, if $\delta M_{ij} = 0$ for $i \neq j$ (that is, $[M]$ is a diagonal matrix), then the least squares system has dimension $N_e^2 \times N$. In this case, to induce an overdetermined system requires a minimum of

$$N_e \geq \sqrt{N} \tag{49}$$

measured modes. Similarly, in the case of the stiffness matrix, if the condition that $[K]$ is a tridiagonal matrix is imposed, at least

$$N_e \geq \sqrt{3N-2} \tag{50}$$

experimental modes are needed to achieve an overdetermined least squares system. The above results will be validated the following sections.

4. Numerical Experiments

To update the analytical mass and stiffness matrices using the proposed routines requires the solution of least squares problems, which can be either overdetermined or underdetermined, depending on the number of measured modes, N_e, used to perform the update. When solving a least squares problem, the CMLIB [13] routine *sglss* was implemented, which is specialized to handle both underdetermined and overdetermined systems of the form $[A]\mathbf{x} = \mathbf{b}$, where $[A]$ is an $m \times n$ matrix and $\mathbf{b}$ is a vector of length m. When the system is overdetermined $(m \geq n)$, the least squares solution is computed by decomposing the matrix $[A]$ into the product of an orthogonal matrix $[Q]$ and an upper triangular matrix $[R]$ (QR factorization). When the system is underdetermined $(m < n)$, the minimum norm solution is computed by factoring the matrix $[A]$ into the product of a lower triangular matrix $[L]$ and an orthogonal matrix $[Q]$ (LQ factorization). If matrix $[A]$ is determined to be rank deficient, that is, if the rank of $[A]$ is less than min(m,n), then again the minimum norm least squares solution is computed.

4.1. Model Updating Based on a Complete Set of Measured Modes

Consider the system of Fig. 1 with $N = 25$. We first examine the effectiveness of our model updating techniques for the case where the number of measured modes, N_e, is equal to the total degrees-of-freedom of the system, N (that is, the set of measured modes is complete). The analytical masses and stiffnesses are given by $m_o = 2.00$ kg and $k_o = 5.00$ N/m. The actual masses and stiffnesses are listed in Tables 1 and 2, respectively.

For the system of Fig. 1, because the mass matrix $[M]$ is diagonal, updating the mass matrix requires finding the solution of an $N_e^2 \times N$ least squares problem. Because the stiffness matrix is tri-diagonal, updating the stiffness matrix requires finding the solution of an $N_e^2 \times (3N-2)$

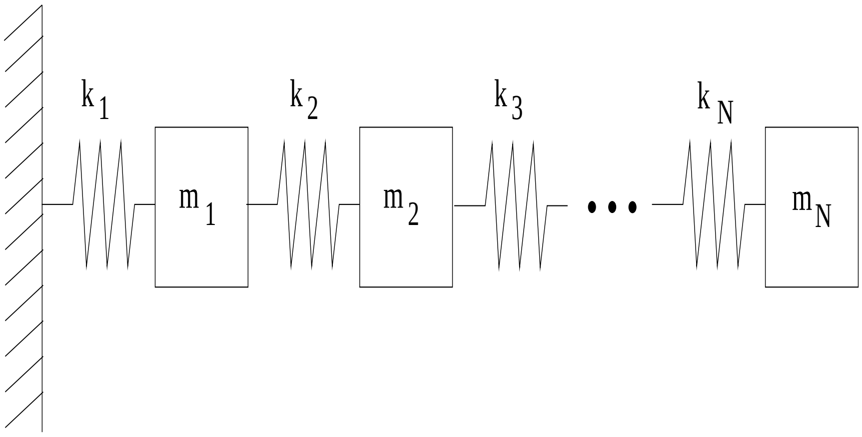

Figure 1. Simple chain of coupled oscillators.

least squares problem. In general, these systems may be overdetermined or underdetermined, depending on whether N_e^2 is greater than or less than N in the case of mass updating, and on whether N_e^2 is greater than or less than $(3N-2)$ in the case of stiffness updating. For this set of experiments, because $N_e = N$, both the mass update system and the stiffness update system will be overdetermined.

In Table 1 we compare the actual and the updated masses for the system of Fig. 1. For the purpose of numerical simulations, the ith actual mass, m_i, and the analytical mass, m_o, are related as follows:

$$m_i = m_o(1 + \Delta m_i) \tag{51}$$

where Δm_i represents the percent deviation of the ith actual mass from its nominal analytical value. The Δm_i's are randomly chosen (with uniform distribution), and they have a mean and standard deviation of -0.06 and 0.28, respectively. To perform the mass updating algorithm, lumped masses of magnitude 0.2 kg are added to masses (or nodes) 5, 10, 15, 20 and 25. The added masses are all an order of magnitude smaller than the nominal analytical masses. Note

Table I. The actual and the updated masses (in kg), for $N_e = N = 25$. The analytical masses are $m_o = 2.000$ kg.

m_{actual}	m_{update}	m_{actual}	m_{update}
m_1 =1.2942	1.2942	m_{14} =2.6722	2.6722
m_2 =2.1831	2.1831	m_{15} =1.3355	1.3355
m_3 =1.3117	1.3117	m_{16} =2.6680	2.6680
m_4 =2.6581	2.6581	m_{17} =1.3899	1.3899
m_5 =2.4371	2.4371	m_{18} =1.8632	1.8632
m_6 =1.7651	1.7651	m_{19} =1.6231	1.6231
m_7 =2.8502	2.8502	m_{20} =1.1578	1.1578
m_8 =1.7984	1.7984	m_{21} =1.1439	1.1439
m_9 =1.7793	1.7792	m_{22} =1.8189	1.8189
m_{10} =2.7588	2.7588	m_{23} =1.2320	1.2320
m_{11} =2.1221	2.1221	m_{24} =1.6389	1.6389
m_{12} =1.1613	1.1613	m_{25} =2.2254	2.2254
m_{13} =2.0234	2.0234	—–	—–

how well the updated masses correspond to the actual values, despite the large deviations of the actual masses from the analytical values.

Table 2 shows the actual and the updated stiffnesses for the system of Fig. 1. As in the case of the masses, the numerical choice for the actual stiffnesses, k_i, and the analytic stiffnesses,

Table II. The actual and the updated stiffnesses (in N/m), for $N_e = N = 25$. The analytical masses are $k_o = 5.000$ N/m.

k_{actual}	k_{update}	k_{actual}	k_{update}
k_1 =4.1400	4.1398	k_{14} =5.9877	5.9877
k_2 =6.8802	6.8804	k_{15} =5.4973	5.4973
k_3 =5.6052	5.6056	k_{16} =5.9483	5.9483
k_4 =6.5108	6.5109	k_{17} =5.0320	5.0320
k_5 =2.9343	2.9343	k_{18} =6.3608	6.3608
k_6 =7.1326	7.1326	k_{19} =6.2726	6.2726
k_7 =3.3072	3.3072	k_{20} =7.0572	7.0572
k_8 =3.2986	3.2986	k_{21} =6.6026	6.6026
k_9 =6.2021	6.2020	k_{22} =4.9326	4.9326
k_{10} =6.6399	6.6398	k_{23} =6.3932	6.3932
k_{11} =5.9489	5.9488	k_{24} =5.8004	5.8003
k_{12} =6.3203	6.3202	k_{25} =6.5935	6.5935
k_{13} =3.3357	3.3356	—–	—–

k_o, are chosen to have the following relationship:

$$k_i = k_o(1 + \Delta k_i) \tag{52}$$

where Δk_i represents the percent deviation of the ith actual stiffness from its nominal analytical value. The Δk_i's are randomly chosen (with uniform distribution), and they have a mean and standard deviation of 0.13 and 0.25, respectively. In this case, too, the updated

Table III. The analytical, actual and updated eigenvalues (in $1/\text{sec}^2$), for the analytical, actual and updated system parameters of Tables 4.1 and 4.1

$\lambda_{\text{analytical}}$	λ_{actual}	λ_{update}	$\lambda_{\text{analytical}}$	λ_{actual}	λ_{update}
$\lambda_1 = 0.0095$	0.0105	0.0105	$\lambda_{14} = 5.4613$	6.6635	6.6635
$\lambda_2 = 0.0851$	0.0954	0.0954	$\lambda_{15} = 6.0697$	7.7364	7.7365
$\lambda_3 = 0.2353$	0.2580	0.2580	$\lambda_{16} = 6.6618$	8.8005	8.8005
$\lambda_4 = 0.4577$	0.4674	0.4674	$\lambda_{17} = 7.2287$	9.8139	9.8139
$\lambda_5 = 0.7489$	0.8441	0.8441	$\lambda_{18} = 7.7618$	9.9112	9.9112
$\lambda_6 = 1.1046$	1.1282	1.1282	$\lambda_{19} = 8.2531$	10.7379	10.7378
$\lambda_7 = 1.5193$	1.7577	1.7577	$\lambda_{20} = 8.6950$	11.3051	11.3051
$\lambda_8 = 1.9868$	2.1220	2.1220	$\lambda_{21} = 9.0810$	12.2057	12.2056
$\lambda_9 = 2.5000$	2.7864	2.7864	$\lambda_{22} = 9.4051$	12.9773	12.9774
$\lambda_{10} = 3.0511$	3.2055	3.2056	$\lambda_{23} = 9.6624$	13.2552	13.2555
$\lambda_{11} = 3.6317$	3.5768	3.5768	$\lambda_{24} = 9.8490$	14.8620	14.8619
$\lambda_{12} = 4.2330$	4.9515	4.9515	$\lambda_{25} = 9.9621$	18.8955	18.8955
$\lambda_{13} = 4.8460$	5.4920	5.4920	—	—	—

stiffness values track the actual stiffnesses very closely, even though the deviations between the actual and the analytical stiffnesses are large.

In Table 3, we compare the analytical eigenvalues for the system of Fig. 1 with those of the actual and the updated systems. The analytical eigenvalues of the system are found by solving Eq. (1), the actual eigenvalues are obtained from a modal survey, and the updated eigenvalues are calculated by solving

$$[K_{\text{update}}]\mathbf{x}_{\text{update}} = \lambda_{\text{update}}[M_{\text{update}}]\mathbf{x}_{\text{update}} \tag{53}$$

where $[M_{\text{update}}]$ and $[K_{\text{update}}]$ are the updated mass and stiffness matrices (see Tables 1 and 2 for the updated system parameters), and $(\lambda_{\text{update}}, \mathbf{x}_{\text{update}})$ are the updated eigenvalues and eigenvectors. From Table 3, note the excellent agreement between the updated and the actual (or the measured) eigenvalues of the system, despite the large differences between the actual and the initial analytical eigenvalues. For $i = 1, \cdots, N$, $(\lambda_{\text{actual}} - \lambda_{\text{update}})_i$ is consistently two to four orders of magnitude smaller than the corresponding $(\lambda_{\text{actual}} - \lambda_{\text{analytical}})_i$, which clearly indicates the drastic improvement in the eigenvalues of the updated model over the initial analytical system.

When the set of measured modes of vibration is complete, *i.e.*, when $N_e = N$, the updating algorithms can be used to accurately correct the analytical mass and stiffness matrices. Because the proposed updating algorithms allow the well known and readily available connectivity information to be enforced, our mass and stiffness updating schemes generate matrices that not only produce excellent numerical agreement between the physical and the updated system, but also preserve the physical configuration of the structure exactly.

Finally, while 5 lumped masses were used to generate the results of Tables 1 and 2, fewer lumped masses can also be used to execute the updating algorithms. For example, when only 2 lumped masses of magnitude 0.2 kg are added at nodes 1 and 25, the resulting updated masses are found to be nearly as accurate as those of Table 1. However, the proposed stiffness updating algorithm returns stiffnesses that are not nearly as accurate as those of Table 2. Nevertheless, the resulting updated stiffnesses are still much closer to the actual values than the initial analytical ones. Clearly the number, placement and magnitude of the added lumped masses affect the quality of the updates. Their influences on the proposed updating algorithms will be detailed in Section 4.4.

4.2. Model Updating Based on an Incomplete Set of Measured Modes

The results of section 4.1 were obtained for the ideal case when the set of measured modes is complete. Note, however, that because of physical limitations or time and cost constraints, the set of measured modes is often incomplete. Thus in practice, N_e is almost always less than N. Of considerable interest, then, is the effect of the number of measured modes, N_e, on the quality of the updated system matrices.

Theoretically, N_e can include any modes of vibration of the system. However, because the lower modes are typically easier to measure experimentally than the higher modes, the parameter N_e will be used in the subsequent analysis to represent the lowest N_e measured modes of the structure. Thus, when $N_e = 5$, the first 5 measured modes will be used to perform the update.

To quantify the accuracy of the mass updating algorithm, we measure the relative error in the updated masses with the following relation:

$$\epsilon_m = \frac{|\mathbf{m}_{\text{update}} - \mathbf{m}_{\text{actual}}|}{|\mathbf{m}_{\text{actual}}|} \tag{54}$$

where $\mathbf{m}_{\text{update}}$ and $\mathbf{m}_{\text{actual}}$ are vectors of length N whose elements are the updated and the actual lumped masses, respectively, and $|\mathbf{a}|$ represents the Euclidean norm of the vector $\mathbf{a}$. To illustrate the improvement of the updated masses over their initial analytical values, the following relative error parameter for the analytical masses is also introduced:

$$(\epsilon_m)_o = \frac{|\mathbf{m}_{\text{analytical}} - \mathbf{m}_{\text{actual}}|}{|\mathbf{m}_{\text{actual}}|} \tag{55}$$

Similar expressions can be defined for the stiffness and the eigenvalue error parameters, denoted by ϵ_k and ϵ_λ, respectively.

An error parameter can also be defined for the modal matrix of the system. It is common practice to check for the correctness of the modal matrix by resorting to the orthogonality characteristics of the normal modes. If the actual modal matrix, $[X]$, is properly normalized, then it is orthogonal with respect to the actual mass matrix, $[M]$, such that

$$[X]^T[M][X] = [I] \tag{56}$$

where $[I]$ is the identity. Because the updated modal matrix is approximate, replacing the actual modal matrix, $[X]$, by the updated modal matrix, $[X_{\text{update}}]$, in Eq. (56) yields

$$[X_{\text{update}}]^T[M][X_{\text{update}}] = [I_{\text{update}}] \tag{57}$$

where $[X_{\text{update}}]$ is normalized so that the diagonal elements of $[I_{\text{update}}]$ are identically one. In general, $[I_{\text{update}}]$ is a full matrix, but if the update is good, all the off diagonal terms will be small. The following can be used to describe the accuracy of the modal matrix quantitatively:

$$\epsilon_X = ||[I] - [I_{\text{update}}]|| \tag{58}$$

As before, to gauge the improvement of the updated modal matrix over the initial analytical modal matrix, the following error parameter for the analytical modal matrix is also introduced:

$$(\epsilon_X)_o = ||[I] - [X_{\text{analytic}}]^T[M][X_{\text{analytic}}]|| \tag{59}$$

where $[X_{\text{analytic}}]$ represents the normalized analytical modal matrix of the system.

For an updated model to be judged better than the initial analytical model, we must have $\epsilon_m < (\epsilon_m)_o$, $\epsilon_k < (\epsilon_k)_o$, $\epsilon_\lambda < (\epsilon_\lambda)_o$ and $\epsilon_X < (\epsilon_X)_o$. For an updated model to be considered accurate, ϵ_m, ϵ_k, ϵ_λ and ϵ_X must be sufficiently small. Finally, the smaller the error parameters, the better the updated model.

Figure 2 shows the variations of ϵ_m and ϵ_k for the system parameters of Tables 1 and 2 as a function of N_e. Also shown are the corresponding $(\epsilon_m)_o$ and $(\epsilon_k)_o$, which are independent of N_e and are given by the horizontal lines. Note the improvement in the updated parameters as N_e increases. The experimental results are consistent with physical intuition: the larger the knowledge space or the more information we can gather about the physical system, the better our updated model becomes.

The curves of Fig. 2 reveal the fewest N_e needed in order to achieve a certain level of accuracy. Therefore, these curves can be used to determine the smallest N_e that should be obtained for the purpose of performing the update. Interestingly, each curve reaches a saturation point beyond which additional information does not lead to significant improvement in the corrected model. The results clearly indicate that there is a minimum N_e that needs to be obtained in order to ensure sufficient accuracy in the adjusted model.

Figure 3 shows the resulting error parameters for the updated eigenvalues and modal matrix, ϵ_λ and ϵ_X, as a function of N_e. Note how well the modes of vibration of the updated system track the actual system, especially as N_e becomes large. Thus, the proposed updating algorithms return system matrices that yield modes of vibration that are much closer to the measured data than those of the initial analytical model.

4.3. The Minimum N_e Needed to Ensure Sufficient Accuracy

An overdetermined system of full rank has a unique least squares solution. An underdetermined system of full rank will admit a unique least squares solution only when an additional constraint is imposed, such as the minimum norm constraint. In our application, when the sparsity information of the mass or stiffness matrix is enforced, the system may become rank-deficient.

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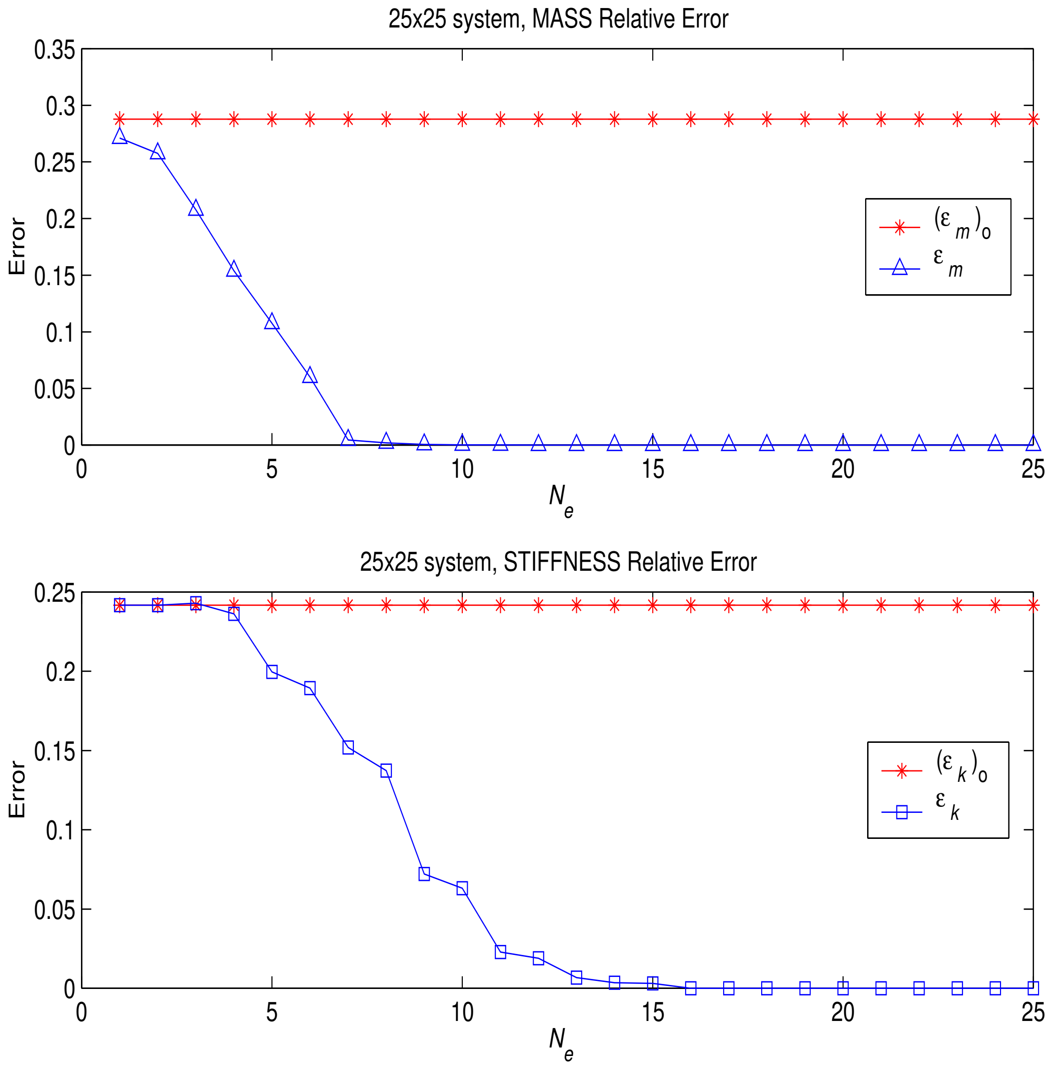

Figure 2. The mass and stiffness error parameters, ϵ_m and ϵ_k, as a function of N_e, for the system parameters of Tables 1 and 2. Lumped elements are added to masses 5, 10, 15, 20 and 25. The horizontal lines represent the mass and stiffness error parameters of the analytical model, $(\epsilon_m)_o$ and $(\epsilon_k)_o$.

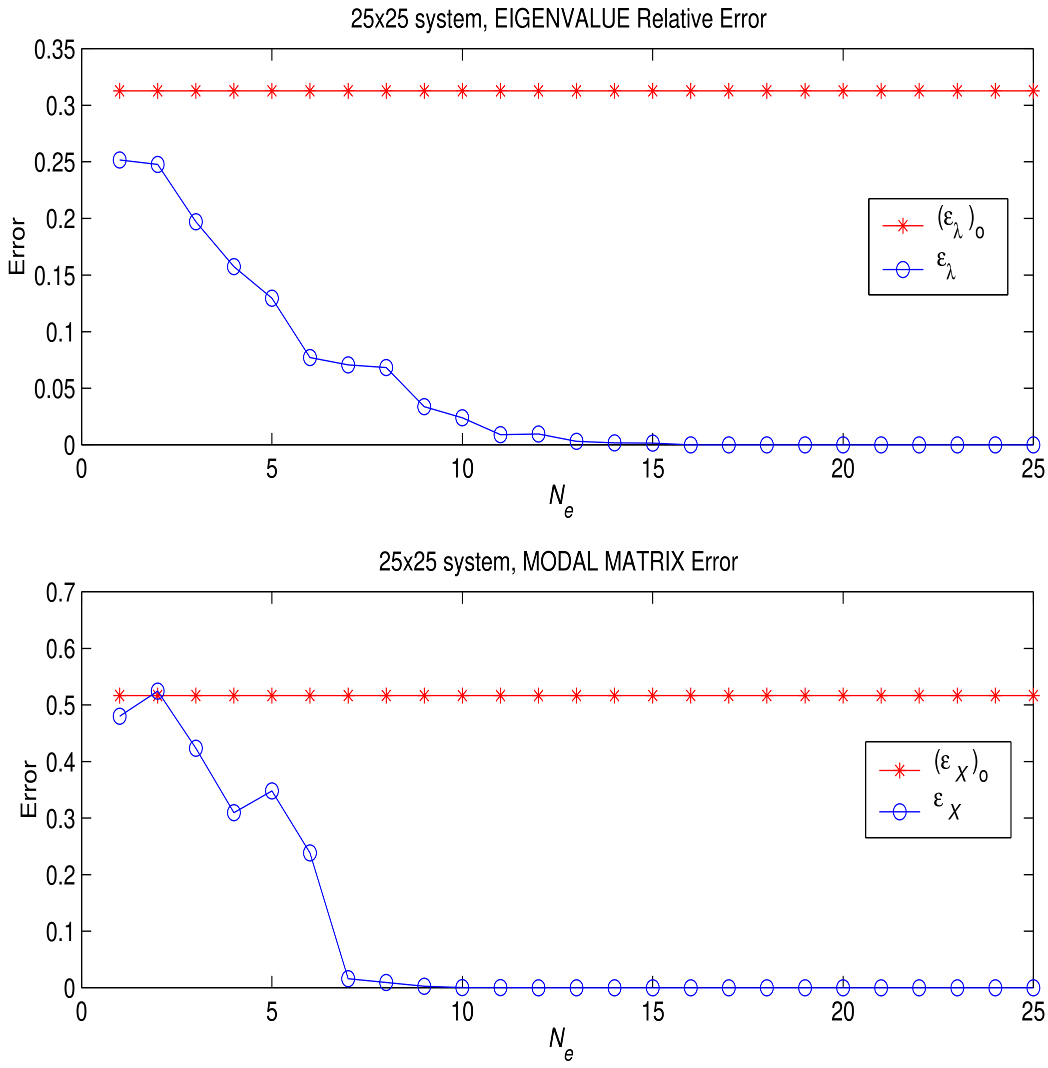

Figure 3. The eigenvalue and modal matrix error parameters, ϵ_λ and ϵ_X, as a function of N_e, for the system parameters of Tables 1 and 2. Lumped elements are added to masses 5, 10, 15, 20 and 25. The horizontal lines represent the mass and stiffness error parameters of the analytical model, $(\epsilon_\lambda)_o$ and $(\epsilon_X)_o$.

From numerical experiments, it was observed that more accurate solutions are obtained when the least squares system is overdetermined as opposed to underdetermined, even if the systems are rank-deficient. The numerical results from section 4.2 have direct implications about the minimum number of experimentally determined modes one should measure in order to ensure sufficient accuracy of the updated solution.

For instance, to update a diagonal mass matrix so that the resulting least squares problem becomes overdetermined, we would need at least $N_e \geq \sqrt{N}$ measured modes. Similarly, to update a tri-diagonal stiffness matrix so that the resulting least squares problem is overdetermined, we would need at least $N_e \geq \sqrt{3N-2}$ measured modes.

The above criteria regarding the fewest number of measured modes needed to perform the update are supported by various numerical experiments. For $N = 25$, our heuristic indicates that at least 5 measured modes are required to sufficiently update the mass matrix, and at least 9 measured modes are needed to correct the stiffness matrix. Naturally, because the number of measured modes used to perform the update directly affects the errors, ϵ_m and ϵ_k, the minimum number of measured modes that we should use to perform the update also depends on the size of error that we are willing to tolerate. In the case of the experiments of Section 4.2, the minimum number of measured modes suggested by the heuristics for the mass update ($N_e = 5$) leads to $\epsilon_m \approx 0.11$ (see Fig. 2), and for the stiffness update ($N_e = 9$) results in $\epsilon_k \approx 0.07$ (see Fig. 2), both of which are substantially lower than their corresponding analytical error parameters.

It should be emphasized that the criteria regarding the fewest N_e needed to perform the update are formulated empirically. Thus, depending on the system parameters, and, as mentioned previously, required error tolerances, sometimes more and other times fewer measured modes are required to ensure accuracy of the updated model. For example, for the mass parameters of Fig. 2, 7 measured modes are sufficient to correct the mass matrix to achieve an error of less than 0.025 ($\epsilon_m < 0.025$). For the stiffness values of Fig. 2, 13 measured modes are sufficient to update the stiffness matrix to achieve an error of less than 0.025 ($\epsilon_k < 0.025$).

4.4. The Effects of Locations, Magnitudes and the Number of Added Elements

From numerical experiments, it was observed that the location, magnitude and the number of added lumped masses may affect the numerical stability of the updating algorithms. The objective of the following discussion is to formulate an expression that can be used as a measure for determining whether the resulting mass-modified system allow for a numerically stable updated solution. Additionally, we will develop an expression that will provide some guidance in choosing the required magnitude, placement and the number of added lumped masses in order to execute the updating algorithms.

A frequently encountered scenario in structural dynamics is determining the changes in the eigensolution of a system after certain modifications are introduced. If the changes made are small, then the initial modal characteristics can be used as a basis from which to extract the new eigensolution of the modified system without performing a new and possibly costly analysis. Using perturbation theory for the eigenvalues and assuming the unperturbed eigenvalues to be distinct, it can be shown (see references [14] and [15]) that the jth first-order perturbed eigenvalue of a slightly modified structural system is given by:

$$\rho_j = \rho_{oj} + \mathbf{y}_{oj}^T \left([\delta\mathcal{K}] - \rho_{oj}[\delta\mathcal{M}]\right)\mathbf{y}_{oj} \tag{60}$$

where ρ_j represents the jth eigenvalue of the perturbed system, ρ_{oj} is the jth unperturbed eigenvalue, $\mathbf{y}_{oj}$ is the jth unperturbed eigenvector, and $[\delta\mathcal{M}]$ and $[\delta\mathcal{K}]$ are the first-order perturbation mass and stiffness matrices, respectively.

Consider now the case of correcting or updating the mass matrix of the system using the mass updating algorithm developed in Section 2.1 to solve solve Eq. (13). This method requires a modal survey of the mass-modified system to be performed. Assuming that the added masses are small compared to the analytical masses of the system, then the unperturbed and the perturbed systems correspond to the actual and the mass-modified systems, respectively. Using the nomenclature introduced earlier, we have for this case $[\delta\mathcal{M}] = [M_a]$ (matrix of added masses), $[\delta\mathcal{K}] = [0]$, $\rho_{oj} = \lambda_j$ (the jth measured eigenvalue of the actual system) and $\mathbf{y}_{oj} = \mathbf{x}_j$ (the jth measured eigenvector of the actual system). Then Eq. (60) can be rewritten as

$$\lambda_{aj} = \lambda_j - \lambda_j \mathbf{x}_j^T [M_a] \mathbf{x}_j \tag{61}$$

where λ_{aj} denotes the jth eigenvalue of the perturbed or the mass-modified system. Equation (10) shows that the mass updating algorithm is stable as long as

$$\lambda_i - \lambda_{aj} \neq 0, \quad \text{for} \;\; i, j = 1, \cdots, N_e \tag{62}$$

That is, we require all the measured eigenvalues for the mass-modified system to be distinct from those of the initial structure. Using Eq. (62), each time we add a different set of masses to the structure, we have to remeasure eigenvalues of the new system in order to check our stability criterion. We therefore look for an equivalent equation which directly relates the actual elements of the added mass system to the stability of the numerical solution. Substituting Eq. (61) into Eq. (62) we have

$$\lambda_i - \lambda_j + \lambda_j \mathbf{x}_j^T [M_a] \mathbf{x}_j \neq 0, \quad \text{for} \;\; i, j = 1, \cdots, N_e \tag{63}$$

Rearranging Eq. (63), it becomes

$$\mathbf{x}_j^T [M_a] \mathbf{x}_j \neq 1 - \frac{\lambda_i}{\lambda_j}, \quad \text{for} \;\; i, j = 1, \cdots, N_e \tag{64}$$

Expanding the left hand side of Eq. (64) yields

$$\sum_{r=1}^{N}\sum_{s=1}^{N}(M_a)_{rs}(x_j)_r(x_j)_s \neq 1-\frac{\lambda_i}{\lambda_j}, \quad \text{for} \;\; i,j=1,\cdots,N_e \tag{65}$$

where $(M_a)_{rs}$ represents the (r,s)th element of the added mass matrix $[M_a]$, and $(x_j)_r$ denotes the rth element of eigenvector $\mathbf{x}_j$ (the jth measured eigenvector of the actual system). When the left and right hand sides of Eq. (65) are nearly identical, the mass updating algorithm becomes unstable. To ensure numerical stability, the location, magnitude and the number of added masses must be chosen so that the inequality of Eq. (65) is maintained. Thus, Eq. (65) allows one to select the size, location and the number of added masses that should be employed.

Assuming $[M_a]$ to be diagonal, Eq. (65) simplifies to

$$\sum_{r=1}^{N}(M_a)_{rr}(x_j)_r^2 \neq 1-\frac{\lambda_i}{\lambda_j}, \quad \text{for} \;\; i,j=1,\cdots,N_e \tag{66}$$

Equation (66) can be used to make an a priori determination of the number, placement and size of the added masses. This can be accomplished by recasting Eq. (66) as a least-squares problem of the form $[A]\mathbf{x}=\mathbf{b}$, where $[A]$ and $\mathbf{b}$ are knowns and of dimension $N_e^2 \times N$ and length N_e^2, respectively, and vector $\mathbf{x}$ is the unknown, consisting of $[(M_a)_{rr}]_{r=1,\cdots,N}$. Computation of the solution of the resulting least squares problem may be very expensive, however, especially for large N and N_e. For our purposes, because Eq. (62) is much easier to check once all eigenvalue measurements have been taken, we choose to make use of Eq. (62) as an a posteriori measure of the goodness of the number, placement and size of the added masses. Our experiments are intended to highlight the fact that mass placement, number and size do indeed affect the accuracy of our solution. In all cases considered, we have ensured that Eq. (62) is satisfied.

In applying the mass updating scheme, the mechanism that leads to instability will be more complicated and more difficult to show analytically because the perturbed eigenvalues of the mass-modified system include higher-order terms. Nevertheless, the above perturbation analysis readily reveals that the magnitude, location and the number of added masses do in fact affect the stability of the updating algorithm.

To illustrate how the added lumped masses affect the quality of the update, consider the actual system whose masses and stiffnesses are given by those of Tables 1 and 2. Instead of adding 5 lumped masses to execute the updating algorithms as we did to obtain the results of Fig. 2, consider adding only 2 masses of magnitude 0.2 kg each at nodes 3 and 21. Figure 4 shows the variations of the resulting relative mass error parameter, ϵ_m, and the resulting relative stiffness error parameter, ϵ_k, as a function of N_e. Note that ϵ_m decreases initially with increasing N_e until $N_e = 4$, but increases to above $(\epsilon_m)_o$ suddenly for $N_e = 5$ and $N_e = 6$. For $N_e \geq 7$, ϵ_m becomes nearly zero. Similarly, note that ϵ_k decreases with increasing N_e until $N_e = 9$, increases slightly for N_e equal to 10 and 11, and then drops quickly down to nearly zero for larger values of N_e. Because the relative solution error for the least squares problem depends on the condition number of the linear system (recall the discussions on overdetermined and underdetermined systems in Sections 3.1 and 3.2) we speculate that the regions in which ϵ_m and ϵ_k are above $(\epsilon_m)_o$ and $(\epsilon_k)_o$ correspond to those in which the condition number of the least squares matrix, cond($[A]$), becomes large. The sensitivity of an underdetermined system depends on cond($[A]$), and the sensitivity of an overdetermined system is generally dominated by $(\text{cond}([A]))^2$, so we introduce an instability measure which equals the logarithm of cond($[A]$) if the system is underdetermined, and equals the logarithm of cond($[A]^2$) if the system is overdetermined. Figure 5 shows the variation of the instability number as a function of N_e. When $N = 25$, the mass updating algorithm is underdetermined for $N_e < 5$, and overdetermined for $N_e \geq 5$; similarly, the stiffness updating algorithm is underdetermined for

$N_e < 9$, and overdetermined for $N_e \geq 9$. We point out that a large condition number implies that the *upper bound* on the numerical error may be large, but does not necessarily imply that the error itself is large. On the other hand, a very small condition number does imply that the numerical error itself will also be small. The variation in the instability number as a function of N_e shown in Fig. 5 does in fact track fairly well the accuracy of the computed solutions shown in Fig. 4, in the sense of the upper bound. For this set of system parameters, a matrix with the logarithm of the condition number close to 10^{10} or higher has the potential to give rise to numerical errors.

Figure 6 shows the variations of the eigenvalue and modal matrix error parameters, ϵ_λ and ϵ_X, as a function of N_e, for the case of adding 2 lumped masses at nodes 3 and 21. We note that both the eigenvalue and modal matrix error parameters increase approximately where the mass matrix itself suffers from error. In this set of experiments, stiffness errors were not sufficiently pronounced to allow themselves to be evidenced in these plots.

For the set of system parameters of Tables 1 and 2, we have seen that as few as 2 added lumped masses can be used to execute the update, provided the masses are strategically placed. However, even when the updating schemes are numerically stable for 2 added masses, the updated mass and stiffness parameters are not as accurate as those obtained when 5 added masses are used to perform the update. Thus, there appears to be a trade-off between numerical accuracy and experimental effort. When more lumped masses are attached, more work and longer down-time will result; however, the resulting updated model will be more accurate. Conversely, when fewer lumped masses are added, less work will be required but at the expense of less accuracy. To validate the above conjecture, lumped masses (of magnitude 0.2 kg) are added to every single node. Figures 7 and 8 show the resulting variations of ϵ_m,

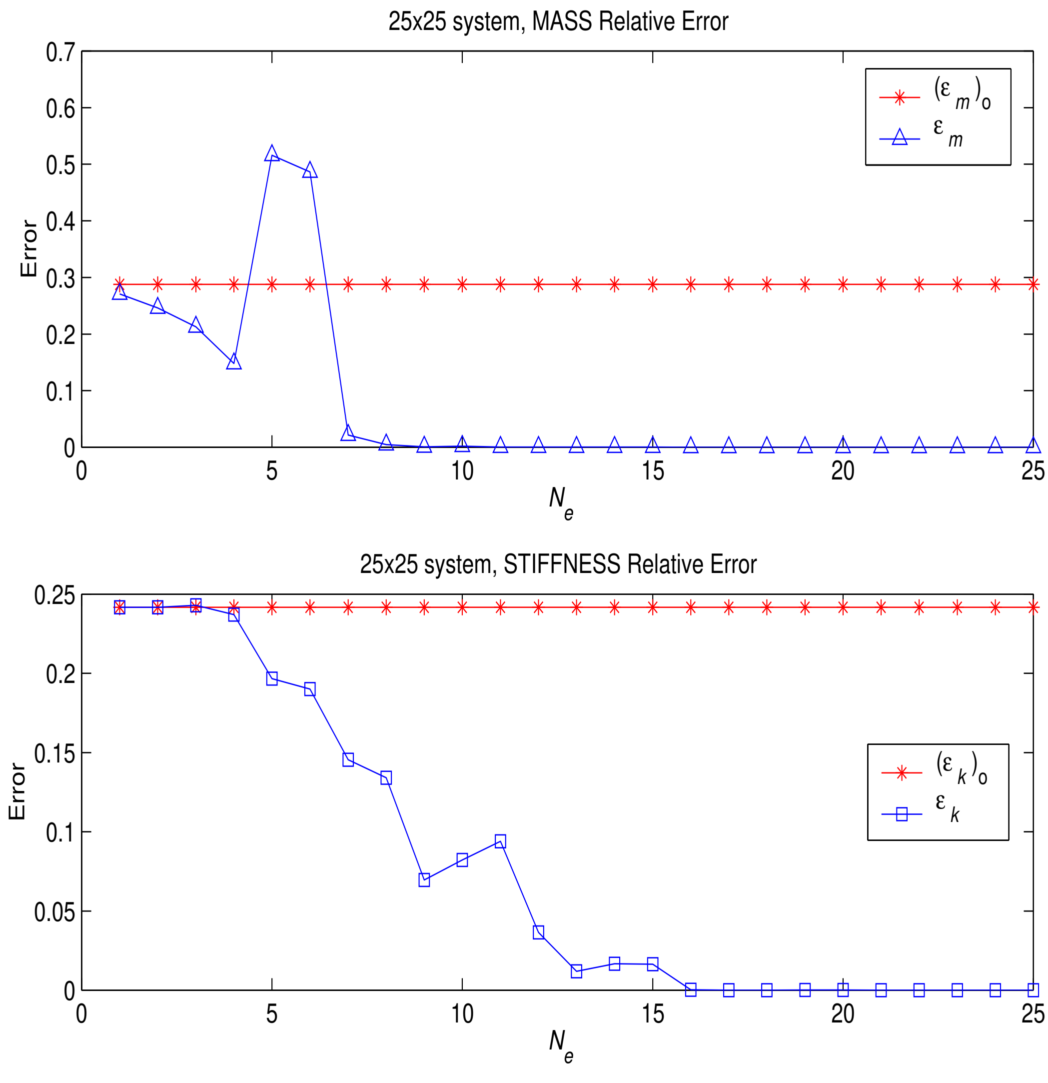

Figure 4. The mass and stiffness error parameters, ϵ_m and ϵ_k, as a function of N_e, for the system parameters of Tables 1 and 2. Lumped elements are added to masses 3 and 21. The horizontal lines represent the mass and stiffness error parameters of the analytical model, $(\epsilon_m)_o$ and $(\epsilon_k)_o$.

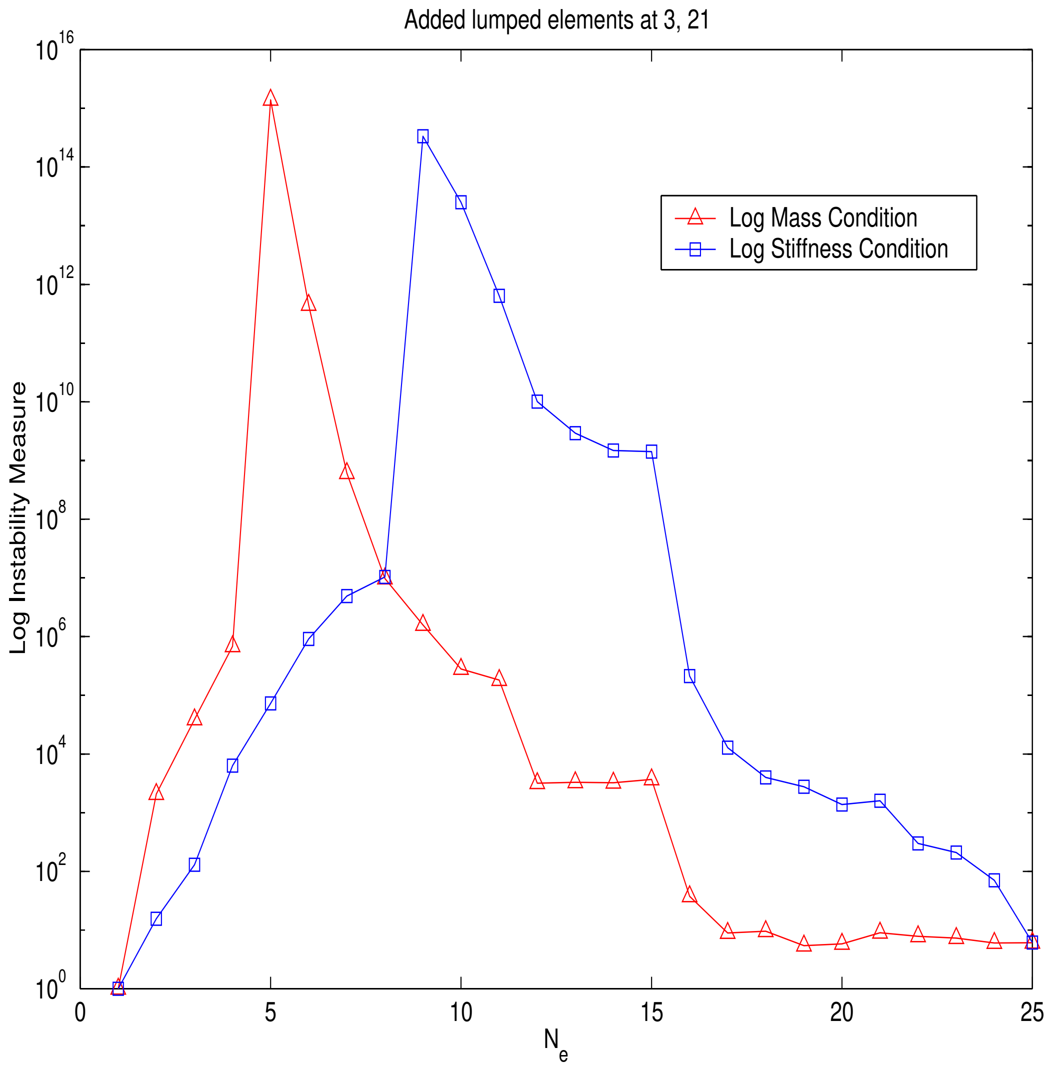

Figure 5. The instability measures for the mass and stiffness updates as a function of N_e, for the system parameters of Tables 1 and 2. Lumped elements are added to masses 3 and 21.

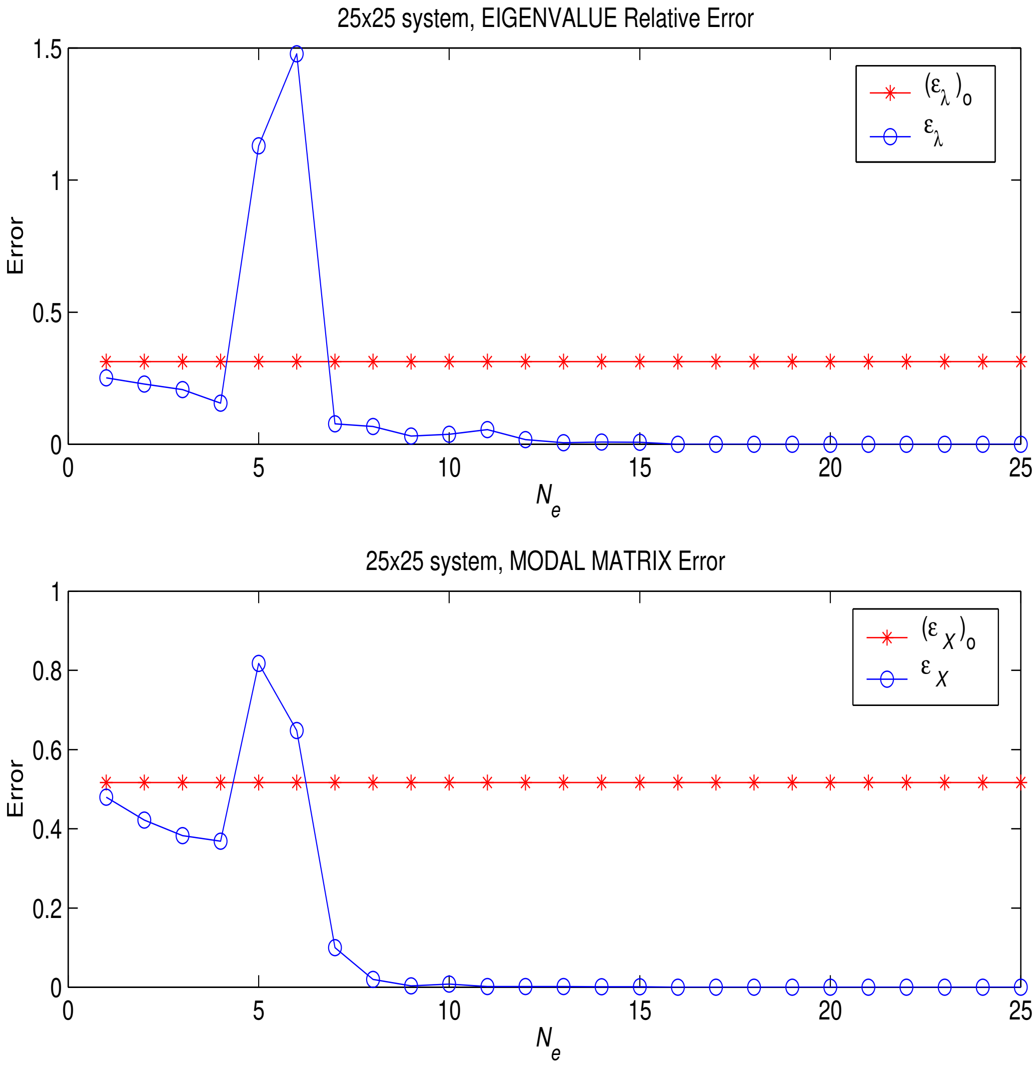

Figure 6. The eigenvalue and modal matrix error parameters, ϵ_λ and ϵ_X, as a function of N_e, for the system parameters of Tables 1 and 2. Lumped elements are added to masses 3 and 21. The horizontal lines represent the mass and stiffness error parameters of the analytical model, $(\epsilon_\lambda)_o$ and $(\epsilon_X)_o$.

ϵ_k, ϵ_λ and ϵ_X as a function of N_e. Note that for this case of 25 added lumped masses, the error parameters ϵ_m, ϵ_k, ϵ_λ and ϵ_X decrease at a faster rate compared to those of Figs. 2 and 3, implying that the updated system parameters converge to the actual values more rapidly. Thus, the above trade-off hypothesis is validated.

The results of Figs. 2, 4 and 7 indicate that for the system of Fig. 1, fewer modes are required to update the mass matrix than the stiffness matrix. This can be explained by noting that to render the least squares problem overdetermined for the mass matrix, at least $\sqrt{N}$ measured modes are required, while to render the problem overdetermined for the stiffness matrix, at least $\sqrt{3N-2}$ measured modes are needed.

5. Conclusion

New mass and stiffness updating algorithms are developed. Using the original test data and the newly acquired mass-modified modes of vibration, the mass and stiffness matrices of the analytical model can be accurately corrected. By manipulating the matrix equations in such a way that the unknown correction mass and stiffness matrices appear as column vectors, the connectivity information can be easily implemented, thus preserving the physical configuration of the system and reducing the amount of computational effort required to correct the analytical model. In addition, the structure of the least squares problems reveals the minimum number of modes that would be needed to measure in order to ensure a sufficiently accurate updated model.

When the set of measured modes is complete, the updating routines return updated mass and stiffness matrices that are nearly identical to those of the actual system. While the level of accuracy diminishes as the number of measured modes is decreased, the resulting updated

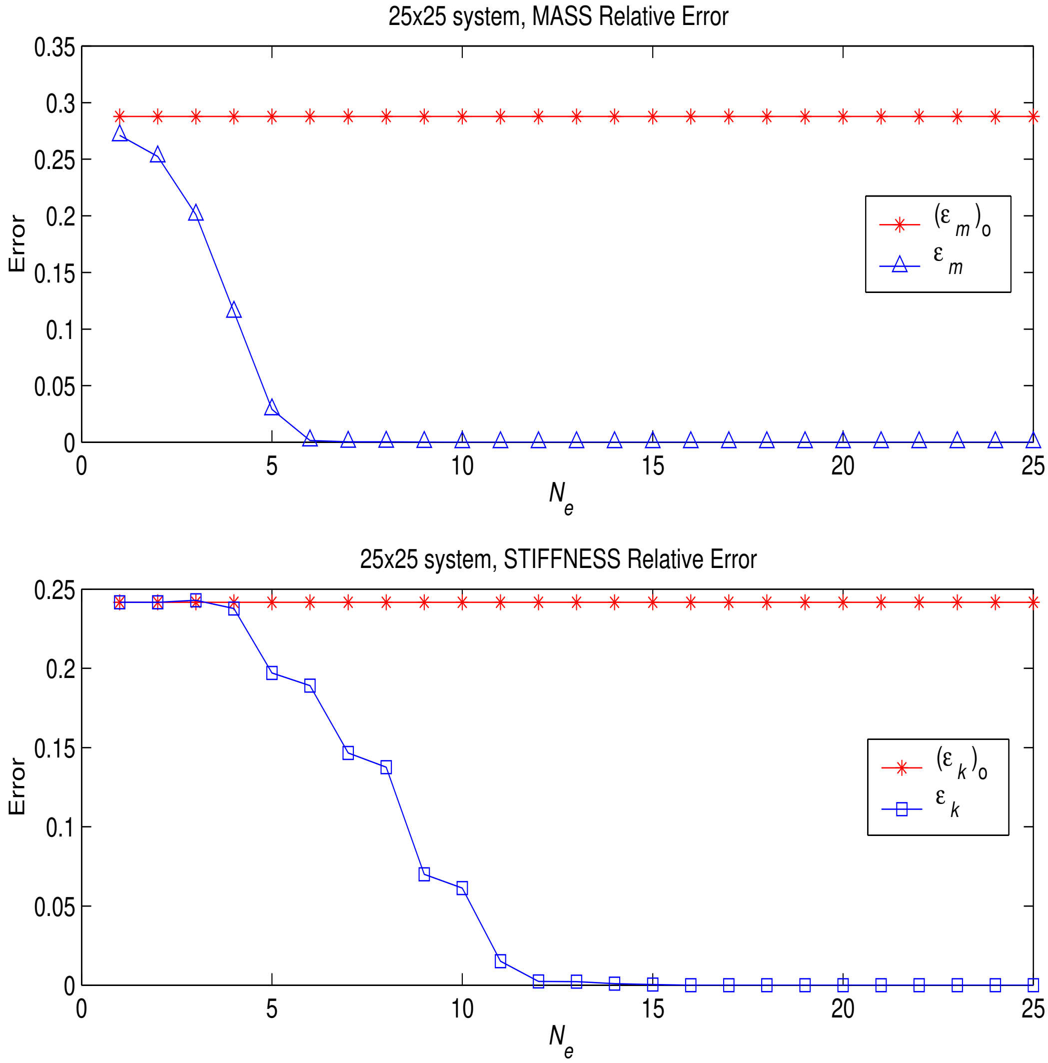

Figure 7. The mass and stiffness error parameters, ϵ_m and ϵ_k, as a function of N_e, for the system parameters of Tables 1 and 2. Lumped elements are added to every mass of the system. The horizontal lines represent the mass and stiffness error parameters of the analytical model, $(\epsilon_m)_o$ and $(\epsilon_k)_o$.

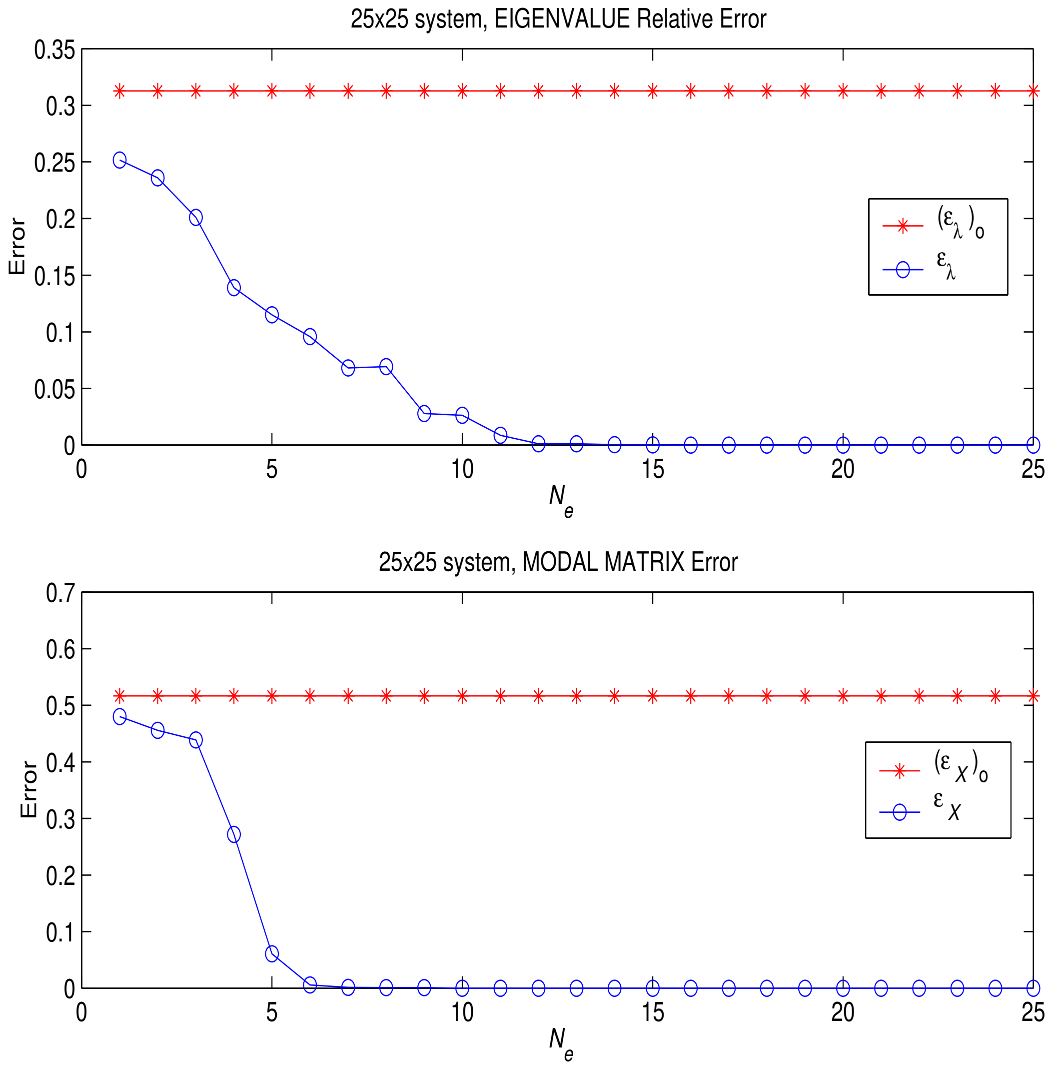

Figure 8. The eigenvalue and modal matrix error parameters, ϵ_λ and ϵ_X, as a function of N_e, for the system parameters of Tables 1 and 2. Lumped elements are added to every mass of the system. The horizontal lines represent the mass and stiffness error parameters of the analytical model, $(\epsilon_\lambda)_o$ and $(\epsilon_X)_o$.

model that is obtained is still more accurate than the initial analytical model, even for a limited number of measured modes. To ensure the numerical stability of the updating schemes, any masses can be added to the actual system as long as the measured eigenvalues of the actual system and the mass-modified systems are distinct. This implies that the magnitude of the added masses can be small, and the number of added masses can be few. The proposed mass and stiffness updating algorithms do require additional work and cause down-time, because the modes of vibration for the mass-modified system need to be measured. The additional effort, however, is a relatively small price to pay for the ability to correct the analytical model accurately.

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APPENDIX

Because the modal matrices $[X]$ and $[X_a]$ are always of full rank, we claim that matrix $[G]$ will also be of full rank. Consider the following triple product which has the same form as the left hand sides of Eqs. (12) and (24):

$$[X]^T[\delta Y][W] \tag{67}$$

The known matrices $[X]$ and $[W]$ are of dimension $N \times N_e$, and the unknown matrix $[\delta Y]$ is of dimension $N \times N$. Let the (i, j)th element of $[X]$, $[W]$ and $[\delta Y]$ be represented by x_{ij}, w_{ij} and δy_{ij}, respectively. Let the jth column vector of $[X]$ and $[W]$ be denoted by $\mathbf{x}_j$ and $\mathbf{w}_j$ $(j = 1, \cdots, N_e)$, respectively. Expanding and manipulating Eq. (67) so that the elements of the unknown matrix appear as an unknown vector, we get

$$[G]\,\delta\mathbf{y} \tag{68}$$

where $\delta\mathbf{y}$, of length N^2, is given by

$$\delta\mathbf{y} = [\delta y_{11} \cdots \delta y_{1N} \mid \delta y_{21} \cdots \delta y_{2N} \mid \cdots \mid \delta y_{N1} \cdots \delta y_{NN}]^T \tag{69}$$

Element by element calculations reveal that matrix $[G]$ is of the form

$$[G] = \begin{bmatrix} [G_1] \\ \vdots \\ [G_i] \\ \vdots \\ [G_{N_e}] \end{bmatrix} \tag{70}$$

where each block $[G_i]$, of dimension $N_e \times N^2$, is given by

$$[G_i] = \begin{bmatrix} x_{1i}\mathbf{w}_1^T & x_{2i}\mathbf{w}_1^T & x_{3i}\mathbf{w}_1^T & \cdots & x_{Ni}\mathbf{w}_1^T \\ x_{1i}\mathbf{w}_2^T & x_{2i}\mathbf{w}_2^T & x_{3i}\mathbf{w}_2^T & \cdots & x_{Ni}\mathbf{w}_2^T \\ \vdots & \vdots & \vdots & & \vdots \\ x_{1i}\mathbf{w}_{N_e}^T & x_{2i}\mathbf{w}_{N_e}^T & x_{3i}\mathbf{w}_{N_e}^T & \cdots & x_{Ni}\mathbf{w}_{N_e}^T \end{bmatrix} \tag{71}$$

We claim that $[G]$ has full rank, and we formalize this in the following theorem.

Theorem A: *Given $N \times N_e$ real matrices $[X]$ and $[W]$ of full rank, if matrix $[G]$, of dimension $N_e^2 \times N^2$, is constructed from $[X]$ and $[W]$ as in Eqs. (70) and (71), then $[G]$ also has full rank.*

Proof: Because $[W]$ has full rank, the set of vectors $\mathbf{w}_j$ $(j = 1, 2, \cdots, N_e)$ is linearly independent, and therefore block $[G_k]$ (which contains rows $(k-1)N_e + 1$ through kN_e) is also linearly independent, for $k = 1, \ldots, N_e$. We must next examine the possibility of linear dependence among rows in different blocks, and in particular, among rows numbered k, $k + N_e$, $k + 2N_e$, $\ldots$, $k + (N_e - 1)N_e$, for $k = 1, \ldots, N_e$.

Without loss of generality, assume that there is linear dependence to be found for $k = 1$, which would imply that there exist scalars α_j (for $j = 1, \ldots, N_e - 1$) so that we can write row 1 of block 1 as a linear combination of the first rows from subsequent blocks as follows:

$$\begin{aligned}
[x_{11}\mathbf{w}_1^T \;\; x_{21}\mathbf{w}_1^T \;\; x_{31}\mathbf{w}_1^T \;\; \ldots \;\; x_{N1}\mathbf{w}_1^T] &= \alpha_1[x_{12}\mathbf{w}_1^T \;\; x_{22}\mathbf{w}_1^T \;\; x_{32}\mathbf{w}_1^T \;\; \ldots \;\; x_{N2}\mathbf{w}_1^T] \\
&+ \alpha_2[x_{13}\mathbf{w}_1^T \;\; x_{23}\mathbf{w}_1^T \;\; x_{33}\mathbf{w}_1^T \;\; \ldots \;\; x_{N3}\mathbf{w}_1^T] \\
&+ \cdots \qquad (72) \\
&+ \alpha_{N_e-1}[x_{1N_e}\mathbf{w}_1^T \;\; x_{2N_e}\mathbf{w}_1^T \;\; x_{3N_e}\mathbf{w}_1^T \;\; \ldots \;\; x_{NN_e}\mathbf{w}_1^T]
\end{aligned}$$

This is equivalent to writing

$$\begin{aligned}
x_{11}\mathbf{w}_1^T &= (\alpha_1 x_{12} + \alpha_2 x_{13} + \cdots + \alpha_{N_e-1} x_{1N_e})\mathbf{w}_1^T \\
x_{21}\mathbf{w}_1^T &= (\alpha_1 x_{22} + \alpha_2 x_{23} + \cdots + \alpha_{N_e-1} x_{2N_e})\mathbf{w}_1^T \qquad (73) \\
&\vdots \\
x_{N1}\mathbf{w}_1^T &= (\alpha_1 x_{N2} + \alpha_2 x_{N3} + \cdots + \alpha_{N_e-1} x_{NN_e})\mathbf{w}_1^T
\end{aligned}$$

This implies that

$$\mathbf{x}_1 = \alpha_1\mathbf{x}_2 + \alpha_2\mathbf{x}_3 + \cdots + \alpha_{N_e-1}\mathbf{x}_{N_e} \qquad (74)$$

Equation (74) says that the first column of $[X]$ can be expressed as a linear combination of the last $N_e - 1$ columns of $[X]$. This is clearly a contradiction, because $[X]$ is given to have full rank. Therefore, our initial supposition that linear dependence can be found among the rows of $[G]$ is shown to be impossible, and it follows that $[G]$ must have full rank.