ON A CERTAIN PARAMETER OF THE DISCRETIZED EXTENDED LINEAR-QUADRATIC PROBLEM OF OPTIMAL CONTROL

Ciyou Zhu *

Mathematics and Computer Science Division, Argonne National Laboratory
9700 South Cass Avenue, Argonne, IL 60439

July 1993 ; revised May 1994

Abstract. The number $\gamma := \|\hat{Q}^{-\frac{1}{2}}\hat{R}\hat{P}^{-\frac{1}{2}}\|$ is an important parameter for the extended linear-quadratic programming (ELQP) problem associated with the Lagrangian $L(\hat{u},\hat{v}) = \hat{p}\cdot\hat{u} + \frac{1}{2}\hat{u}\cdot\hat{P}\hat{u} + \hat{q}\cdot\hat{v} - \frac{1}{2}\hat{v}\cdot\hat{Q}\hat{v} - \hat{v}\cdot\hat{R}\hat{u}$ over polyhedral sets $\hat{U}\times\hat{V}$. Some fundamental properties of the problem, as well as the convergence rates of certain newly developed algorithms for large-scale ELQP, are all related to γ.

In this paper, we derive an estimate of γ for the ELQP problems resulting from discretization of an optimal control problem. We prove that the parameter γ of the discretized problem is bounded independently of the number of subintervals in the discretization.

Keywords. Extended linear-quadratic programming, Minimax problem, Optimal control, Primal-dual projected gradient algorithm

* This research was supported by the Office of Scientific Computing, U. S. Department of Energy, under Contract W-31-109-Eng-38 and the National Science Foundation under Contract ASC-9213149.

1. Introduction.

The extended linear-quadratic programming (ELQP), in its standard minimax form, is to find a saddle point of the Lagrangian

$$L(\hat{u},\hat{v}) = \hat{p}\cdot\hat{u} + \tfrac{1}{2}\hat{u}\cdot\hat{P}\hat{u} + \hat{q}\cdot\hat{v} - \tfrac{1}{2}\hat{v}\cdot\hat{Q}\hat{v} - \hat{v}\cdot\hat{R}\hat{u} \quad \text{over} \quad \hat{U}\times\hat{V}, \tag{1.1}$$

where $\hat{U}$ and $\hat{V}$ are polyhedral sets in $\mathbb{R}^{\hat{k}}$ and $\mathbb{R}^{\hat{l}}$, respectively, and $\hat{P}\in\mathbb{R}^{\hat{k}\times\hat{k}}$ and $\hat{Q}\in\mathbb{R}^{\hat{l}\times\hat{l}}$ are symmetric positive semidefinite matrices [1]. The associated primal and dual problems are

$(\hat{\mathcal{P}})$ minimize $f(\hat{u})$ over all $\hat{u}\in\hat{U}$, where $f(\hat{u}) := \sup_{\hat{v}\in\hat{V}} L(\hat{u},\hat{v})$,

$(\hat{\mathcal{Q}})$ maximize $g(\hat{v})$ over all $\hat{v}\in\hat{V}$, where $g(\hat{v}) := \inf_{\hat{u}\in\hat{U}} L(\hat{u},\hat{v})$.

The problem is called *fully quadratic* if both P and Q are positive definite.

The number

$$\gamma := \|\hat{Q}^{-\frac{1}{2}}\hat{R}\hat{P}^{-\frac{1}{2}}\| \tag{1.2}$$

introduced by Rockafellar [2] is an important parameter for the problem in the fully quadratic case. (We use the Euclidean norm for vectors and the associated operator norm for matrices unless otherwise specified.) It serves as a Lipschitz constant for the mappings $F:\mathbb{R}^{\hat{k}}\to\mathbb{R}^{\hat{l}}$ and $G:\mathbb{R}^{\hat{l}}\to\mathbb{R}^{\hat{k}}$ defined as

$$F(\hat{u}) = \operatorname*{argmax}_{\hat{v}\in\hat{V}} L(\hat{u},\hat{v}) \quad \text{and} \quad G(\hat{v}) = \operatorname*{argmin}_{\hat{u}\in\hat{U}} L(\hat{u},\hat{v}),$$

respectively [2]. It also plays a central role in the convergence results of several newly developed algorithms for large scale ELQP problems.

Let $\varepsilon_\nu = f(u^\nu) - g(v^\nu)$ be the ν-th duality gap related to the primal-dual pair of iterates (u^ν, v^ν). Rockafellar proved that the sequence $\{(u^\nu,v^\nu)\}$ generated by the finite-envelope algorithm [2] satisfies

$$\frac{\varepsilon_{\nu+1}}{\varepsilon_\nu} \le 1 - \frac{1}{4(1+\gamma^2)}. \tag{1.3}$$

In [10], Zhu proved that the sequence $\{(u^\nu,v^\nu)\}$ generated by certain variants of the primal-dual steepest descent algorithm (PDSD) developed in [9] satisfies

$$\frac{\varepsilon_{\nu+1}}{\varepsilon_\nu} \le \begin{cases} \frac{\gamma^2}{2-\gamma^2} & \text{if } 0\le\gamma^2<\frac{1}{2}, \\ 1-\frac{1}{2\gamma^2+0.5} & \text{if } \gamma^2\ge\frac{1}{2}. \end{cases} \tag{1.4}$$

One of the variants uses a "fixed step length" strategy [10], where the step lengths are also related to γ. In [6, 7], S. J. Wright described interior point algorithms for linear complementarity problems (LCP). If the ELQP is formulated as an LCP and if the standard conjugate gradient algorithm is used to solve the resulting linear equations, the convergence rate for the "inner iterations" [8] will be

$$\frac{\varepsilon_{\nu+1}}{\varepsilon_\nu} \leq 1 - \frac{2}{1+(1+\gamma^2)^{\frac{1}{2}}}. \tag{1.5}$$

The right-hand sides of (1.3)–(1.5) all depend on the parameter γ of the problem. The smaller the value of γ, the faster the convergence for these algorithms.

In this paper, we consider the ELQP resulting from discretizing a continuous-time optimal control problem with time-independent data. As we show in Section 2, the matrix $\hat{R}$ for such problem consists of a large number of nonzero blocks, each of which is a product of infinite series in terms of the matrices in the original continuous-time problem. It is usually impractical to compute γ from the definition (1.2). Actually, a primary goal in algorithm design is to avoid computations involving the $\hat{R}$ matrix, because of its size and density. All three of the above mentioned algorithms for the discretized ELQP could be implemented in such a way that this goal is reached by taking advantage of the discretized system dynamics in their computations [3,4,8].

Mathematically, however an unanswered question is the dependence of γ on the data of the original continuous-time problem and on the number of subintervals used in the discretization. Zhu and Rockafellar [9] observe that the number of iterations needed for their algorithms to converge remains essentially unchanged as the discretization is refined. This observation suggests strongly that the value of γ approaches a constant, or is at least bounded above, as the number of subintervals increases. In this paper, we will prove this conjecture on γ. In Section 2, we derive expressions for the matrices $\hat{P}$, $\hat{Q}$, and $\hat{R}$ in the Lagrangian (1.1) for the discretized problem. In Section 3, we give an estimate of γ in terms of the matrices in the original continuous-time extended linear-quadratic problem of optimal control — an estimate that is independent of the mesh width.

2. Data Matrices in the Lagrangian for the Discretized Problem.

The continuous-time extended linear-quadratic problem of optimal control (with time-independent data and normalized time interval) is

$$(\mathcal{P}^{\text{cont}}) \quad \text{minimize } \mathcal{F}(u_e, u) = \int_0^1 [p{\cdot}u(t) + \tfrac{1}{2}u(t){\cdot}Pu(t) - c{\cdot}x(t)]dt + [p_e{\cdot}u_e + \tfrac{1}{2}u_e{\cdot}P_e u_e - c_e{\cdot}x(1)] + \int_0^1 \rho_{V,Q}\big(q - Cx(t) - Du(t)\big)dt + \rho_{V_e,Q_e}\big(q_e - C_e x(1) - D_e u\big)$$

over the state trajectory

$$\dot{x}(t) = Ax(t) + Bu(t) + b \quad \text{a.e.}, \quad x(0) = B_e u_e + b_e \quad (x(t) \in \mathbb{R}^m),$$

with the control space

$$\mathcal{U} = \{(u_e, u) \in \mathbb{R}^{k_e} \times \mathcal{L}_k^\infty[0,1] \,\big|\, u_e \in U_e, u(t) \in U \text{ a.e. }\}$$

(Rockafellar [1]). Here U, U_e, V, and U_e are polyhedral convex sets, and P, P_e, Q, and Q_e are symmetric positive semidefinite matrices. Each ρ term, defined as

$$\rho_{V,Q}(s) = \sup_{v \in V}\{s{\cdot}v - \tfrac{1}{2}v{\cdot}Qv\},$$

is a lower semicontinuous convex piecewise linear-quadratic function [1, Proposition 2.3]. The dual problem is

$$(\mathcal{Q}^{\text{cont}}) \quad \text{maximize } \mathcal{G}(v, v_e) = \int_0^1 [q{\cdot}v(t) - \tfrac{1}{2}v(t){\cdot}Qv(t) - b{\cdot}y(t)]dt + [q_e{\cdot}v_e + \tfrac{1}{2}v_e{\cdot}Q_e v_e - b_e{\cdot}y(0)] - \int_0^1 \rho_{U,P}\big(B^T y(t) + D^T v(t) - p\big)dt - \rho_{U_e,P_e}\big(B_e^T y(0) + D_e^T v - p_e\big)$$

over the state trajectory

$$-\dot{y}(t) = A^T y(t) + C^T v(t) + c \quad \text{a.e.}, \quad y(1) = C_e^T v_e + c_e \quad (y(t) \in \mathbb{R}^m),$$

with the control space

$$\mathcal{V} = \{(v, v_e) \in \mathcal{L}_l^\infty[0,1] \times \mathbb{R}^{l_e} \,\big|\, v(t) \in V \text{ a.e.}, \ v_e \in V_e\},$$

where

$$\rho_{U,P}(r) = \sup_{u\in U}\{r{\cdot}u - \tfrac{1}{2}u{\cdot}Pu\}.$$

Problems ($\mathcal{P}^{\rm cont}$) and ($\mathcal{Q}^{\rm cont}$) differ from the conventional linear-quadratic models in optimal control in that they allow for piecewise linear-quadratic penalty terms in the objective functionals, as well as constraints on the controls. See Rockafellar [1] for a detailed presentation.

Problems ($\mathcal{P}^{\rm cont}$) and ($\mathcal{Q}^{\rm cont}$) are equivalent to a saddle point problem under certain *finiteness conditions* (which will be satisfied if, for example, the matrices P, Q, P_e, and Q_e, are positive definite) [1, Theorem 6.1 and Corollary 6.4]. The saddle point problem is

$$(\mathcal{S}^{\rm cont}) \qquad \text{minimax}_{(u_e,u)\in\mathcal{U},(v,v_e)\in\mathcal{V}}\, \mathcal{J}(u_e,u;v,v_e),$$

where

$$\mathcal{J}(u_e,u;v,v_e) = \int_0^1 J(u(t),v(t))dt + J_e(u_e,v_e) - \langle (u_e,u);(v,v_e)\rangle,$$

with

$$J(u,v) = p{\cdot}u + \tfrac{1}{2}u{\cdot}Pu + q{\cdot}v - \tfrac{1}{2}v{\cdot}Qv - v{\cdot}Du \;\text{ for }\; u\in\mathbb{R}^k, v\in\mathbb{R}^l$$
$$J_e(u_e,v_e) = p_e{\cdot}u_e + \tfrac{1}{2}u_e{\cdot}P_eu_e + q_e{\cdot}v_e - \tfrac{1}{2}v_e{\cdot}Q_ev_e \;\text{ for }\; u_e\in\mathbb{R}^{k_e}, v\in\mathbb{R}^{l_e},$$

and

$$\begin{aligned}\langle (u_e,u);(v,v_e)\rangle &= \int_0^1 x(t){\cdot}(C^Tv(t)+c)dt + x(1){\cdot}(C_e^Tv_e+c_e)\\ &= \int_0^1 y(t){\cdot}(Bu(t)+b)dt + y(0){\cdot}(B_eu_e+b_e).\end{aligned}$$

A numerical solution of ($\mathcal{S}^{\rm cont}$) can be obtained by discretizing the problem into the following approximate version [4, 5]:

$$(\mathcal{S}^{\rm cont}_{\rm n}) \qquad \text{minimax}_{\mathcal{U}_n\times\mathcal{V}_n}\, \mathcal{J}(u_e,u;v,v_e),$$

where

$$\mathcal{U}_n = \{(u_e,u)\in\mathcal{U}\,\big|\, u(t) \text{ is constant on } \left(\frac{\tau-1}{n},\frac{\tau}{n}\right),\ \tau=1,\ldots,n\},$$
$$\mathcal{V}_n = \{(v,v_e)\in\mathcal{V}\,\big|\, v(t) \text{ is constant on } \left(\frac{\tau-1}{n},\frac{\tau}{n}\right),\ \tau=1,\ldots,n\}.$$

If we denote the constant values of $u(t)$ and $v(t)$ on $\left(\frac{\tau-1}{n}, \frac{\tau}{n}\right)$ by u_τ and v_τ, respectively, for $\tau = 1, \ldots, n$, problem $(\mathcal{S}_{\mathrm{n}}^{\mathrm{cont}})$ can be written in the form of a finite-dimensional discrete-time saddle point problem as

$$(\mathcal{S}_{\mathrm{n}}^{\mathrm{disc}}) \qquad \operatorname{minimax}_{U_n \times V_n} \mathcal{J}_n(u_e, u_1, \ldots, u_n;\ v_1, \ldots, v_n, v_e),$$

where

$$\begin{aligned} &\mathcal{J}_n(u_e, u_1, \ldots, u_n;\ v_1, \ldots, v_n, v_e) \\ &\quad = \sum_{\tau=1}^{n} J_n(u_\tau, v_\tau) + J_e(u_e, v_e) - \langle (u_e, u_1, \ldots, u_n);\ (v_1, \ldots, v_n, v_e) \rangle_n, \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} J_n(u_\tau, v_\tau) &= p_n \cdot u_\tau + q_n \cdot v_\tau + \tfrac{1}{2} u_\tau \cdot P_n u_\tau - \tfrac{1}{2} v_\tau \cdot Q_n v_\tau - v_\tau \cdot D_n u_\tau + d_n, \\ J_e(u_e, v_e) &= p_e \cdot u_e + q_e \cdot v_e + \tfrac{1}{2} u_e \cdot P_e u_e - \tfrac{1}{2} v_e \cdot Q_e v_e, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\langle (u_e, u_1, \ldots, u_n); (v_1, \ldots, v_n, v_e) \rangle_n \\ &\quad = \sum_{\tau=1}^{n} y_{\tau+1} \cdot (B_n u_\tau + b_n) + y_1 \cdot (B_e u_e + b_e) \\ &\quad = \sum_{\tau=1}^{n} x_{\tau-1} \cdot (C_n^T v_\tau + c_n) + x_n \cdot (C_e^T v_e + c_e). \end{aligned} \tag{2.3}$$

The trajectories are given by the discretized system dynamics

$$\begin{aligned} x_\tau &= A_n x_{\tau-1} + B_n u_\tau + b_n \text{ for } \tau = 1, \ldots, n, \quad x_0 = B_e u_e + b_e \quad (x_\tau \in \mathbb{R}^m), \\ y_\tau &= A_n^T y_{\tau+1} + C_n^T v_\tau + c_n \text{ for } \tau = 1, \ldots, n, \quad y_{n+1} = C_e^T v_e + c_e \quad (y_\tau \in \mathbb{R}^m), \end{aligned} \tag{2.4}$$

where we impose

$$\begin{aligned} (u_e, u_1, \ldots, u_n) &\in U_n := U_e \times (U)^n \subseteq \mathbb{R}^{k_e} \times (\mathbb{R}^k)^n, \\ (v_1, \ldots, v_n, v_e) &\in V_n := (V)^n \times V_e \subseteq (\mathbb{R}^l)^n \times \mathbb{R}^{l_e}. \end{aligned}$$

The transformation of the data is

$$A_n = I + M_n A, \tag{2.5a}$$

$$B_n = M_n B, \qquad b_n = M_n b, \tag{2.5b}$$

$$C_n = CM_n, \qquad c_n = M_n^T c, \tag{2.5c}$$

$$D_n = \frac{1}{n}D + CS_nB, \qquad d_n = -c \cdot S_n b, \tag{2.5d}$$

$$P_n = \frac{1}{n}P, \qquad p_n = \frac{1}{n}p - B^T S_n^T c, \tag{2.5p}$$

$$Q_n = \frac{1}{n}Q, \qquad q_n = \frac{1}{n}q - CS_n b, \tag{2.5q}$$

where

$$S_n = \sum_{i=2}^{\infty} \frac{1}{i!}\left(\frac{1}{n}\right)^i A^{i-2}, \qquad M_n = \frac{1}{n}I + AS_n. \tag{2.6}$$

(Wright [4, 5]). The associated primal and dual problems are

($\mathcal{P}_{\rm n}^{\rm disc}$) minimize $f(u)$ over $u \in U_n$ where

$$f(u_e, u_1, \ldots, u_n) := \max_{v \in V_n} \mathcal{J}_n(u_e, u_1, \ldots, u_n;\, v_1, \ldots, v_n, v_e)$$

and

($\mathcal{Q}_{\rm n}^{\rm disc}$) maximize $g(v)$ over $v \in V_n$ where

$$g(v_1, \ldots, v_n, v_e) := \min_{u \in U_n} \mathcal{J}_n(u_e, u_1, \ldots, u_n;\, v_1, \ldots, v_n, v_e).$$

Problems ($\mathcal{P}_{\rm n}^{\rm disc}$) and ($\mathcal{Q}_{\rm n}^{\rm disc}$) are ELQP in the multistage format, which could be solved directly by the techniques mentioned in Section 1 without forming the huge $\hat{R}$ matrix in the Lagrangian (1.1) of its standard form. However, in order to get an expression for γ in terms of the matrices A, B, C, D, P, and Q from the continuous-time problem, we eliminate the state variables x_τ and y_τ in the expression of $\mathcal{J}_n$. From the discretized system dynamics (2.4), we obtain

$$\begin{bmatrix} x_e \\ x_1 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} B_e & & & & \\ A_nB_e & B_n & & & \\ \cdot & \cdot & \cdots & & \\ \cdot & \cdot & \cdots & \cdot & \\ A_n^nB_e & A_n^{n-1}B_n & \cdots & A_nB_n & B_n \end{bmatrix} \begin{bmatrix} u_e \\ u_1 \\ \cdot \\ \cdot \\ u_n \end{bmatrix} + \begin{bmatrix} b_e \\ A_nb_e + b_n \\ \cdot \\ \cdot \\ A_n^nb_e + A_n^{n-1}b_n + \ldots + b_n \end{bmatrix}$$

$$= \begin{bmatrix} I & & & & \\ A_n & I & & & \\ \cdot & \cdot & \cdots & & \\ \cdot & \cdot & \cdots & \cdot & \\ A_n^n & A_n^{n-1} & \cdots & A_n & I \end{bmatrix} \left(\begin{bmatrix} B_eu_e \\ B_nu_1 \\ \cdot \\ \cdot \\ B_nu_n \end{bmatrix} + \begin{bmatrix} b_e \\ b_n \\ \cdot \\ \cdot \\ b_n \end{bmatrix} \right). \tag{2.7}$$

By substituting (2.7) in the second expression of (2.3), we obtain

$$
\begin{aligned}
&\langle (u_e, u_1, \ldots, u_n); (v_1, \ldots, v_n, v_e) \rangle_n \\
&= \begin{bmatrix} c_n \\ \cdot \\ \cdot \\ c_n \\ c_e \end{bmatrix} \cdot \begin{bmatrix} x_e \\ x_1 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \\ v_e \end{bmatrix} \cdot \begin{bmatrix} C_n & & & \\ & \cdots & & \\ & & C_n & \\ & & & C_e \end{bmatrix} \begin{bmatrix} x_e \\ x_1 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} c_n \\ \cdot \\ \cdot \\ c_n \\ c_e \end{bmatrix} \cdot \begin{bmatrix} B_e & & & & \\ A_n B_e & B_n & & & \\ \cdot & \cdot & \cdots & & \\ \cdot & \cdot & \cdots & \cdot & \\ A_n^n B_e & A_n^{n-1} B_n & \cdots & A_n B_n & B_n \end{bmatrix} \begin{bmatrix} u_e \\ u_1 \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \\
&\quad + \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \\ v_e \end{bmatrix} \cdot \begin{bmatrix} C_n B_e & & & & \\ C_n A_n B_e & C_n B_n & & & \\ \cdot & \cdot & \cdots & & \\ \cdot & \cdot & \cdots & \cdot & \\ C_e A_n^n B_e & C_e A_n^{n-1} B_n & \cdots & C_e A_n B_n & C_e B_n \end{bmatrix} \begin{bmatrix} u_e \\ u_1 \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \\
&\quad + \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \\ v_e \end{bmatrix} \cdot \begin{bmatrix} C_n & & & & \\ C_n A_n & C_n & & & \\ \cdot & \cdot & \cdots & & \\ \cdot & \cdot & \cdots & \cdot & \\ C_e A_n^n & C_e A_n^{n-1} & \cdots & C_e A_n & C_e \end{bmatrix} \begin{bmatrix} b_e \\ b_n \\ \cdot \\ \cdot \\ b_n \end{bmatrix} \\
&\quad + \begin{bmatrix} c_n \\ \cdot \\ \cdot \\ c_n \\ c_e \end{bmatrix} \cdot \begin{bmatrix} I & & & & \\ A_n & I & & & \\ \cdot & \cdot & \cdots & & \\ \cdot & \cdot & \cdots & \cdot & \\ A_n^n & A_n^{n-1} & \cdots & A_n & I \end{bmatrix} \begin{bmatrix} b_e \\ b_n \\ \cdot \\ \cdot \\ b_n \end{bmatrix}. \qquad (2.8)
\end{aligned}
$$

Similarly, we have from (2.2) that

$$
\begin{aligned}
&\sum_{\tau=1}^{n} J_n(u_\tau, v_\tau) + J_e(u_e, v_e) \\
&= \begin{bmatrix} p_e \\ p_n \\ \cdot \\ \cdot \\ p_n \end{bmatrix} \cdot \begin{bmatrix} u_e \\ u_1 \\ \cdot \\ \cdot \\ u_n \end{bmatrix} + \begin{bmatrix} q_n \\ \cdot \\ \cdot \\ q_n \\ q_e \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \\ v_e \end{bmatrix} + \frac{1}{2} \begin{bmatrix} u_e \\ u_1 \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \cdot \begin{bmatrix} P_e & & & \\ & P_n & & \\ & & \cdots & \\ & & & P_n \end{bmatrix} \begin{bmatrix} u_e \\ u_1 \\ \cdot \\ \cdot \\ u_n \end{bmatrix}
\end{aligned}
$$

$$-\frac{1}{2}\begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \\ v_e \end{bmatrix} \cdot \begin{bmatrix} Q_n & & & \\ & \cdots & & \\ & & Q_n & \\ & & & Q_e \end{bmatrix} \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \\ v_e \end{bmatrix}$$

$$-\begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ v_n \\ v_e \end{bmatrix} \cdot \begin{bmatrix} 0 & D_n & & \\ & \cdots & \cdots & \\ & & 0 & D_n \\ & & & 0 \end{bmatrix} \begin{bmatrix} u_e \\ u_1 \\ \cdot \\ \cdot \\ u_n \end{bmatrix} \tag{2.9}$$

By substituting (2.8) and (2.9) in (2.1), we obtain equality of $\mathcal{J}_n$ with the Lagrangian $L(\hat{u}, \hat{v})$ in (1.1) by noting the identities

$$\hat{P} = \mathrm{diag}[P_e, P_n, \cdots, P_n], \qquad \hat{Q} = \mathrm{diag}[Q_n, \cdots, Q_n, Q_e], \tag{2.10}$$

$$\hat{R} = \begin{bmatrix} C_nIB_e & D_n & & & & & \\ C_nA_nB_e & C_nIB_n & D_n & & & & \\ C_nA_n^2B_e & C_nA_nB_n & C_nIB_n & D_n & & & \\ \cdot & \cdot & \cdot & \cdot & \cdots & & \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \\ C_nA_n^{n-1}B_e & C_nA_n^{n-2}B_n & \cdot & \cdot & \cdots & C_nIB_n & D_n \\ C_eA_n^nB_e & C_eA_n^{n-1}B_n & \cdot & \cdot & \cdots & C_eA_nB_n & C_eIB_n \end{bmatrix}, \tag{2.11}$$

$$\hat{p} = \begin{bmatrix} p_e \\ p_n \\ \cdot \\ \cdot \\ p_n \end{bmatrix} - \begin{bmatrix} B_e^Tc_n + \ldots + B_e^T(A_n^T)^{n-1}c_n + B_e^T(A_n^T)^nc_e \\ B_n^Tc_n + \ldots + B_n^T(A_n^T)^{n-2}c_n + B_n^T(A_n^T)^{n-1}c_e \\ \cdot \\ \cdot \\ B_n^Tc_n + B_n^TA_n^Tc_e \\ B_n^Tc_e \end{bmatrix},$$

$$\hat{q} = \begin{bmatrix} q_n \\ \cdot \\ \cdot \\ q_n \\ q_e \end{bmatrix} - \begin{bmatrix} C_nb_e \\ C_nA_nb_e + C_nb_n \\ \cdot \\ \cdot \\ C_nA_n^{n-1}b_e + C_nA_n^{n-2}b_n + \ldots + C_nb_n \\ C_eA_n^nb_e + C_eA_n^{n-1}b_n + \ldots + C_eb_n \end{bmatrix},$$

with

$$\hat{u} = (u_e, u_1, \cdots, u_n), \qquad \hat{v} = (v_1, \cdots, v_n, v_e).$$

(The additive constants in $\mathcal{J}_n$ are dropped since they play no role in the problem.)

3. Estimation of the Parameter γ of the Discretized Problem.

In this section, we prove the following estimate of the parameter γ of the discretized problem in terms of the continuous-time problem data.

Theorem 3.1. *Suppose the matrices P, Q, P_e, and Q_e in the continuous-time extended linear-quadratic problems ($\mathcal{P}^{\rm cont}$) and ($\mathcal{Q}^{\rm cont}$) of optimal control are positive definite. Then the parameter γ_n of the discretized versions ($\mathcal{P}_{\rm n}^{\rm disc}$) and ($\mathcal{Q}_{\rm n}^{\rm disc}$) satisfies*

$$\begin{aligned}\gamma_n \le\ & \|Q^{-\frac{1}{2}}C\|\|B_eP_e^{-\frac{1}{2}}\|e^{(n-1)\|A\|/n} \\ & + \|Q_e^{-\frac{1}{2}}C_e\|\|BP^{-\frac{1}{2}}\|e^{(n-1)\|A\|/n} + \|Q_e^{-\frac{1}{2}}C_e\|\|B_eP_e^{-\frac{1}{2}}\|e^{\|A\|} \\ & + \|Q^{-\frac{1}{2}}DP^{-\frac{1}{2}}\| + \|Q^{-\frac{1}{2}}C\|\|BP^{-\frac{1}{2}}\|\frac{1-e^{(n-1)\|A\|/n}}{n(1-e^{\|A\|/n})} + O(n^{-1}), \end{aligned} \tag{3.1}$$

where n is the number of equal-length subintervals used in the discretization. Moreover,

$$\begin{aligned}\limsup_{n\to\infty}\gamma_n \le\ & \|Q^{-\frac{1}{2}}C\|\|B_eP_e^{-\frac{1}{2}}\|e^{\|A\|} \\ & + \|Q_e^{-\frac{1}{2}}C_e\|\|BP^{-\frac{1}{2}}\|e^{\|A\|} + \|Q_e^{-\frac{1}{2}}C_e\|\|B_eP_e^{-\frac{1}{2}}\|e^{\|A\|} \\ & + \|Q^{-\frac{1}{2}}DP^{-\frac{1}{2}}\| + \|Q^{-\frac{1}{2}}C\|\|BP^{-\frac{1}{2}}\|\frac{e^{\|A\|}-1}{\|A\|}. \end{aligned} \tag{3.2}$$

Two conclusions follow immediately from Theorem 3.1:

(i) The parameter γ_n of the discretized problem, as a function of n, is bounded above when $n \to \infty$. Hence for the algorithms with their convergence rates having an upper bound determined solely by γ_n, the number of iterations needed for convergence should remain essentially the same as n increases. If, in addition, the algorithm has only $O(n)$ operations in each iteration, then the total CPU time needed for convergence should be proportional to n. These results are consistent with the observations of Zhu and Rockafellar [9].

(ii) The only part of the original data that has an exponential contribution to γ_n is the norm of the matrix A in the system dynamics as the coefficient of the state variables. The norms of all the other matrices contribute linearly.

Before proving the theorem, we first state two simple propositions. The proofs of these propositions are elementary and therefore skipped.

Proposition 3.2. *Suppose matrix E can be partitioned as*

$$E = \begin{bmatrix} E_{11} & E_{12} & \cdots & E_{1s} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ E_{r1} & E_{r2} & \cdots & E_{rs} \end{bmatrix}.$$

Then

$$\|E\| \leq \sum_{i=1}^{r} \sum_{j=1}^{s} \|E_{ij}\|.$$

Proposition 3.3. *Suppose matrix E is of block diagonal form*

$$E = \mathrm{diag}[E_1, E_2, \ldots E_r].$$

Then

$$\|E\| = \max_{1 \leq i \leq r} \{\|E_i\|\}.$$

Proof of the Theorem. Let $\Gamma := \hat{Q}^{-\frac{1}{2}} \hat{R} \hat{P}^{-\frac{1}{2}}$. Then $\gamma_n = \|\Gamma\|$. Observe that the matrix $\hat{R}$ in (2.11) is block lower Hessenberg, while the matrices $\hat{P}$ and $\hat{Q}$ in (2.10) are block diagonal with the corresponding block structure. Hence the matrix $\Gamma = \hat{Q}^{-\frac{1}{2}} \hat{R} \hat{P}^{-\frac{1}{2}}$ is also of block lower Hessenberg with the same block structure as that of $\hat{R}$. Partition Γ as

$$\Gamma = \begin{bmatrix} \Gamma_{ne} & \Gamma_{nn} \\ \Gamma_{ee} & \Gamma_{en} \end{bmatrix}, \tag{3.3}$$

where

$$\Gamma_{ne} = \begin{bmatrix} Q_n^{-\frac{1}{2}} C_n I B_e P_e^{-\frac{1}{2}} \\ Q_n^{-\frac{1}{2}} C_n A_n B_e P_e^{-\frac{1}{2}} \\ \cdot \\ \cdot \\ \cdot \\ Q_n^{-\frac{1}{2}} C_n A_n^{n-1} B_e P_e^{-\frac{1}{2}} \end{bmatrix}, \; \Gamma_{ee} = \left[Q_e^{-\frac{1}{2}} C_e A_n^n B_e P_e^{-\frac{1}{2}} \right], \tag{3.4}$$

$$\Gamma_{en} = \left[Q_e^{-\frac{1}{2}} C_e A_n^{n-1} B_n P_n^{-\frac{1}{2}} \quad \cdots \quad Q_e^{-\frac{1}{2}} C_e A_n B_n P_n^{-\frac{1}{2}} \quad Q_e^{-\frac{1}{2}} C_e I B_n P_n^{-\frac{1}{2}} \right], \tag{3.5}$$

and

$$\Gamma_{nn} = \tag{3.6}$$

$$\begin{bmatrix} Q_n^{-\frac{1}{2}} D_n P_n^{-\frac{1}{2}} & & & \\ Q_n^{-\frac{1}{2}} C_n I B_n P_n^{-\frac{1}{2}} & Q_n^{-\frac{1}{2}} D_n P_n^{-\frac{1}{2}} & & \\ \cdot & \cdot & \cdots & \\ \cdot & \cdot & \cdots & \cdot \\ Q_n^{-\frac{1}{2}} C_n A_n^{n-2} B_n P^{-\frac{1}{2}} & Q_n^{-\frac{1}{2}} C_n A_n^{n-3} B_n P^{-\frac{1}{2}} & \cdots & Q_n^{-\frac{1}{2}} C_n I B_n P_n^{-\frac{1}{2}} & Q_n^{-\frac{1}{2}} D_n P_n^{-\frac{1}{2}} \end{bmatrix}.$$

It follows from (2.6) that

$$S_n = \sum_{i=2}^{\infty} \frac{1}{i!}\left(\frac{1}{n}\right)^i A^{i-2} = \frac{1}{2n^2}I + O(n^{-3}), \tag{3.7a}$$

$$M_n = \frac{1}{n}I + AS_n = \frac{1}{n}I + O(n^{-2}). \tag{3.7b}$$

Hence by equations (2.5), we have the following first order approximations for the matrices in the discretized problem:

$$A_n = I + M_n A = \sum_{i=0}^{\infty} \frac{1}{i!}\left(\frac{A}{n}\right)^i = e^{A/n}, \tag{3.8a}$$

$$B_n = M_n B = \frac{1}{n}B + O(n^{-2}), \tag{3.8b}$$

$$C_n = CM_n = \frac{1}{n}C + O(n^{-2}), \tag{3.8c}$$

$$D_n = \frac{1}{n}D + CS_nB = \frac{1}{n}D + O(n^{-2}), \tag{3.8d}$$

$$P_n = \frac{1}{n}P, \tag{3.8p}$$

$$Q_n = \frac{1}{n}Q, \tag{3.8q}$$

Applying Proposition 3.2 to the partitioned form of Γ in (3.3), we have

$$\|\Gamma\| \le \|\Gamma_{ne}\| + \|\Gamma_{ee}\| + \|\Gamma_{en}\| + \|\Gamma_{nn}\|. \tag{3.9}$$

Now we estimate the norms on the right-hand side of (3.9). The matrix Γ_{ne} in (3.4) can be written as

$$\Gamma_{ne} = \operatorname{diag}[Q_n^{-\frac{1}{2}}C_nB_eP_e^{-\frac{1}{2}}, Q_n^{-\frac{1}{2}}C_nA_nB_eP_e^{-\frac{1}{2}}, \ldots, Q_n^{-\frac{1}{2}}C_nA_n^{n-1}B_eP_e^{-\frac{1}{2}}][I \ \cdots \ I]^T. \tag{3.10}$$

However

$$\begin{aligned} &\|\operatorname{diag}[Q_n^{-\frac{1}{2}}C_nB_eP_e^{-\frac{1}{2}}, Q_n^{-\frac{1}{2}}C_nA_nB_eP_e^{-\frac{1}{2}}, \ldots, Q_n^{-\frac{1}{2}}C_nA_n^{n-1}B_eP_e^{-\frac{1}{2}}]\| \\ &= \max_{0\le i\le n-1}\{\|Q_n^{-\frac{1}{2}}C_nA_n^iB_eP_e^{-\frac{1}{2}}\|\} \\ &\le \max_{0\le i\le n-1}\{\|Q_n^{-\frac{1}{2}}C_n\|\|A_n^i\|\|B_eP_e^{-\frac{1}{2}}\|\} \\ &\le \|Q_n^{-\frac{1}{2}}C_n\|\|B_eP_e^{-\frac{1}{2}}\| \max_{0\le i\le n-1}\|A_n\|^i\} \\ &\le \|Q_n^{-\frac{1}{2}}C_n\|\|B_eP_e^{-\frac{1}{2}}\| \max\{1, \|A_n\|^{n-1}\} \end{aligned} \tag{3.11}$$

by Proposition 3.3. It follows from (3.8a) and (3.8c) that

$$\|A_n\|^{n-1} \le e^{(n-1)\|A\|/n}, \tag{3.12}$$

$$\|Q_n^{-\frac{1}{2}} C_n\| = n^{-1/2}\|Q^{-\frac{1}{2}}C\| + O(n^{-3/2}). \tag{3.13}$$

Substituting (3.12) and (3.13) in (3.11), and using $\|[I \ \cdots \ I]^T\| = n^{\frac{1}{2}}$, we obtain

$$\begin{aligned}\|\Gamma_{ne}\| &\le \big(\|Q^{-\frac{1}{2}}C\| + O(n^{-1})\big)\|B_e P_e^{-\frac{1}{2}}\| \max\{1, e^{(n-1)\|A\|/n}\},\\ &\le \|Q^{-\frac{1}{2}}C\|\|B_e P_e^{-\frac{1}{2}}\| e^{(n-1)\|A\|/n} + O(n^{-1}).\end{aligned} \tag{3.14}$$

We can show in a similar way that

$$\|\Gamma_{en}\| \le \|Q_e^{-\frac{1}{2}}C_e\|\|BP^{-\frac{1}{2}}\| e^{(n-1)\|A\|/n} + O(n^{-1}). \tag{3.15}$$

For the matrix Γ_{ee}, it is obvious by (3.8a) that

$$\|\Gamma_{ee}\| = \|Q_e^{-\frac{1}{2}} C_e e^A B_e P_e^{-\frac{1}{2}}\| \le \|Q_e^{-\frac{1}{2}}C_e\|\|B_e P_e^{-\frac{1}{2}}\| e^{\|A\|}. \tag{3.16}$$

Next, we estimate $\|\Gamma_{nn}\|$. Let $\Gamma_{nn}^{(0)}$ be the matrix obtained by zeroing out all the blocks of Γ_{nn} except the diagonal blocks. Let $\Gamma_{nn}^{(i)}$, $i = 1, \ldots, n-1$, be the matrix obtained by zeroing out all the blocks of Γ_{nn} except the blocks on the ith subdiagonal. Then by Propositions 3.2 and 3.3, we have

$$\|\Gamma_{nn}^{(0)}\| = \|Q_n^{-\frac{1}{2}} D_n P_n^{-\frac{1}{2}}\| = \|Q^{-\frac{1}{2}} D P^{-\frac{1}{2}}\| + O(n^{-1})$$

and

$$\|\Gamma_{nn}^{(i)}\| = \|Q_n^{-\frac{1}{2}} C_n A_n^{i-1} B_n P_n^{-\frac{1}{2}}\| = \frac{1}{n}\|Q^{-\frac{1}{2}} C e^{(i-1)A/n} B P^{-\frac{1}{2}}\| + O(n^{-2})$$

for $i = 1, \ldots, n-1$, where the first-order approximations in (3.8) are used to get the right-hand sides. Hence

$$\begin{aligned}\|\Gamma_{nn}\| &\le \|\Gamma_{nn}^{(0)}\| + \sum_{i=1}^{n-1} \|\Gamma_{nn}^{(i)}\|\\ &= \|Q^{-\frac{1}{2}} D P^{-\frac{1}{2}}\| + \frac{1}{n}\sum_{i=1}^{n-1} \|Q^{-\frac{1}{2}} C e^{(i-1)A/n} B P^{-\frac{1}{2}}\| + O(n^{-1})\\ &\le \|Q^{-\frac{1}{2}} D P^{-\frac{1}{2}}\| + \frac{1}{n}\|Q^{-\frac{1}{2}}C\|\|BP^{-\frac{1}{2}}\| \sum_{i=1}^{n-1} \|e^{(i-1)A/n}\| + O(n^{-1}).\end{aligned}$$

But

$$\sum_{i=1}^{n-1} \|e^{(i-1)A/n}\| \leq \sum_{i=1}^{n-1} e^{(i-1)\|A\|/n} = \frac{1 - e^{(n-1)\|A\|/n}}{1 - e^{\|A\|/n}}.$$

Therefore

$$\|\Gamma_{nn}\| \leq \|Q^{-\frac{1}{2}} D P^{-\frac{1}{2}}\| + \|Q^{-\frac{1}{2}} C\| \|B P^{-\frac{1}{2}}\| \frac{1 - e^{(n-1)\|A\|/n}}{n(1 - e^{\|A\|/n})} + O(n^{-1}). \qquad (3.17)$$

Substituting (3.14), (3.15), (3.16), and (3.17) in (3.9), we get the inequality (3.1) in Theorem 3.1. Taking lim sup on both sides of (3.1), we obtain (3.2). $\square$

In all the above discussions, we assume time-independent data in the optimal control problem. As a final remark, we point out that it would not be difficult to derive a similar bound for the ELQP arising from the Euler difference scheme when the optimal control data is time-dependent. Actually the proof of Theorem 3.1, with minor adaptations, still works in this latter case if the data elements in

$$A(t), B(t), C(t), D(t), b(t), c(t), P(t), Q(t), p(t), q(t), U(t), V(t)$$

are all Lipschitzian in t. A detailed exposition for this kind of time-dependent case will be presented elsewhere.

Acknowledgments. The author is indebted to S. J. Wright and the associate editor for their helpful comments and suggestions, and to an anonymous referee for remarks on the time-dependent case. The final remark in the last section was due to this referee.

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