Lattice Rules by Component Scaling*

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Abstract

We introduce a theory of rectangular scaling of integer lattices. This may be used to construct families of lattices. We determine the relation between the Zaremba index $\rho(\Lambda)$ of various members of the same family. It appears that if one member of a family has a high index, some of the other family members of higher order may have extraordinarily high indices.

We have applied a technique based on this theory to lists of good lattices available to us. This has enabled us to construct lists of excellent previously-unknown lattices of high order in three and four dimensions and of moderate order in five dimensions.

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1 Background

The purpose of this paper is to find s-dimensional integer lattices Λ that have relatively high Zaremba indices or figures of merit. This index may be defined in terms of absolute values of the nonzero components of a lattice point $\mathbf{x} = (x_1, x_2, ..., x_s)$.

Definition 1 *The product coordinate distance function of* $\mathbf{x}$ *is*

$$\rho(\mathbf{x}) = \bar{x}_1\bar{x}_2...\bar{x}_s, \tag{1.1}$$

where

$$\overline{x_i} = \max(|\ x_i\ |, 1). \tag{1.2}$$

In terms of this we have the following definition.

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Definition 2 *The Zaremba index or figure of merit, $\rho(\Lambda)$, of an s-dimensional integer lattice is*

$$\rho(\Lambda) = \min_{\mathbf{x}\in\Lambda;\mathbf{x}\neq\mathbf{0}} \rho(\mathbf{x}). \tag{1.3}$$

Note that all lattice points of an integer lattice have integer components. Thus $\rho(\mathbf{x})$ and $\rho(\Lambda)$ are positive integers. In Section 3 we shall generalize this definition to other point sets.

The Zaremba index, $\rho(\Lambda)$, is a recognized measure of efficiency of the multidimensional quadrature rule based on Λ_Q, the lattice reciprocal to Λ. This lattice rule employs all N points of Λ_Q lying in $[0,1)^s$, N being the order of Λ. Thus, attention has been devoted to searching in $\mathcal{L}_s(N)$, the set of s-dimensional integer lattices of order N, with a view to finding optimal lattices of this order, that is, lattices Λ' for which $\rho(\Lambda')$ coincides with

$$\rho_s(N) = \max_{\Lambda\in\mathcal{L}_s(N)} \rho(\Lambda). \tag{1.4}$$

It is convenient to introduce a "measure of goodness" of a lattice using which one can compare lattices having different values of N. Our measure is based on the Zaremba conjecture that suggests that there exists a constant z_s such that

$$\rho_s(N) \leq z_s \frac{N}{(\log N)^{s-2}}; \quad s \geq 2. \tag{1.5}$$

For $s = 2$ the conjecture has been proved by Zaremba [Zar66]. Following Kedem and Zaremba [KZ74], we assign to every lattice a value defined by

$$z(\Lambda) = \frac{\rho(\Lambda)(\log N)^{s-2}}{N}. \tag{1.6}$$

This is of course nothing more than a scaled version of $\rho(\Lambda)$. However, this value is useful when examining a list that contains lattices of different orders N to recognize quickly those lattices which have an outstanding value of ρ.

This paper is not directly concerned with the conjectures on which such estimates are based. We note, however, that there exists a bound on $\rho_s(N)$ of order $O(N/(\log N)^{s-1})$ [Zar74] and that both the conjecture and bound are in the context of number theoretic rules; that is they are restricted to lattice rules of rank 1.

For an account of the general theory, of which this conjecture forms part, we refer the reader to recent papers by Niederreiter [Nie88],[Nie92], who has extended the theory to cover lattice rules of general rank. This developing theory is mainly devoted to existence proofs and asymptotic bounds. The present paper, on the other hand, is devoted to providing concrete examples of lattices whose reciprocal may be used to construct cost-effective lattice rules. These examples seem to confirm the theory and are in compliance with the truth of the conjecture.

In our searches [LS91] and [LS92], each integer lattice Λ is represented by an $s \times s$ generator matrix B. All elements of Λ are integer weighted sums of the rows of B, and Λ is of order $N = |\det B|$. The lattice Λ_Q on which the corresponding lattice rule is based has a generator matrix $A = (B^{-1})^T$.

2 Component Scaled Lattices

Theorem 3 *Given s nonzero real numbers $k_1, k_2, ..., k_s$ and a lattice Λ, there exists a lattice Λ' such that*

$$\mathbf{p} = (p_1, p_2, ..., p_s) \in \Lambda \Leftrightarrow \mathbf{p}' = (k_1 p_1, k_2 p_2, ..., k_s p_s) \in \Lambda'. \tag{2.1}$$

The proof is almost trivial, whatever definition of a lattice is invoked.

Definition 4 *Λ' in the theorem is termed a* **rectangularly scaled version** *of Λ, obtained by using an s-dimensional* **scaling factor** *$(k_1, k_2, ..., k_s)$ or a* **scaling matrix** *$K = diag(k_1, k_2, ..., k_s)$. $N_K = \mid \det(K) \mid = \mid \prod_{i=1}^{s} k_i \mid$ is called the* **order** *of this scaling.*

Rectangular scaling of a lattice has several trivial and obvious properties. In particular, a set of successive scaling operations is itself a scaling operation, and the scaling operation is commutative. If B is a generator matrix for Λ, then BK is one for Λ'. When Λ and Λ' are scaled versions of one another, so are their reciprocal lattices, $\Lambda^{\perp}$ and $\Lambda^{\perp'}$; the scaling matrices involved are inverses of one another.

It appears that, when one confines oneself to the set of integer lattices, one may construct distinct families of lattices, in which each member is a rectangularly scaled version of every other member. Each family is specified by a unique family root lattice Λ.

Definition 5 *A* **family root** *lattice is one whose generator matrix, B, has columns each of whose greatest common divisor is 1.*

Other members of the family are precisely those whose generator matrices are $B' = B\,\mathrm{diag}(k_1, k_2, ..., k_s)$ with k_i integer. To determine to which family some integer lattice Λ' belongs one takes its generator matrix B' and calculates the greatest common divisors, $h_1, h_2, ..., h_s$ of its columns. Then the matrix $B = B'\,\mathrm{diag}(h_1^{-1}, h_2^{-1}, ..., h_s^{-1})$ is a generator matrix of the family root lattice that generates the family to which Λ' belongs.

We are interested in the relation between $\rho(\Lambda)$ and $\rho(\Lambda')$.

We consider first the scaling of only one coordinate using a scaling factor $(k_1, 1, ..., 1)$ with $k_1 > 1$. As mentioned before, corresponding to every point $\mathbf{x} = (x_1, x_2, ..., x_s)$ of Λ is a point $\mathbf{x}' = (x_1', x_2', ..., x_s')$. Applying Definition 1, we find

$$\rho(\mathbf{x}') = k_1 \rho(\mathbf{x}) \quad when\ x_1 \neq 0$$
$$\rho(\mathbf{x}') = \rho(\mathbf{x}) \quad when\ x_1 = 0.$$

It follows from this and Definition 2 that

$$\rho(\Lambda) \leq \rho(\Lambda') \leq k_1 \rho(\Lambda). \tag{2.2}$$

The possibility of successive scaling in each coordinate in turn and the commutative property of the scaling operation, allows us to state the following theorem.

Theorem 6 *Let Λ' be a rectangularly scaled version of Λ obtained by using a scaling factor $\mathbf{k} = (k_1, k_2, ..., k_s)$ with each $k_i \geq 1$. Then*

$$N' = k_1 k_2 ... k_s N \tag{2.3}$$

and

$$\rho(\Lambda) \leq \rho(\Lambda') \leq k_1 k_2 ... k_s \rho(\Lambda). \tag{2.4}$$

We note that, under the hypotheses of this theorem, $\rho(\Lambda')/N'$ cannot exceed $\rho(\Lambda)/N$. However, since

$$z(\Lambda) = \frac{\rho(\Lambda)(\log N)^{s-2}}{N}, \tag{2.5}$$

we see that if, in fact, $\rho(\Lambda')/N' = \rho(\Lambda)/N$, then the z-value of Λ' is greater than the z-value of Λ; in this case, if Λ is a "good" lattice, then Λ' is better. Because of this, the present authors decided to carry out scaling of lattices already known to be good lattices, to see whether we could discover some better lattices or "good" lattices of higher order.

In its simplest terms, the idea developed in this paper is to take a set of lattices that are known to be good, scale them in various ways and inspect the scaled lattices (which are generally of higher order) to see whether any of them are good. In some cases, if we are lucky, we may find that the value of $z(\Lambda')$ is close to or even exceeds the upper limit of those already known. In this very fortunate case the new lattice is relatively as good or even better than the original lattice and has a higher order.

The underlying philosophy of this approach is that a calculation of this sort, while nontrivial, is orders of magnitude shorter than for example a direct search to find $\rho(N')$. If in a minor proportion of the calculations, say one in a thousand, we find a good lattice, the whole calculation can be termed a success.

The organization of this search requires some care. One can find lattices with arbitrarily high indices $\rho(\Lambda)$ by making N sufficiently large. To see this simply consider the scaled versions of the unit lattice Λ_0. The lattice $\Lambda' = k\Lambda_0$ (which can be obtained from Λ_0 using $\mathbf{k} = (k, k, \ldots, k)$) has $\rho(\Lambda) = k$ and $N(\Lambda) = k^s$.

In providing guidelines for the scope of the search, the following theorem is helpful.

Theorem 7 *Under the hypothesis of the previous theorem,*

$$\rho(\Lambda') \leq N(N'/N)^{1/s}. \tag{2.6}$$

Proof. All integer lattices of order N contain the sublattice $N\Lambda_0$ (where Λ_0 is the unit lattice). Thus Λ' contains each of the points $(k_1 N, 0, 0, \ldots, 0)$,$(0, k_2 N, 0, \ldots, 0)$,... $(0, 0, \ldots, k_s N)$ and it follows that

$$\rho(\Lambda') \leq N k_i \qquad i = 1, 2, \ldots, s. \tag{2.7}$$

Since all k_i are positive integers, we may take the geometric mean of the s equations in (2.7) and using (2.3) obtain (2.6). □

In order to make the search finite, we choose a lower bound $\overline{\overline{z}}$ (generally $(2/3)\overline{z}_s$ of Section 4) and limit the search to lattices Λ' for which $z(\Lambda') > \overline{\overline{z}}$. ¿From the theorem we see that

$$z(\Lambda') \leq (N/N')^{1-1/s} \log^{s-2} N' \tag{2.8}$$

and so is bounded by a quantity that approaches zero with large N'. Thus, since N' is restricted to integer multiples of N, the number of family members to be treated is finite. In fact, elementary manipulation yields the following lemma.

Lemma 8 *$z(\Lambda') < \overline{\overline{z}}$ when*

$$N'/\log^{s-1} N' > N/\overline{\overline{z}} \tag{2.9}$$

and

$$\overline{\overline{z}} \log N' > 1. \tag{2.10}$$

In practice (2.10) is satisfied trivially and so we may restrict our search to values of N' violating (2.9).

3 Scaling an Individual Lattice

In Sections 4, 5, and 6 we shall describe and analyze results obtained by scaling lists of lattices, all of which are reasonably good lattices. In this section we present a more detailed theory about rectangular scaling. The thrust of this section is to provide information, using which, $\rho(\Lambda')$ for a family of lattices may be readily calculated. Readers interested principally in the results of our numerical experiments may omit this section in a first reading. Without loss in generality we shall assume as before that the lattice Λ is a family root lattice (see Definition 5) and that Λ' is a scaled version obtained using a scaling factor $\mathbf{k}$, all of whose components are positive integers.

The behavior of $\rho(\Lambda')$ as a function of $\mathbf{k}$ is given by the function in (3.1) below.

Theorem 9 *Under the hypotheses of the preceding theorems, there exist $2^s - 1$ positive integers $A, A_1, \ldots, A_{23\ldots s}$, which depend only on Λ such that*

$$\frac{\rho(\Lambda')}{N'} = \frac{1}{N} \min\left(A, \frac{A_1}{k_1}, \frac{A_2}{k_2}, \frac{A_3}{k_3}, \ldots, \frac{A_{12}}{k_1 k_2}, \frac{A_{13}}{k_1 k_3}, \ldots, \frac{A_{23}}{k_2 k_3}, \ldots\right). \tag{3.1}$$

Note that these denominators comprise all distinct products of up to $s-1$ distinct components of $\mathbf{k}$. There is no term in $(k_1, k_2, \ldots, k_s)^{-1}$.

Note also that (3.1), while containing many inequalities, actually defines a function of $\mathbf{k}$. The rest of this section is devoted to establishing Theorem 9 and to showing how to calculate a set of constants $A_{i,j,\ldots}$ from a generator matrix B of Λ. It will appear that each coefficient $A_{i,j,\ldots}$ can be conveniently defined in terms of functions of the form $\rho(S)$, where S is a specified set of points belonging to Λ_0 and

$$\rho(S) = \min_{\mathbf{x}\in S;\mathbf{x}\neq\mathbf{0}} \rho(\mathbf{x}), \tag{3.2}$$

where $\rho(\mathbf{x})$ as defined in Section 1 is the absolute value of the product of the nonzero components of $\mathbf{x}$. This notation is a natural generalization of that introduced in Definition 2.

We now partition the elements of Λ into 2^s distinct sets. We distinguish these using an s-component binary index $\mathbf{u} = (u_1, u_2, \ldots, u_s)$, that is, one in which each component is either 1 or 0.

Definition 10 *Let Λ be an s-dimensional integer lattice and $\mathbf{t}$ an s-component binary index. Then*

$$\Gamma^{(\mathbf{t})} = \{\mathbf{x} | \mathbf{x} \in \Lambda \text{ and } x_i = 0 \text{ when } t_i = 0 \text{ and } x_i \neq 0 \text{ when } t_i \neq 0\}. \tag{3.3}$$

Note that $\Gamma^{(\mathbf{t})}$ is not a lattice (except technically for $\Gamma^{(0,0,\ldots,0)}$ which is the single point $(0,0,\ldots,0)$). These 2^s distinct sets form a partition of Λ; that is,

$$\Lambda = \bigcup_{\substack{u_i = 0,1 \\ 1 \leq i \leq s}} \Gamma^{(\mathbf{u})}. \tag{3.4}$$

This partition has been constructed with the following property in view.

Lemma 11 *When $(x_1, x_2, \ldots, x_s) \in \Gamma^{(\mathbf{u})}$,*

$$\rho(k_1 x_1, k_2 x_2, \ldots, k_s x_s) = k^{(\mathbf{u})} \rho(x_1, x_2, \ldots, x_s), \tag{3.5}$$

where

$$k^{(\mathbf{u})} = k_1^{u_1} k_2^{u_2} \ldots k_s^{u_s} \tag{3.6}$$

is the product of those components k_i of $\mathbf{k}$ for which $u_i = 1$.

Theorem 12 *Under the hypotheses of Theorem 6,*

$$\rho(\Lambda') = \min_{\substack{u_i = 0,1 \\ 1 \leq i \leq s \\ \mathbf{u} \neq 0}} (k^{(\mathbf{u})} \rho(\Gamma^{(\mathbf{u})})), \tag{3.7}$$

these quantities being defined in (3.6), (3.2), and (3.3).

Proof. The theorem follows because

$$\begin{aligned} \rho(\Lambda') &= \min_{\substack{\mathbf{x} \in \Lambda \\ \mathbf{x} \neq 0}} \rho(k_1 x_1, k_2 x_2, \ldots, k_s x_s) \\ &= \min_{\substack{u_i = 0,1 \\ 1 \leq i \leq s \\ \mathbf{u} \neq 0}} \min_{\mathbf{x} \in \Gamma^{(\mathbf{u})}} \rho(k_1 x_1, k_2 x_2, \ldots, k_s x_s). \end{aligned} \tag{3.8}$$

The first equality above follows from the definition of ρ and of the scaled lattice. The second follows from the partition (3.4) above. When we apply successively Lemma 11 and (3.2), we find that the expression on the right in (3.8) reduces to the right-hand side of (3.7). $\square$

Theorem 9 may be obtained from this theorem by simply dividing by $N' = k_1 k_2 \ldots k_s N$ and recognizing that, when $\mathbf{u} \neq \mathbf{0}$, the set $\Gamma^{(\mathbf{u})}$ is not empty and $\rho(\Gamma^{(\mathbf{u})})$ is a positive integer.

One readily identifies

$$A_{i,j\ldots} = \rho(\Gamma^{(\mathbf{u})}), \tag{3.9}$$

where $\mathbf{u}$ is the binary index that has zeros in positions corresponding to $i, j \ldots$, the subscripts of A, and units elsewhere.

Since the point set $\Gamma^{(\mathbf{u})}$ contains the point $(u_1 N, u_2 N, \ldots, u_s N)$, it follows that when $\mathbf{u} \neq \mathbf{0}$,

$$1 \leq \rho(\Gamma^{(\mathbf{u})}) \leq N^{u_1+u_2+\ldots+u_s} \tag{3.10}$$

and (3.7) supports the $2^s - 1$ inequalities

$$\rho(\Lambda') \leq (k_1 N)^{u_1} (k_2 N)^{u_2} \ldots (k_s N)^{u_s} \qquad \mathbf{u} \neq 0, \quad u_i = 0, 1 \quad i = 1, 2, \ldots, s. \tag{3.11}$$

There is a somewhat unexpected reformulation of Theorem 12. We recall that the points of $\Gamma^{(\mathbf{u})}$ of Definition 10 do not form a lattice. We may, however, form a lattice $\Lambda^{(\mathbf{u})}$ from the points of $\Gamma^{(\mathbf{u})}$ by adding all points of the form $\mathbf{x} \pm \mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \Gamma^{(\mathbf{u})}$, and iterating. This turns out to be a $(u_1 + u_2 + \ldots + u_s)$-dimensional projection of Λ, defined by the following.

Definition 13 *Let Λ be an s-dimensional lattice and $\mathbf{t}$ an s-component binary index. Then*

$$\Lambda^{(\mathbf{t})} = \{\mathbf{x} | \mathbf{x} \in \Lambda \text{ and } x_i = 0 \text{ when } t_i = 0\}. \tag{3.12}$$

It follows quite simply that partition (3.4) of Λ induces a similar partition of $\Lambda^{(\mathbf{t})}$, namely,

$$\Lambda^{(\mathbf{t})} = \bigcup_{\substack{0 \leq u_i \leq t_i \\ 1 \leq i \leq s}} \Gamma^{(\mathbf{u})} \tag{3.13}$$

from which it follows that

$$\rho(\Lambda^{(\mathbf{t})}) = \min_{\substack{0 \leq u_i \leq t_i \\ 1 \leq i \leq s \\ \mathbf{u} \neq 0}} \rho(\Gamma^{(\mathbf{u})}). \tag{3.14}$$

Lemma 14 *For a given s-dimensional lattice Λ and s-dimensional binary index $\mathbf{t}$,*

$$\min_{\substack{0 \le u_i \le t_i \\ 1 \le i \le s \\ \mathbf{u} \neq \mathbf{0}}} k^{(\mathbf{u})} \rho(\Gamma^{(\mathbf{u})}) = \min \left(k^{(\mathbf{t})} \rho(\Lambda^{(\mathbf{t})}), \min_{\substack{0 \le u_i \le t_i \\ 1 \le i \le s \\ \mathbf{u} \neq \mathbf{t} \\ \mathbf{u} \neq \mathbf{0}}} k^{(\mathbf{u})} \rho(\Gamma^{(\mathbf{u})}) \right), \qquad (3.15)$$

where $\Lambda^{(\mathbf{t})}, \Gamma^{(\mathbf{u})}$ are defined in terms of Λ in Definitions 13 and 10, and ρ is defined in (3.2).

The reader will recognize that the two sides of Equation (3.15) differ only in that a single term has been changed.

Proof. To establish the lemma, we take the right-hand side of (3.15) and replace the cofactor of $k^{(\mathbf{t})}$ by the expression given in (3.14). This procedure leaves us with an expression involving two somewhat similar sets of terms. By inspection we see that, except for the principal term in which $\mathbf{u} = \mathbf{t}$ there are a pair of terms corresponding to each $\mathbf{u}$, one of which has a factor $k^{(\mathbf{t})}$ and the other $k^{(\mathbf{u})}$. In all cases $k^{(\mathbf{u})} \le k^{(\mathbf{t})}$ and the first term can be discarded. Doing this leaves the expression on the left-hand side of (3.15) and so establishes the lemma. $\Box$

Theorem 15

$$\rho(\Lambda') = \min_{\substack{u_i = 0,1 \\ 1 \le i \le s \\ \mathbf{u} \neq 0}} (k^{(\mathbf{u})} \rho(S^{(\mathbf{u})})),$$

where S stands for Γ or Λ and may be chosen variously in each of $2^s - 1$ terms.

Proof. One may successively apply the lemma to the right-hand side of (3.7). Each application alters one Γ to S. The lemma must be applied in a proper order. Any ordering in which all terms having $\Sigma t_i = d$ are treated before any having $\Sigma t_i > d$ with $d = 1, 2, \dots, s$ is suitable. $\Box$

Theorem 15 sets the stage for the calculation of $\rho(\Lambda')$ in the situation in which Λ is defined by a generator matrix B in *utlf* (Hermite Normal Form) and in which software is available to calculate $\rho(\Lambda)$ for up to s-dimensional lattices from its B matrix. The problem is to identify a generator matrix of $\Lambda^{(\mathbf{t})}$.

Let B be in *utlf* and the binary index vector $\mathbf{t} = (0, 0, \dots, 0, 1, 1, \dots, 1)$ be a string of $s - \sigma$ zeros followed by a string of σ ones with, of course, $\sigma = \sum_{i=1}^{s} t_i$. In this case it is almost self-evident that a generator matrix of $\Lambda^{(\mathbf{t})}$ is obtained by replacing the first

$s-\sigma$ rows of B by zeros. Thus, $\rho(\Lambda^{(\mathbf{t})})$ may be obtained by applying the software to the σ-dimensional lattice whose generator is the $\sigma \times \sigma$ lower right-hand minor of B.

When $\mathbf{t} = (t_1, t_2, \ldots, t_s)$ is not of that form, we exploit the circumstance that ρ is invariant under permutations of the coordinate system. Thus, let P be an $s \times s$ permutation matrix, and set $\mathbf{t} = P\mathbf{t}$, $\overline{B} = BP$, and let $\overline{\Lambda}$ be the lattice whose generator matrix is $\overline{B}$. Then $\rho(\Lambda^{(\mathbf{t})}) = \rho(\overline{\Lambda}^{(\mathbf{t})})$. Thus, one finds the permutation P which takes $\mathbf{t}$ into $\mathbf{t}'$ of form $(0,0,0,1,\ldots,1)$, applies it to the columns of B to obtain $\overline{B}$, and then puts $\overline{B}$ in *utlf.* This problem is now reduced to the one described in the preceding paragraph.

In our numerical calculations, in pilot schemes we calculated each $\rho(\Lambda')$ individually using our own software. However, applying the results of the previous two paragraphs led to a much faster code. For each root lattice Λ we calculated $2^s - 1$ constants required in (3.1). This involved calculating only one s-dimensional figure of merit $A = \rho(\Lambda)$, the other constants $A_{i,j,\ldots}$ being lower-dimensional figures of merit. Then we relied on (3.1) to calculate $z(\Lambda')$ for all lattices Λ' in which we were interested. These included at most those with N' violating (2.9).

4 The Highlight Lists

Applying the technique of Section 3, we have found apparently endless lists of lattices, hundreds of which are excellent or interesting by previously acceptable standards. In order not to overwhelm the reader, we are presenting our results in two parts. In this section we present two "highlight" lists. These include three- and four-dimensional lattices with exceptionally high z-values and also lattices with moderate z-values but exceptionally high values of N.

In Section 5 we shall give in more detail some of the actual results and explain precisely how they were obtained; then in Section 6 we shall comment on some aspects of these results.

To provide criteria for our lists, we have defined an s-dimensional benchmark lattice as follows:

Definition 16 *The s-dimensional lattice $\overline{\Lambda}_s$ of order 2^{s+1} whose generator matrix in utlf is*

$$B(\overline{\Lambda}_s) = \begin{pmatrix} 2 & 0 & \ldots & 0 & 2 \\ 0 & 2 & \ldots & 0 & 2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 2 & 2 \\ 0 & 0 & \ldots & 0 & 4 \end{pmatrix} \tag{4.1}$$

is termed the s-dimensional benchmark lattice.

Clearly, $\rho(\overline{\Lambda}_s) = 4$ and

$$\overline{z}_s = z(\overline{\Lambda}_s) = \frac{1}{2}\left(\frac{(s+1)\log 2}{2}\right)^{(s-2)}. \tag{4.2}$$

The authors have introduced this benchmark simply because it is convenient in the context of discussing our lists of lattice rules. No intrinsic mathematical property is implied or conjectured.

The highlight lists include:

1. all s-dimensional lattices Λ known to us having $z(\Lambda) \geq \overline{z}_s$, and

2. All s-dimensional lattices Λ known to us satisfying both

   - $z(\Lambda) > 2/3\overline{z}_s$ and
   - $z(\Lambda) > z(\tilde{\Lambda})$ for all $\tilde{\Lambda}$ known to us whose order, $\tilde{N}$, exceeds N, the order of Λ.

Tables 1–8 and A1 and A2 contain lists of lattices. Each line corresponds to a single lattice Λ. The $s(s+1)/2$ entries which follow N and ρ are elements of an upper triangular matrix B. This is the upper triangular lattice form or Hermite Normal form of any generator matrix of Λ (see the remarks at the end of Section 1). Then comes the rank of the corresponding lattice rule. (See [SL89] for a full discussion of rank.) In this column an entry 0 indicates rank 1 simple (see [LS92]), and an entry 1 indicates rank 1 not simple.

In Tables 1 and 2, we identify the list from which this lattice was taken. These lists are specified in Section 5; the abbreviations are B = Blue, G = Green, SG = Scaled Green, and SB = Scaled Blue.

The authors must emphasize that these are lists of lattices that *happen to be known to us at this time*. In Section 6 we shall discuss the question of how many other lattices there may be that belong on such a list but have not been encountered yet. Only for $N < \overline{N}$ (= 4000 for $s = 3$ and 600 for $s = 4$) are these lists complete.

It is of interest to note the extremely disparate values of N involved. ¿From complete lists of optimal lattices of order up to $\overline{N}$, we obtain excellent lattices of order up to say $5\overline{N}$, a few of these being better than any found previously. After this the list degrades in quality only slowly, containing lattices of good (but not top) quality up to order $50\overline{N}$.

The tail of the list is unlikely to include any optimal lattices at all. However, for these extraordinarily high orders, an example of a lattice of moderate quality is of some interest.

Undoubtedly, the most outstanding lattices on these lists are
(i) a three-dimensional lattice having $N = 9760$, $\rho = 864$, and $z = 0.81319$, and
(ii) a four-dimensional lattice having $N = 8992$, $\rho = 212$, and $z = 1.95413$.

The results of scaling the short list [BP85] of five-dimensional rank 1 simple lattices were relatively unexciting. Possible reasons for this are mentioned in Section 5. We found no lattices whose z-values exceeded $\overline{z}_5$, and only 25 whose z-values exceeded $2/3\ \overline{z}_5$. We have listed in Table 8 all the lattices known to us whose z-values exceed $2/3\ \overline{z}_5$.

Table 1. *A Highlight List of Three-Dimensional Lattices*

N	ρ	b_{11}	b_{12}	b_{13}	b_{22}	b_{23}	b_{33}	rank	z	source
16	4	2	0	2	2	2	4	3	0.69315	G
1672	160	2	0	90	2	130	418	3	0.71022	G
2352	216	2	0	111	2	171	588	2	0.71293	G
3069	270	1	1	464	3	168	1023	2	0.70637	G
4704	390	2	0	228	3	148	784	2	0.70109	SG,SB
4880	432	1	0	638	2	1002	2440	2	0.75183	SG,SB
4900	400	1	0	452	2	748	2450	2	0.69363	SG
8922	700	1	0	823	2	1362	4461	1	0.71367	SB
9760	864	2	0	638	2	1002	2440	3	0.81319	SG,SB
9800	800	2	0	452	2	748	2450	3	0.75022	SG
17844	1356	1	0	2656	2	4062	8922	2	0.74392	SB
19416	1404	1	0	1431	2	3540	9708	2	0.71399	SG
20008	1440	2	0	1314	2	2048	5002	3	0.71279	SG
45576	2968	2	0	1658	2	4192	11394	3	0.69857	SB
48264	2864	2	0	1820	3	1184	8044	2	0.63995	SG
67410	3822	2	0	2469	3	1548	11235	2	0.63040	SG
67527	3762	3	0	1971	3	3072	7503	3	0.61952	SG
68238	3678	2	0	3441	3	5328	11373	2	0.59994	SG
90984	4680	2	0	5368	3	3680	15164	2	0.58734	SG
109050	5310	2	0	7940	3	3080	18175	1	0.56482	SG
130860	5808	2	0	9528	3	3696	21810	2	0.52292	SG
153819	6678	3	0	2487	3	6288	17091	3	0.51852	SB
160064	6688	4	0	2628	4	4096	10004	3	0.50070	SG
179760	7424	3	0	3292	4	2064	14980	2	0.49970	SG
227460	8880	3	0	6710	4	4600	18955	1	0.48155	SG

Table 2. *A Highlight List of Four-Dimensional Lattices*

N	ρ	b_{11}	b_{12}	b_{13}	b_{14}	b_{22}	b_{23}	b_{24}	b_{33}	b_{34}	b_{44}	rank	z	
32	4	2	0	0	2	2	0	2	2	2	4	4	1.50142	G
928	32	2	0	0	34	2	0	44	2	52	116	4	1.61001	SG
992	32	2	0	0	20	2	0	46	2	54	124	4	1.53568	SG
1008	32	2	0	0	16	2	2	8	6	12	42	4	1.51832	SG
1008	32	2	0	0	16	2	2	8	6	30	42	4	1.51832	SG
1354	40	1	0	0	492	1	0	550	1	658	1354	0	1.53607	B
1748	48	1	0	0	286	1	0	360	1	472	1748	0	1.53074	B
2097	54	1	0	0	435	1	0	936	1	1035	2097	0	1.50633	B
2112	55	1	0	0	100	1	0	162	1	830	2112	0	1.52617	B
2215	60	1	0	0	257	1	0	448	1	558	2215	0	1.60730	B
2248	60	1	0	0	106	1	0	178	2	442	1124	2	1.58980	SG
2320	64	2	0	0	34	2	0	56	2	82	290	4	1.65661	SG
2477	63	1	0	0	128	1	0	701	1	915	2477	0	1.55328	B
2570	65	1	0	0	787	1	0	1138	1	1246	2570	0	1.55921	B
2686	66	1	0	0	852	1	0	1142	1	1218	2686	0	1.53190	B
2730	68	1	0	0	170	1	0	452	1	1328	2730	0	1.55928	B
2836	72	1	0	0	418	1	0	1010	1	1290	2836	0	1.60464	B
3298	84	1	0	0	535	1	0	701	1	937	3298	0	1.67153	B
4496	106	1	0	0	106	2	0	178	2	442	1124	3	1.66790	SG,SB
8992	212	2	0	0	106	2	0	178	2	442	1124	4	1.95413	SG,SB
9908	180	1	0	0	256	1	0	1402	2	1830	4954	2	1.53803	SB
20232	318	2	0	0	159	2	0	267	3	663	1686	2	1.54517	SG,SB
52768	672	2	0	0	1070	2	0	1402	2	1874	6596	4	1.50574	SB
267138	2268	3	0	0	1605	3	0	2103	3	2811	9894	4	1.32561	SB
474912	3024	3	0	0	2140	3	0	3748	4	2804	13192	2	1.08787	SB

5 Lists of Scaled Lattices

In the preceding section we presented two short lists that included the best lattices we have found so far. These were extracted from results that we outline in detail in this section. This is in order that subsequent workers can relate their investigation to ours for purposes that may include confirming or extending our work.

All our work involves taking a list of lattices and treating each member of the list in the way described in Section 3. We now describe the seven different lists that were used as input. The three blue lists (containing only rank 1 simple lattices) have been available in the literature for several years. The two red lists have appeared in the literature only very recently. The two green lists have not been published. Each list is in a format corresponding to that used in Table 1.

1. Three Dimensions

   - **Green list** $N \in [16, 4000]$; contains 6557 entries. These are all the lattices in this range for which $\rho(\Lambda) = \rho_3(N)$. This list is unpublished.
   - **Red list** $N \in [16, 3916]$; contains 68 entries. This is a subset of the green list above, containing entries for N if and only if $\rho_3(N) > \rho(M)$ for all $M < N$. This list is published in [LS91]
   - **Blue list** $N \in [21, 6066]$; contains 101 entries. This is a concatenation of lists published by Maisonneuve [Mai72] and Kedem and Zaremba [KZ74]. It contains only rank-1 simple lattices, assembled from this subclass, using the standard red list convention described above.

2. Four Dimensions

   - **Green list** As above; $N \in [20, 600]$; contains 16127 entries, but is not complete. It contains most lattices for which $\rho(\Lambda) = \rho_4(N)$. If there exist more than ten lattices Λ for a single value of N, some may be missing but at least ten are included. This list is unpublished.
   - **Red list** As above; $N \in [32, 562]$; contains 23 entries. this list is published in [LS92].
   - **Blue list** As above; $N \in [52, 3298]$; contains 47 entries. This is a concatenation of lists published by Maisonneuve [Mai72] and Bourdeau and Pitre [BP85].

3. Five Dimensions

   - **Blue list** $N \in [112, 772]$; contains 9 entries. This list is published in [BP85]. Seven of these are repeated in the first part of Table 8.

Each of these seven lists were processed in the same way. This process produced from the input list, three more lists. These are specified below in the case that the input list is the three-dimensional **green list**.

1. We form first a three-dimensional raw **scaled green list**. For this, we require $\overline{\overline{z}}$, a cut-off value specified in (5.1) below. This list contains each lattice with $z(\Lambda) \geq \overline{\overline{z}}$ obtained by scaling every member of the green list. This huge list includes duplicate entries, and for some values of N, entries with different rho values.

2. From this, by cutting out all duplicates entries and any entries for which there is another lattice of the same order with a higher value of ρ, we produce a **green scaled green list**. We have retained this list in our files.

3. Next we use the standard procedure to produce a **red scaled green list**. This, as usual, retains only lattices on the green scaled green list for which $\rho(\Lambda) > \rho(\tilde{\Lambda})$ for all $\tilde{\Lambda}$ on that list having $\tilde{N} < N$. This list is given in Table 5.

The cut-off values we used were

$$\begin{aligned} \overline{\overline{z}} &= (2/3)\overline{\overline{z}}_3 \simeq 0.46 \\ \overline{\overline{z}} &= (2/3)\overline{\overline{z}}_4 \simeq 1.00 \\ \overline{\overline{z}} &= (5/9)\overline{\overline{z}}_5 \simeq 2.50 \end{aligned} \tag{5.1}$$

in 3, 4, and 5 dimensions, respectively. Table 3 gives some information on the length and the scope of the lists in this section.

Table 3. *Length and Scope of Lists Involved*

Kind of list		Input lists			Green scaled lists			Red scaled lists	
Dim.	Type	Length	N_{min}	N_{max}	Length	N_{min}	N_{max}	Length	Table
3-d	Blue	101	21	6066	177	4044	153819	29	4
	Green	6557	16	4000	4910	4002	227460	80	5
	Red	68	16	3916	42	4032	31328	17	A1
4-d	Blue	46	52	3298	162	624	474912	47	6
	Green	16127	20	600	4750	602	365625	55	7
	Red	23	32	562	51	640	80928	25	A2
5-d[1]	Blue	7	112	772	117	112	15768	17	

[1]See remarks at end of Section 4. We note that, when the input is a green or a red list of lattices with $N \in [N_{\min}, \overline{N}]$, there is no need to remain scaled lattices having $N \leq \overline{N}$ because these lattices, or better ones having the same N' are available by definition on the input list. This is not the case when the input is a blue list. The input blue list comprises excellent lattices, all of which are rank 1 simple. One may well find an interesting lattice of higher rank having $N' \geq N$ but $N' < \overline{N}$. Tables 4 and 6 (in which $\overline{N} = 6066$ and 3298, respectively) contain a handful of such lattices. These are generally of technical interest only. By including them we specify precisely the effect of scaling a blue list.

Table 4. *Red Scaled Blue List in Three Dimensions*

N	ρ	b_{11}	b_{12}	b_{13}	b_{22}	b_{23}	b_{33}	rank	z
4108	270	1	0	556	1	1408	4108	0	0.546880
4142	312	1	0	966	1	1422	4142	0	0.627380
4358	336	1	0	1398	1	1998	4358	0	0.646080
4704	390	2	0	228	3	148	784	2	0.701090
4880	432	1	0	638	2	1002	2440	2	0.751830
5862	450	1	0	538	1	1902	5862	0	0.666040
6066	460	1	0	600	1	1581	6066	0	0.660540
7430	544	1	0	1039	2	1425	3715	1	0.652600
7664	552	1	0	1194	2	1600	3832	2	0.644210
7698	576	1	0	603	2	1701	3849	1	0.669580
7734	588	1	0	600	1	2019	7734	0	0.680710
8922	700	1	0	823	2	1362	4461	1	0.713670
9760	864	2	0	638	2	1002	2440	3	0.813190
14112	908	2	0	454	2	746	3528	3	0.614780
15328	958	2	0	728	2	1778	3832	3	0.602340
15396	1032	2	0	603	2	1701	3849	2	0.646300
17436	1040	1	0	2094	2	1299	8718	1	0.582530
		1	0	1299	2	2094	8718	2	0.582530
17844	1356	1	0	2656	2	4062	8922	2	0.743920
22788	1484	1	0	1658	2	4192	11394	2	0.653430
26766	1646	1	0	2656	3	4062	8922	2	0.626940
31008	1680	2	0	2190	2	3664	7752	3	0.560330
32940	2016	3	0	957	3	1503	3660	3	0.636650
34872	2080	2	0	1299	2	2094	8718	2	0.623870
45576	2968	2	0	1658	2	4192	11394	3	0.698570
69282	3042	3	0	1206	3	3402	7698	3	0.489390
69744	3120	2	0	1732	3	2792	11624	2	0.498910
78080	3584	4	0	1276	4	2004	4880	3	0.517100
102546	4452	2	0	2487	3	6288	17091	2	0.500920
153819	6678	3	0	2487	3	6288	17091	3	0.518520

Table 5. *Red Scaled Green List in Three Dimensions*

N	ρ	b_{11}	b_{12}	b_{13}	b_{22}	b_{23}	b_{33}	rank	z
4002	280	1	0	958	1	1258	4002	0	0.580330
4008	288	2	0	219	2	294	1002	2	0.596120
4044	308	1	0	400	1	1054	4044	0	0.632530
4050	312	1	3	178	9	410	450	1	0.639910
4185	324	3	0	108	3	168	465	3	0.645620
4358	336	1	0	1398	1	1998	4358	0	0.646080
4528	344	2	0	218	2	316	1132	3	0.639530
4580	348	1	0	348	1	2002	4580	0	0.640490
4588	360	1	0	808	2	588	2294	2	0.661560
4704	390	2	0	228	3	148	784	2	0.701090
4880	432	1	0	638	2	1002	2440	2	0.751830
5862	450	1	0	538	1	1902	5862	0	0.666040
6066	460	1	0	600	1	1581	6066	0	0.660540
6198	468	1	0	1203	2	1470	3099	1	0.659340
		1	0	864	2	234	3099	1	0.659340
6322	480	1	0	800	1	2998	6322	0	0.664480
6682	504	1	0	1808	1	2624	6682	0	0.664290
6976	506	1	0	1644	1	3034	6976	0	0.641950
7116	510	1	0	1606	1	2120	7116	0	0.635720
7184	560	1	2	586	4	1544	1796	2	0.692170
7544	572	2	0	336	2	582	1886	3	0.676980
7698	576	1	0	603	2	1701	3849	1	0.669580
7734	588	1	0	600	1	2019	7734	0	0.680710
8391	598	1	0	1635	1	3849	8391	0	0.643890
8628	630	1	0	792	1	3363	8628	0	0.661750
8836	660	1	0	942	2	2126	4418	2	0.678720
9297	702	1	0	864	3	234	3099	2	0.689950
9760	864	2	0	638	2	1002	2440	3	0.813190
12944	936	1	0	954	2	2360	6472	2	0.684670
13524	940	1	0	2488	2	984	6762	2	0.661160
14068	948	1	0	1880	2	800	7034	2	0.643660
14260	980	1	0	996	2	2440	7130	2	0.657360
14820	1032	1	0	1702	2	2650	7410	2	0.668760
15420	1080	1	0	2002	2	3050	7710	2	0.675410
16914	1120	1	0	2973	2	1755	8457	1	0.644690
16926	1152	1	0	1712	3	644	5642	1	0.662680
18372	1160	2	0	1713	2	2049	4593	2	0.619940
18882	1224	2	0	669	3	1449	3147	2	0.638250
19194	1260	1	0	1839	2	1161	9597	1	0.647420
19416	1404	1	0	1431	2	3540	9708	2	0.713990

N	ρ	b_{11}	b_{12}	b_{13}	b_{22}	b_{23}	b_{33}	rank	z
20008	1440	2	0	1314	2	2048	5002	3	0.712790
21810	1470	1	0	4764	2	1848	10905	1	0.673340
22980	1512	1	0	4266	2	771	11490	1	0.660750
24132	1584	2	0	888	2	1365	6033	2	0.662380
25888	1618	1	0	1908	2	4720	12944	2	0.635100
27048	1764	2	0	984	2	2488	6762	3	0.665570
29080	1936	1	0	6352	2	2464	14540	2	0.684240
32176	1992	2	0	1184	2	1820	8044	3	0.642560
32940	2016	3	0	957	3	1503	3660	3	0.636650
33075	2034	3	0	678	3	1122	3675	3	0.639970
34758	2052	2	0	1101	3	2688	5793	2	0.617300
35868	2136	2	0	1167	2	4131	8967	2	0.624550
36198	2148	2	0	1365	3	888	6033	2	0.622880
38388	2232	2	0	1161	2	1839	9597	2	0.613730
39348	2418	2	0	2220	2	2931	9837	2	0.650170
42064	2448	2	0	2164	2	4876	10516	3	0.619620
44940	2548	2	0	1548	2	2469	11235	2	0.607410
		2	0	2112	2	3249	11235	2	0.607410
45492	2664	2	0	2760	2	4026	11373	2	0.628070
48264	2864	2	0	1820	3	1184	8044	2	0.639950
54525	2904	1	0	7940	3	3080	18175	1	0.580880
57582	3006	2	0	1839	3	1161	9597	2	0.572200
60656	3120	2	0	3680	2	5368	15164	3	0.566480
60858	3180	2	0	3732	3	1476	10143	2	0.575630
65430	3186	2	0	4044	3	1788	10905	2	0.539950
67410	3822	2	0	2469	3	1548	11235	2	0.630400
76776	4008	2	0	2452	3	1548	12796	2	0.587220
87240	4248	2	0	6352	3	2464	14540	2	0.553950
90495	4440	3	0	1480	3	2275	10055	2	0.559960
90984	4680	2	0	5368	3	3680	15164	2	0.587340
109050	5310	2	0	7940	3	3080	18175	1	0.564820
120660	5370	3	0	2275	4	1480	10055	1	0.520740
130860	5808	2	0	9528	3	3696	21810	2	0.522920
144792	5976	3	0	2730	4	1776	12066	2	0.490450
151640	6130	2	0	6710	4	4600	18955	2	0.482240
160064	6688	4	0	2628	4	4096	10004	3	0.500700
174480	7104	2	0	9528	4	3696	21810	3	0.491420
179760	7424	3	0	3292	4	2064	14980	2	0.499700
191940	7440	3	0	3065	4	1935	15995	1	0.471540
227460	8880	3	0	6710	4	4600	18955	1	0.481550

Table 6. *Red Scaled Blue List in Four Dimensions*

N	ρ	b_{11}	b_{12}	b_{13}	b_{14}	b_{22}	b_{23}	b_{24}	b_{33}	b_{34}	b_{44}	rank	z
624	16	2	0	0	12	2	0	21	2	27	78	3	1.06215
		2	0	0	9	2	0	15	2	36	78	3	1.06215
708	18	1	0	0	27	2	0	60	2	78	177	2	1.09489
718	22	1	0	0	158	1	0	210	1	234	718	0	1.32521
732	24	1	0	0	248	1	0	294	1	324	732	0	1.42637
932	26	1	0	0	116	1	0	288	1	314	932	0	1.30416
1124	30	1	0	0	106	1	0	178	1	442	1124	0	1.31706
1234	36	1	0	0	170	1	0	306	1	404	1234	0	1.47811
1354	40	1	0	0	492	1	0	550	1	658	1354	0	1.53607
1748	48	1	0	0	286	1	0	360	1	472	1748	0	1.53074
1990	50	1	0	0	256	1	0	584	1	684	1990	0	1.44969
2052	51	1	0	0	184	1	0	282	1	598	2052	0	1.44561
2097	54	1	0	0	435	1	0	936	1	1035	2097	0	1.50633
2112	55	1	0	0	100	1	0	162	1	830	2112	0	1.52617
2248	60	1	0	0	106	1	0	178	2	442	1124	2	1.58980
2686	66	1	0	0	852	1	0	1142	1	1218	2686	0	1.53190
2730	68	1	0	0	170	1	0	452	1	1328	2730	0	1.55928
2836	72	1	0	0	418	1	0	1010	1	1290	2836	0	1.60464
4496	106	1	0	0	106	2	0	178	2	442	1124	3	1.66790
5672	108	1	0	0	194	1	0	718	2	994	2836	2	1.42248
6744	120	1	0	0	159	2	0	267	2	663	1686	2	1.38308
8448	126	1	0	0	830	2	0	100	2	162	2112	3	1.21931
8508	144	1	0	0	627	1	0	1935	2	1515	4254	1	1.38584
8992	212	2	0	0	106	2	0	178	2	442	1124	4	1.95413
16116	218	1	0	0	1278	2	0	1713	2	1827	4029	2	1.26949
20232	318	2	0	0	159	2	0	267	3	663	1686	2	1.54517
26384	336	1	0	0	614	2	0	1402	2	1874	6596	3	1.31989
		1	0	0	1070	2	0	1402	2	1874	6596	3	1.31989
32232	432	2	0	0	1278	2	0	1713	2	1827	4029	3	1.44428
42976	436	2	0	0	1704	2	0	2284	2	2436	5372	4	1.15467
43821	468	3	0	0	156	3	0	228	3	495	1623	4	1.21996
45522	477	3	0	0	159	3	0	267	3	663	1686	4	1.20550
48348	480	2	0	0	1035	2	0	1281	3	882	4029	2	1.15504
		2	0	0	375	2	0	1257	3	585	4029	2	1.15504
52768	672	2	0	0	614	2	0	1402	2	1874	6596	4	1.50574
89046	756	1	0	0	1605	3	0	2103	3	2811	9894	3	1.10276
		1	0	0	79	3	0	2103	3	2811	9894	3	1.10276
108783	972	3	0	0	327	3	0	558	3	1386	4029	4	1.20172
118728	1008	2	0	0	79	2	0	2103	3	2811	9894	2	1.15914
		2	0	0	1605	2	0	2103	3	2811	9894	2	1.15914
		2	0	0	1605	2	0	2811	3	2103	9894	2	1.15914
145044	1080	3	0	0	1176	3	0	1380	3	1708	5372	3	1.05174
178092	1512	2	0	0	1605	3	0	2103	3	2811	9894	3	1.24098
		2	0	0	79	3	0	2103	3	2811	9894	3	1.24098
257856	1728	3	0	0	744	4	0	436	4	1848	5372	3	1.04043
267138	2268	3	0	0	79	3	0	2103	3	2811	9894	3	1.32561
474912	3024	3	0	0	2140	3	0	3748	4	2804	13192	2	1.08787

Table 7. *Red Scaled Green List in Four Dimensions*

N	ρ	b_{11}	b_{12}	b_{13}	b_{14}	b_{22}	b_{23}	b_{24}	b_{33}	b_{34}	b_{44}	rank	z
676	22	1	0	0	50	1	0	158	2	34	338	2	1.38186
		1	0	0	54	1	0	128	2	12	338	2	1.38186
688	24	1	0	0	106	1	0	144	2	22	344	2	1.48920
900	25	1	0	4	9	1	6	25	30	0	30	2	1.28535
928	32	2	0	0	34	2	0	44	2	52	116	4	1.61001
1281	34	1	0	0	54	1	0	129	3	98	427	1	1.35893
1344	36	1	0	0	45	2	0	99	2	162	336	2	1.38989
1556	40	1	0	0	84	1	0	218	2	244	778	2	1.38871
1692	42	1	0	0	96	1	0	412	2	134	846	2	1.37169
1952	48	2	0	0	24	2	0	42	2	56	244	4	1.41160
		2	0	0	24	2	0	40	2	92	244	4	1.41160
2200	50	1	0	2	28	2	2	94	10	20	110	3	1.34617
2248	60	1	0	0	106	1	0	178	2	442	1124	2	1.58980
2320	64	2	0	0	34	2	0	56	2	82	290	4	1.65661
3132	72	2	0	0	51	3	0	66	3	78	174	3	1.48950
4080	80	2	0	0	40	2	0	62	2	154	510	4	1.35530
		2	0	0	40	2	0	62	2	134	510	4	1.35530
4496	106	1	0	0	106	2	0	178	2	442	1124	3	1.66790
6192	108	2	0	0	196	2	0	234	3	168	516	3	1.32960
6736	120	2	0	0	62	2	0	206	2	294	842	4	1.38435
7312	128	2	0	0	218	2	0	340	2	414	914	4	1.38576
7888	144	2	0	0	172	2	0	314	2	382	986	4	1.46987
8992	212	2	0	0	106	2	0	178	2	442	1124	4	1.95413
14112	216	2	0	0	174	2	0	284	3	510	1176	3	1.39735
18816	224	2	0	0	174	2	0	284	4	510	1176	4	1.15326
19632	252	2	0	0	616	2	0	684	3	180	1636	3	1.25424
20232	318	2	0	0	159	2	0	267	3	663	1686	2	1.54517
26622	324	2	0	0	258	3	0	471	3	573	1479	3	1.26360
31590	360	2	0	0	327	3	0	483	3	543	1755	3	1.22327
37017	432	3	0	0	327	3	0	510	3	621	1371	4	1.29135
39933	486	3	0	0	258	3	0	471	3	573	1479	4	1.36616
47385	540	3	0	0	327	3	0	483	3	543	1755	4	1.32089
65808	576	3	0	0	680	3	0	828	4	436	1828	2	1.07736
		3	0	0	436	3	0	680	4	828	1828	2	1.07736
70992	648	3	0	0	344	3	0	628	4	764	1972	2	1.13893
		3	0	0	628	3	0	764	4	344	1972	2	1.13893
81360	720	3	0	0	632	3	0	916	4	392	2260	2	1.13133
87744	768	3	0	0	436	4	0	680	4	828	1828	3	1.13395
94656	792	3	0	0	344	4	0	628	4	764	1972	3	1.09849
108480	864	3	0	0	392	4	0	472	4	1056	2260	3	1.07067
		3	0	0	392	4	0	632	4	916	2260	3	1.07067
112320	960	3	0	0	436	4	0	644	4	724	2340	3	1.15586
116992	1024	4	0	0	436	4	0	680	4	828	1828	4	1.19200
126208	1056	4	0	0	344	4	0	628	4	764	1972	4	1.15434
144640	1152	4	0	0	392	4	0	472	4	1056	2260	4	1.12446
149760	1280	4	0	0	436	4	0	644	4	724	2340	4	1.21376
219375	1500	3	0	0	545	5	0	805	5	905	2925	3	1.03422
228500	1600	4	0	0	545	5	0	850	5	1035	2285	3	1.06614
285625	2000	5	0	0	545	5	0	850	5	1035	2285	4	1.10505
365625	2500	5	0	0	545	5	0	805	5	905	2925	4	1.12191

Table 8. *Five-dimensional Lattices having* $z(\Lambda) > 3.0$

Rank 1 simple lattices in this list having $N \in [112, 772]$ have been taken from [BP85]. The others are scaled versions of these or of the benchmark lattice.

N	ρ	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{22}	b_{23}	b_{24}	b_{25}	b_{33}	b_{34}	b_{35}	b_{44}	b_{45}	b_{55}	rank	z
64	4	2	0	0	0	2	2	0	0	2	2	0	2	2	2	4	5	4.49583
96	4	2	0	0	0	3	2	0	0	3	2	0	3	2	3	6	4	3.96210
128	4	2	0	0	0	4	2	0	0	4	2	0	4	2	4	8	5	3.56961
		1	0	0	0	12	1	0	0	22	1	0	48	1	52	128	0	3.56961
144	4	2	0	0	0	3	2	0	0	3	2	0	3	3	3	96	3	3.40971
160	4	2	0	0	0	5	2	0	0	5	2	0	5	2	5	10	4	3.26808
192	4	2	0	0	0	4	2	0	0	4	2	0	4	3	4	8	4	3.02758
		2	0	0	0	6	2	0	0	6	2	0	6	2	6	12	5	3.02758
		1	0	0	0	18	1	0	0	33	1	0	72	1	78	192	0	3.02758
275	5	1	0	0	0	31	1	0	0	71	1	0	91	1	136	275	0	3.22179
308	6	1	0	0	0	18	1	0	0	62	1	0	70	1	102	308	0	3.66511
438	8	1	0	0	0	38	1	0	0	50	1	0	96	1	168	438	0	4.10962
448	6	1	0	0	0	12	1	0	0	54	2	0	22	2	28	112	3	3.04710
		1	0	0	0	22	1	0	0	54	2	0	12	2	28	112	3	3.04710
512	8	1	0	0	0	22	1	0	0	48	2	0	12	2	52	128	3	3.79336
657	8	1	0	0	0	57	1	0	0	75	1	0	144	1	252	657	0	3.32502
666	9	1	0	0	0	15	1	0	0	42	1	0	175	1	269	666	0	3.71336
		1	0	0	0	57	1	0	0	137	1	0	223	1	240	666	0	3.71336
		1	0	0	0	57	1	0	0	221	1	0	240	1	307	666	0	3.71336
768	8	1	0	0	0	22	1	0	0	48	2	0	12	3	52	128	2	3.05476
		1	0	0	0	22	1	0	0	48	2	0	52	3	12	128	2	3.05476
		1	0	0	0	33	1	0	0	72	2	0	18	2	78	192	3	3.05476
772	10	1	0	0	0	154	1	0	0	170	1	0	230	1	256	772	0	3.80758
924	9	1	0	0	0	93	1	0	0	105	1	0	153	2	27	462	0	3.10161
1158	10	1	0	0	0	231	1	0	0	255	1	0	345	1	384	1158	0	3.03166
1536	12	1	0	0	0	33	2	0	0	18	2	0	72	2	78	192	4	3.08556
1544	12	1	0	0	0	78	1	0	0	264	1	0	378	2	10	772	2	3.07610
2048	16	2	0	0	0	12	2	0	0	22	2	0	48	2	52	128	5	3.46294
		1	0	0	0	44	2	0	0	24	2	0	96	2	104	256	4	3.46294
2560	16	1	0	0	0	55	2	0	0	30	2	0	120	2	130	320	4	3.02077
2664	18	1	0	0	0	442	1	0	0	480	1	0	614	2	114	1332	2	3.31566
3088	20	1	0	0	0	78	1	0	0	378	2	0	10	2	264	772	3	3.36013
4632	24	1	0	0	0	117	1	0	0	567	2	0	15	2	396	1158	2	3.11591
7008	32	2	0	0	0	16	2	0	0	70	2	0	144	2	186	438	5	3.17025

6 Comments about Lists

6.1 Evaluation of Scaled Lists

An immediate question that comes to mind is to what extent any list obtained here compares with the corresponding complete list. The authors believe that, at best, one retains about 70% of a complete list, and that this percentage diminishes to zero as the order N significantly exceeds the order $\overline{N}$ of the input list. The rest of this subsection is devoted to this question.

We carried out some numerical experiments in an environment in which the answer, in the form of a complete red list, is available. We applied our scaling technique to only part of our three-dimensional green list, the first part having $N \leq \overline{N} = 250$. This produced first a long repetitive raw scaled green list and after massaging, as described in Section 5, a green scaled green list containing 450 lattices sharing 326 distinct values of N lying in

[251,13376]. Since we have available a complete green list for $N \in [1, 4000]$, we were able to observe the quality of this particular green scaled green list.

Table 9. *The Three-Dimensional Green Scaled Green List with* $\overline{N} = 250$

Interval	Scaled Lattices	Distinct N-values	N-values for which $\rho_L(N)/\rho_3(N)$ =1	$\epsilon(1, 0.9)$	≤ 0.9
$N \in [251, 500]$	209	1112	94	15	3
$N \in [501, 1000]$	120	100	56	32	12
$N \in [1001, 2000]$	75	69	19	27	23
$N \in [2001, 4000]$	31	30	4	2	24
$N > 4000$	15	15	?	?	?
Total	450	326	173	76	62

Table 9 gives a breakdown of the distribution of lattices in this list and their quality. Here $\rho_L(N)$ is the lower bound on $\rho_3(N)$ based only on the lattices in this list. Examination of this table shows that for values of N near to $\overline{N}$, we seem to be obtaining lattices for about half the values of N. Of these, 80% are optimal, the rest being generally of reasonably high quality. On the other hand, for values of N exceeding $8\overline{N}$ in a range containing 2000 values of N, we have found lattices for only 30 of these values, and only four of these are optimal. Fourteen of these 450 lattices may also be found on the three-dimensional red list which has 45 entries for $N \in [251, 4000]$.

A second numerical experiment concerns a scaled three-dimensional red list. We compare a plot of $\rho_3(N)$ based on a scaled red list with a similar plot based on the complete red list. We consider five such plots. For the complete red list we define a piecewise constant function

$$\tilde{\rho}(N) = \max_{\substack{\Lambda \in \mathcal{L}_3(M) \\ M \leq N}} \rho(\Lambda) \tag{6.1}$$

The discontinuities of $\tilde{\rho}(N)$ occur at values of N appearing on the red list. At such values, $\tilde{\rho}(N-1) < \tilde{\rho}(N)$. One can define the analogous function for a red scaled list based on an input green list for $N \in [1, N_i]$, where $N_i = 125.2^i$.

We have constructed four functions $\tilde{\rho}_i(N)$ $i = 1, 2, 3, 4$. In general, $\tilde{\rho}_i(N) \leq \tilde{\rho}(N)$, but for some values of N this is an equality. We list in Table 10 the percentage of values of N in an interval $(N_j, N_{j+1}]$ $j = 1, 2, 3, 4$ for which $\tilde{\rho}_i(N) = \rho_3(N)$.

Naturally, when $j < i$, this is 100%, and when $j >> i$, this reduces to zero.

Table 10. *Information about Red Scaled Green Lists*

% of N satisfying	$(N_1, N_2]$	$(N_2, N_3]$	$(N_3, N_4]$	$(N_4, N_5]$
$\tilde{\rho}_1(N) = \tilde{\rho}(N)$	51.6	62.4	46.6	0.0
$\tilde{\rho}_2(N) = \tilde{\rho}(N)$		75.0	63.4	38.15
$\tilde{\rho}_3(N) = \tilde{\rho}(N)$			95.2	74.4
$\tilde{\rho}_4(N) = \tilde{\rho}(N)$				85.65

Examination of the complete red lists in three and four dimensions which appeared in [LS92] and [LS91] respectively shows that a proportion that varies unsteadily between 15% and 40% are root lattices, the majority of lattices on these lists being scaled lattices. If this state of affairs were to prevail for higher values of N, then any red scaled list would omit between 15% and 40% of the optimal lattices since it cannot by definition include root lattices.

Finally, we state one further reason that we believe these lists to be incomplete. This one is based on the actual lists, rather than on extrapolation. We have presented separately in Tables 5 and 4 a red scaled green list and the corresponding red scaled blue list. The first contains 80 entries and appears to be an excellent list in many ways, having as far as one can see the same characteristics as the actual red list for $N < 4000$. However, there are some known rules missing. We know this because they appear on the clearly inferior and shorter red scaled blue list. These two lists contain six entries in common. There are eight entries on the red scaled blue list that merit inclusion on the red scaled green list, but are not there. If included, they would in total displace eight entries already there.

One sees that a few missing entries do not alter the overall nature of the list very much. The missing entries are simply replaced by entries representing marginally inferior lattices; the effect on the list as a whole is local. Also, it is not particularly the entries with the highest z-values that seem to be missing.

6.2 Suitable Input for a Scaled List

We have listed the three-dimensional red scaled red list (Table A1) and the red scaled green list (Table 5). Only two elements $N = 4185$ and $N = 4704$ occur on both. Thereafter the red scaled red list deteriorates significantly when compared with the red scaled green list. However, the input red list contains all the really good elements of the input green list. The heuristic conclusion in this case is that, for scaling purposes, one does not want to start with optimal lattices having optimal N values. It appears that one will discover more if one inputs a list of good but not excellent lattices.

All our results appear to support to some extent this conclusion. We have found the red scaled blue list to be intermediate. The blue list being restricted to rank 1 simple lattices is not as good as the red list but seems to provide better scaled lattices. Any conclusion based on our four-dimensional lists must take into account that the blue list includes much higher values of N than the green list.

Theoretical support for this state of affairs can be found in Section 3. There it is noted that, starting with a family root lattice, the effect of scaling is in general to improve the z-value at first but then there is a steady decay in z-value. It is consistent with this situation that, for optimal values of N, the best lattices are not root lattices but are already scaled versions—but not very highly scaled. As mentioned above, the majority of the lattices on our red list are like this, and scaling them is unlikely to provide better ones.

6.3 Comments on Red Lists

It has been traditional to report results of the type treated here using red lists (i.e., lists of optimal lattices). One reason is that it is feasible to publish such a list. A red list contains in one page an excellent selection from a green list of fifty pages. Another reason is linked to the numerical quadrature application in which the cost is taken to be proportional to N the number of function values and the quality of the result to ρ. However, the present authors believe that, for the values of N now reached in three or four dimensions, the red list has become an anachronism. For many purposes a highlight list is adequate. For deeper investigation, the green list is probably needed. And, in applications, questions such as embedding of one rule in another and convenience in locating points using the relevant machine architecture may be much more significant than a small margin in the plot of N versus ρ.

While the red list contains an excellent selection, occasionally good lattices are excluded because they are "in the shadow of" marginally better lattices. An example of this occurs in three dimensions with $N = 9760$ and $N = 9800$. The first has $\rho = 864$ and the second $\rho = 800$, so the second does not occur on a red list. In fact, we know only three lattices with $z(\Lambda) > 0.75$; these are the two mentioned above and one with $N = 4880$. Thus, our red list has omitted what might be considered the third best lattice available. In investigations relating to the distribution of good lattices, one may prefer to know about all good lattices, even if in applications some are not going to be used.

6.4 The Tail of the List

We mentioned towards the end of Section 2 that it is trivial to find infinite sequences of lattices having monotonic increasing $\rho(\Lambda')$ and N'. Thus, an incomplete red list can be extended indefinitely. The lists we have presented have the additional requirement that $z(\Lambda)$ should exceed a specified amount $\overline{\overline{z}}$. The reader should note that this by itself need not render a list finite. In fact, numerical and theoretical evidence suggests the opposite. Our list deteriorates and so is finite simply because it can contain only a subset of lattices, namely, those which are scaled versions of root lattices having $N \leq \overline{N}$. Inequalities (2.8) and (2.9) apply to the scaled versions of each of this finite collection of root lattices, and so to the concatenation from which our lists are formed. It is important to realize that this deterioration is a property of our selection process and has nothing to do with the asymptotic behavior of a complete red list of optimal lattices.

7 Concluding Remarks

The basic contribution of this paper is the introduction of a very simple theory of rectangular scaling of lattices and a description of the behavior of $\rho(\Lambda)$ under such scaling. This theory, described in Sections 2 and 3, remains to be fully exploited. In the rest of this paper we have used it only to provide lists of good lattices from existing lists. We are sure that many more applications of a more detailed and innovative nature may exist.

The rest of this paper is concerned with carrying out this scaling process on lists of lattices. By any measure, this has been very successful, producing a cornucopia of new good lattices. Indeed, so many and varied are the outputs of this process that organization and selection of results for publication has become a problem in itself. This aspect of the work is described and discussed in Sections 5 and 6.

We have uncovered many high-order lattice rules in dimensions 3, 4, and to some extent 5. The best are listed in Tables 1, 2, and 8, respectively. These turn out as might be expected in view of the current advanced theory (see, for example, Niederreiter [Nie92]). It is our hope that these concrete examples will provide a spur to the recognition and practical application of lattice rules in actual scientific projects involving multidimensional quadrature.

Appendix: Red Scaled Red Lists

The two lists in this appendix are included to illustrate the discussion in Section 6.2. At first glance both lists appear reasonable. However, in fact, these lists as a whole are significantly inferior to those given in Tables 4 and 5, and 6 and 7, respectively, though they do contain some very good lattices.

Table A1 *Red Scaled Red List in Three Dimension*

N	ρ	b_{11}	b_{12}	b_{13}	b_{22}	b_{23}	b_{33}	rank	z
4185	324	3	0	108	3	168	465	3	0.645620
4704	390	2	0	228	3	148	784	2	0.701090
5874	414	1	0	1044	2	303	2937	1	0.611650
7056	444	3	0	148	3	228	784	2	0.557620
7248	448	2	0	260	3	376	1208	2	0.549400
7696	480	2	0	466	2	834	1924	3	0.558120
8811	540	1	0	621	3	378	2937	2	0.556720
8820	555	3	0	185	3	285	980	2	0.571660
9408	588	3	0	228	4	148	784	2	0.571830
11748	621	1	0	1392	3	404	3916	1	0.495370
13212	648	2	0	1137	2	1428	3303	2	0.465390
13376	672	4	0	180	4	260	836	3	0.477330
15664	808	2	0	404	2	1392	3916	3	0.498250
17622	909	2	0	621	3	378	2937	2	0.504320
23352	1080	2	0	1304	2	2140	5838	3	0.465190
23496	1212	2	0	1392	3	404	3916	2	0.519160
31328	1440	2	0	828	4	504	3916	3	0.475840

Table A2 *Red Scaled Red List in Four Dimensions*

N	ρ	b_{11}	b_{12}	b_{13}	b_{14}	b_{22}	b_{23}	b_{24}	b_{33}	b_{34}	b_{44}	rank	z
640	16	2	0	2	6	4	0	8	4	16	20	4	1.04376
864	24	1	0	0	42	2	0	69	3	54	144	2	1.26997
		2	0	0	15	2	0	33	3	21	72	2	1.26997
		1	0	0	42	2	0	69	3	12	144	2	1.26997
1124	30	1	0	0	106	1	0	178	1	442	1124	0	1.31706
1944	36	3	0	0	15	3	3	12	6	30	36	4	1.06190
		3	0	0	15	3	0	21	3	33	72	4	1.06190
2164	40	1	0	0	152	1	0	330	2	104	1082	2	1.09017
2248	60	1	0	0	106	1	0	178	2	442	1124	2	1.58980
4328	72	1	0	0	104	2	0	152	2	330	1082	3	1.16625
		1	0	0	104	2	0	242	2	400	1082	3	1.16625
4496	106	1	0	0	106	2	0	178	2	442	1124	3	1.66790
6744	120	1	0	0	159	2	0	267	2	663	1686	2	1.38308
8656	144	2	0	0	104	2	0	152	2	330	1082	4	1.36734
8992	212	2	0	0	106	2	0	178	2	442	1124	4	1.95413
19476	216	2	0	0	156	2	0	228	3	495	1623	2	1.08193
		2	0	0	156	2	0	495	3	228	1623	2	1.08193
20232	318	2	0	0	159	2	0	267	3	663	1686	2	1.54517
35968	356	2	0	0	212	2	0	356	4	884	2248	4	1.08922
38952	360	2	0	0	208	3	0	304	3	660	2164	2	1.03259
40464	424	2	0	0	212	3	0	356	3	884	2248	2	1.17917
43821	468	3	0	0	156	3	0	228	3	495	1623	4	1.21996
45522	477	3	0	0	159	3	0	267	3	663	1686	4	1.20550
77904	624	3	0	0	208	3	0	304	4	660	2164	2	1.01613
80928	636	3	0	0	212	3	0	356	4	884	2248	2	1.00373

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