ON THE BEST CONSTANT FOR THE INEQUALITY*

$$\int_0^\infty y'^{\,2} \le K \left(\int_0^\infty |y| \right) \max_{(0,\infty)} |y''|$$

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ABSTRACT

In this paper we determine that the best constant of the inequality $\int_0^\infty y'^{\,2} \le K \left(\int_0^\infty |y| \right) \max_{(0,\infty)} |y''|$ is $4\sqrt{3}/3$. Our approach consists of reducing the problem to various equivalent inequalities on a finite interval and determining necessary conditions on the extremals. The best constant is shown to satisfy an algebraic equation that is solved exactly with the help of MAPLE. The best constants for several similar inequalities are also determined.

*This work was supported by the Office of Scientific Computing, U.S. Department of Energy, under Contract W-31-109-Eng-38.

1. Introduction and Notation

This paper is concerned with the determination of the best constant K_1 for the inequality

$$\int_0^\infty y'^2 \le K_1 \left(\int_0^\infty |y|\right) \max_{(0,\infty)} |y''| , \tag{1.1}$$

where y, y' are locally absolutely continuous ("$\mathrm{AC}_{\mathrm{loc}}$") real functions such that the right-hand side of the inequality is finite. Equation (1.1) is a limiting case of the inequality

$$\int_0^\infty y'^2 \le K_p \left(\int_0^\infty |y|^p\right)^{1/p} \left(\int_0^\infty |y''|^{p'}\right)^{1/p'} , \tag{1.2}$$

where $1 \le p \le \infty$ and $1/p + 1/p' = 1$. Equation (1.2) in turn is a special case of of a higher-order multiplicative inequality known as Gabushin's inequality, which states that

$$\left(\int_0^\infty |y^{(k)}|^q\right)^{1/q} \le K_{q,p,r,m,k} \left(\int_0^\infty |y|^p\right)^{\alpha/p} \left(\int_0^\infty |y^{(m)}|^r\right)^{\beta/r} , \tag{1.3}$$

where $y^{(m-1)} \in \mathrm{AC}_{\mathrm{loc}}$, $0 \le k \le m-1$, $1 \le q, p, r \le \infty$ and

$$\begin{aligned} \frac{m}{q} &\le \frac{m-k}{p} + \frac{j}{r} \\ \alpha &= \frac{m-k-1/r+1/q}{m-1/r+1/p} \\ \beta &= 1-\alpha . \end{aligned}$$

The best constants K_p for the family of inequalities (1.2) were first studied by Everitt and Giertz [5]. Thus far, the value of K_p has been discovered in only two cases: $p = 2$ is the classic inequality of Hardy and Littlewood [4, Theorem 239] and $K_2 = 2$; $p = \infty$ is a special case of an inequality considered by Arestov—here K_∞ is also 2 (cf. [10]).

For general p and in most cases of Gabushin's inequality (1.3) the determination of K_p, the existence and determination of extremals, or even questions one might reasonably ask about the behavior of K_p as a function of p — Is K_p continuous, differentiable, monotone, etc.? — are difficult and mostly unsolved problems. The situation does not improve much if one considers the problem of finding good (i.e., small) upper bounds for K_p (a lower bound on K_p is plainly 1). Using numerical methods, Everitt and Giertz estimated K_p in the range $1.1 \le p \le 2$; examples of the bounds they obtained are $K_{1.1} \le 13.53$, $K_{1.5} \le 6.98$, and $K_2 \le 5$. More recently in [4], Brown and Hinton found numerical evidence that $K_1 \leq 4.1$, that K_p is bounded above by a monotonically decreasing function $\Theta(p)$ such that $\lim_{p\to\infty} \Theta(p) = 2$. Further discussion of some of these examples as well as a surveys of known results concerning best constants and extremals of other instances of (1.3) may be found in [10],[11].

Here, although we leave the general problem of estimating K_p or $K_{q,p,r,m,k}$ unsolved, we obtain exact results for two cases. We prove first the following theorem.

Theorem 1. $K_1 = 4\sqrt{3}/3$. *Moreover a unique extremal exists for inequality (1.2), which is a certain quadratic polynomial.*

In the final section of the paper, we show how the methods of Theorem 1 may be applied to determine $K_{2p,p,\infty,2,1}$. In subsequent work we hope to show that these methods can be further extended to other cases of (1.2) or (1.3). To establish some motivation for our efforts, we remark that estimates of the best constant of this and other types of multiplicative inequalities can be used to establish nonoscillation criteria as well as spectral lower bounds for differential operators (cf. [1], [6], [12]).

Let I be $[0,\infty)$ or any finite closed interval $[a,b]$, and denote by $W(I)$ the space of functions y defined on I such that y' is locally absolutely continuous in I, and $y \in L^1(I), y' \in L^2(I)$, and $y'' \in L^\infty(I)$. For a given subfamily of functions $Z \subset W(I)$ and a function $z \in Z$, we define

$$Q(z) = \frac{\int_I z'^{\,2}}{\left(\int_I |z|\right) \max\limits_I |z''|},$$

and

$$K(Z) = \sup_{z\in Z} Q(z).$$

We often abbreviate $\int_I f(x)\,dx$ by $\int_I f$, where the variable x involved is understood.

In the special case when $I = [0,\infty)$ and $Z = W(I)$, $K(Z) = K_1$ is the best constant for the inequality under study; we will often abbreviate it simply by K.

Let y be a function defined on some interval I and $J \subset I$ be a subinterval. We use y_J to denote the restriction of y on J. Finally, we will frequently speak of functions that have been originally defined on some finite interval I as if they were defined on a different finite interval I', with the understanding that we actually refer to these functions after a suitable translation and scaling. For instance, let Z be a subset of $W([0,1])$ and y be defined on $[2,4]$. We will say that $y \in Z$, where in reality we mean that $y(\frac{x-2}{2}) \in Z$.

2. Two Finite-Interval Equivalent Problems

Finite-interval equivalent problems were first used to study Landau inequalities on $(-\infty, \infty)$ in [8], and the method was later extended to inequalities on $[0, \infty)$ in [9]. A comprehensive account of these results can be found in [11]. In this section we derive two finite-interval equivalent problems for the inequality being studied. Further reductions will be given in the next section.

Lemma 1. *Let $I = [0, \infty)$ and $y \in W(I)$ be such that $Q(y) \geq K - \epsilon$. Then there exists a function $z \in W(I)$ of compact support such that*

$$Q(z) \geq Q(y) - \epsilon \geq K - 2\epsilon \,. \tag{2.1}$$

Proof. This lemma can be easily proved by using the classical technique of approximating the given function by its convolution with an approximate identity. We give below a more elementary proof.

Let ρ be a C^∞ function with support in $[0, 2)$ such that $0 \leq \rho(t) \leq 1$ and $\rho(t) = 1$ if $t \in [0, 1]$. Set $z = \rho(t/t_0)y$ for some fixed $t_0 > 0$ which we will determine. It follows from the definition of z and the fact $y \in L^1(I)$, that for any given $\epsilon_1 > 0$, we can choose t_0 sufficiently large so that $||y - z||_{1,[t_0,\infty)} \leq \epsilon_1$. Since

$$z'(t) = \rho(t/t_0)y'(t) + \rho'(t/t_0)y(t)/t_0 \,, \tag{2.2}$$

we can also choose t_0 sufficiently large that

$$||\rho(t/t_0)z'||_{2,[t_0,\infty)} \leq \epsilon_2 \,,$$

for any given ϵ_2. On the other hand it is possible to prove (cf. [2, Lemma 2.1]) that there is a constant C independent of y and δ such that if $t \in [t_0, t_0 + \delta]$, then

$$|y(t)| \leq C_1 \left(\delta^{-1} \int_{t_0}^{t_0+\delta} |y| + \int_{t_0}^{t_0+\delta} |y'| \right) . \tag{2.3}$$

Hence, for $t \in [t_0, \infty)$,

$$|y(t)| \leq C_1 \left(\delta^{-1} \int_I |y| + \delta^{1/2} \left(\int_I y'^{\,2} \right)^{1/2} \right) . \tag{2.4}$$

Since δ is arbitrary, minimization of the right side of this inequality as a function of δ gives the multiplicative inequality

$$|y(t)| \leq C_2 \left(\int_I |y| \right)^{1/3} \left(\int_I y'^{\,2} \right)^{1/3} . \tag{2.5}$$

It follows from substituting (2.5) into (2.2) that given ϵ_3 we can choose t_0 so large that $||z' - y'||_{2,[t_0,\infty)} \leq \epsilon_3$. Finally, consider

$$z''(t) = \rho(t/t_0)y'' + 2\rho'(t/t_0)y'(t)/t_0 + \rho''(t/t_0)y(t)/t_0^2\,. \qquad (2)$$

In the same way as in the derivation of (2.3) and (2.4), one can show first that

$$|y'(t)| \leq C_3 \left(\delta^{-2} \int_I |y| + \delta ||y''||_{\infty,I}\right)$$

and then that

$$|y'(t)| \leq C_4 \left(\int_I |y|\right)^{1/3} ||y''||_{\infty,I}^{2/3}\,. \qquad (2.7)$$

Substituting the bounds (2.5) and (2.7) into (2.6) and taking t_0 sufficiently large yield that $||z''||_{\infty,I} \leq ||y''||_{\infty,I} + \epsilon_3$. If y is chosen so that $Q(y) > K - \epsilon$, a straightforward calculation allows us to find ϵ_1, ϵ_2, ϵ_3 so that

$$\frac{\int_I y'^{\,2} - \epsilon_1^2}{\left(\int_I |y| + \epsilon_2\right)\left(||y''||_{\;\infty,I} + \epsilon_3\right)} \geq Q(y) - \epsilon\,.$$

The conclusion (2.1) follows because

$$Q(z) \geq \frac{\int_I y'^{\,2} - \epsilon_1^2}{\left(\int_I |y| + \epsilon_2\right)\left(||y''||_{\;infty,I} + \epsilon_3\right)}\,.$$

■

Lemma 2. *Define*

$$Z_1 = \{z \in W([0,1]) : z(0) \geq 0,\ z(1) = z'(1) = 0\}$$

and

$$Z_2 = \{z \in W([0,1]) : z(1) \geq 0,\ z(0) = z'(0) = 0\}\,.$$

Then $K = K(Z_1) = K(Z_2)$.

Proof. Let $y \in W([0,\infty)$ and z be as asserted in Lemma 1 such that

$$Q(z) \geq Q(y) - \epsilon \geq K - 2\epsilon\,. \qquad (2.8)$$

By replacing z by $-z$ if necessary, we may assume without loss of generality that $z(0) \geq 0$. After scaling we can consider z to be in Z_1. Hence

$$K(Z_1) \geq Q(z) \geq K - 2\epsilon\,.$$

Letting $\epsilon \to 0$, we get $K(Z_1) \geq K$. Conversely, if $z \in Z_1$, we extend it to be zero in $[1, \infty)$. The extended function, which we call $\hat{z}$, belongs to $W([0,\infty))$. Now $K \geq Q(\hat{z}) = Q(z)$. Taking the supremum over Z_1 gives $K \geq K(Z_1)$ so that $K = K(Z_1)$.

Functions in Z_2 are merely reflections of those in Z_1, and they have the same quotient values; hence $K(Z_2) = K(Z_1)$. ■

3. Further Reductions

Our basic strategy in determining K will be to reduce the problem of finding $K(Z_2)$ to a sequence of simpler finite-interval equivalent problems. The reductions are effected either by throwing away a large subclass of functions in the previous problem or by finding another family of functions that satisfy more conditions without affecting the value of K. These procedures can be justified by the obvious facts stated in the following lemma.

Lemma 3. *Given Y and $Z \subset W(I)$, if for every function $z \in Z$ we can find a $y \in Y$ such that $Q(y) \geq Q(z)$, and for every $y \in Y$ we can find a $z \in Z$ such that $Q(y) \leq Q(z)$, then*

$$K(Y) = K(Z)\,.$$

In particular, given $Y \subset Z$, if for every $z \in Z$ we can find a $y \in Y$ such that $Q(y) \geq Q(z)$, then $K(Y) = K(Z)$. Alternatively, if we can show that for every $z \in Z \setminus Y$, $Q(z) < K(Z)$, then $K(Y) = K(Z)$.

One technique to produce from a given function z another function y with a greater quotient value is to restrict z to a suitable subinterval, using the next lemma.

Lemma 4. *Let z be a function defined on I which is the sum of two disjoint subintervals I_1 and I_2, and let z_1 and z_2 be the restrictions of z on I_1 and I_2, respectively. Then*

$$Q(z) \leq \max\{Q(z_1), Q(z_2)\}\,.$$

In general, if $I = \cup_{n=1}^{\infty} I_n$, where I_n are mutually disjoint, and for each restriction z_n of z onto I_n, $Q(z_n) \leq q$, then $Q(z) \leq q$.

Proof. Let

$$q = \max\{Q(z_1), Q(z_2)\}\,.$$

Then

$$\int_{I_1} z_1'^{\,2} \le q \left(\int_{I_1} |z_1| \right) \max_{I_1} |z_1''| \le q \left(\int_{I_1} |z_1| \right) \max_{I} |z''| \,, \tag{3.1}$$

and

$$\int_{I_2} z_2'^{\,2} \le q \left(\int_{I_2} |z_2| \right) \max_{I_2} |z_2''| \le q \left(\int_{I_2} |z_2| \right) \max_{I} |z''| \,. \tag{3.2}$$

Adding (3.1) and (3.2) gives

$$\int_I z'^{\,2} \le q \left(\int_I |z| \right) \max_I |z''| \,,$$

which implies that $Q(z) \le q$. A trivial modification of this argument handles the case $I = \cup_{n=1}^{\infty} I_n$. ■

The next lemma gives a lower bound on K. Any function that gives a quotient value less than this lower bound can therefore be thrown away, a fact to be used in our first reduction (Lemma 6).

Lemma 5. $K > 2$.

Proof. It is easy to see that $K \ge 2$ since $z = x^2$ is in Z_2 and $Q(z) = 2$. To get $Q(z) > 2$, we substitute the more complicated test function (which, after a horizontal scaling, can be considered to be in Z_2)

$$z = \begin{cases} -x^2, & \text{in } [0,1] \\ x^2 - 4x + 2, & \text{in } [1,6] \,. \end{cases}$$

Direct computation gives

$$K > Q(z) = \frac{66}{23 + 4\sqrt{2}} > 2.303 \,.$$

■

Lemma 6. *Define*

$$Z_3 = \{ z \in Z_2 : z(1) > 0,\ z'(1) > 0 \} \,.$$

Then $K(Z_3) = K$.

Proof. *Case 1:* Suppose $z \in Z_2$ and $z(1) = 0$. Then

$$\int_0^1 z'^{\,2} = -\int_0^1 z z'' = \left| \int_0^1 z z'' \right| \le \left(\int_0^1 |z| \right) \max_{[0,1]} |z''| \,.$$

This gives $Q(z) \leq 1$. Since $K > 2$, by the previous lemma, we need not consider z of this type.
Case 2: Suppose $z(1) > 0$ but $z'(1) \leq 0$. Then

$$\int_0^1 z'^{\,2} = z(1)z'(1) - \int_0^1 zz''\,.$$

By the definition of Z_3, the term $z(1)z'(1) \leq 0$, so

$$-\int_0^1 zz'' = \int_0^1 z'^{\,2} + |z(1)z'(1)| \geq 0\,.$$

But

$$-\int_0^1 zz'' = \left|\int_0^1 zz''\right| \leq \left(\int_0^1 |z|\right) \max_{[0,1]} |z''|\,.$$

As in Case 1, this implies that $Q(z) \leq 1$, so that we can throw away z with these boundary conditions also. ■

In several other occasions in the sequel, we shall employ the same arguments used above in establishing Case 2.

Lemma 7. *Define*

$$Z_4 = \{z \in Z_3 : z \geq 0\}$$

and

$$Z_5 = \{z \in Z_4 : z'(x) > 0, \text{ for all } x > 0\}\,.$$

Then $K(Z_4) = K(Z_5) = 2$.

As a consequence, upon defining

$$Z_6 = Z_3 \setminus Z_4\,,$$

we have

$$K = K(Z_6). \tag{3.3}$$

Proof. The function $z = x^2$ belongs to Z_5. Hence $K(Z_5) \geq Q(z) = 2$. That $K(Z_5) \leq K(Z_4)$ follows from the fact that $Z_5 \subset Z_4$. Let $z \in Z_4 \setminus Z_5$. Then $z'(c) = 0$ for some $c > 0$. Since $z'(1) > 0$ (definition of Z_3), z is increasing near $x = 1$. Let τ be the last critical point of z, i.e.. $z'(\tau) = 0$, but $z' > 0$ in $(\tau, 1)$. By Lemma 6, Case 2, $Q(z_{[0,\tau]}) \leq 1$. If $Q(z) \leq 2$, we can throw it away. Suppose $Q(z) > 2$. Set $u = z_{[\tau,1]}$. By Lemma 4,

$$2 < Q(z) \leq \max\left\{Q(z_{[0,\tau]}), Q(u)\right\}\,.$$

Since we have just shown that $Q(z_{[0,\tau]}) \leq 1$,

$$Q(u) \geq Q(z)\,.$$

If we translate the graph of u, namely, setting $v(x) = u(x) - u(\tau)$, we obtain that $v \in Z_5$. Furthermore, since $||v||_{1,[\tau,1]} \leq ||u||_{1,[\tau,1]}$,

$$Q(v) > Q(u) \geq Q(z)\,.$$

Consequently, $K(Z_5) = K(Z_4)$. It remains to show that $Q(z) \leq 2$ for all $z \in Z_5$. Without loss of generality, we take $\max|z''| = 1$. By choosing z as the independent variable and noting that functions in Z_5 are monotonically increasing, and $z' = 0$ at $z = 0$, we obtain

$$(z')^2 = \int_0^z \frac{d(z'^{\,2})}{dz}\,dz = \int_0^z 2z''\,dz \leq \int_0^z 2\,dz = 2z\,.$$

Therefore

$$z' \leq \sqrt{2z} \quad \text{and} \quad \frac{z}{z'} \geq \sqrt{\frac{z}{2}}\,.$$

It follows that

$$\int_0^1 z'^{\,2}\,dx = \int_0^{z(0)} z'\,dz \leq \int_0^{z(0)} \sqrt{2z}\,dz \tag{3.4}$$

and

$$\int_0^1 |z|\,dx = \int_0^{z(0)} \frac{z}{z'} \geq \int_0^{z(0)} \sqrt{\frac{z}{2}}\,dz\,. \tag{3.5}$$

That $Q(z) \leq 2$ follows if we divide (3.4) by (3.5).

To recapitulate, identity (3.3) means that we need only consider functions z such that $z(0) = z'(0)$, $z(1) > 0$, $z'(1) > 0$, and z changes sign.

The change of variable argument used in the last part of the above proof can be modified to give a comparison result that we shall need in Lemma 11 below.

Lemma 8. *Suppose we start out with a function u, defined on I, that is monotone on a subinterval $I_1 = [a,b]$ and we modify u to obtain a new function v, by changing only the portion of u on I_1 to another function. The replacement function will span a shorter subinterval $I_2 = [a,c]$ ($c \leq b$), and so we translate the portion of u to the right of $[a,b]$ to close up the gap. We require that the new function v remain twice differentiable and that $\max|u''| = \max|v''|$. Then if for all $x \in [a,b]$ and $t \in [a,c]$ such that $u(x) = v(t)$, we have*

$$u'(x) \leq v'(t)\,,$$

it follows that

$$Q(u) \le Q(v)\,.$$

Proof. We need only establish this for the case that $u, v \ge 0$ in I_1 and I_2. In the general case, we can apply the result to the positive and negative parts of the functions separately and then add the inequalities. If u and v are the new independent variables, then

$$\int_{I_1} u\,dx = \int_{u(a)}^{u(b)} \frac{u}{u'}\,du \ge \int_{v(c)}^{v(d)} \frac{v}{v'}\,dv = \int_{I_2} v\,dx$$
$$\int_{I_1} u'^2\,dx = \int_{u(a)}^{u(b)} u'\,du \le \int_{v(c)}^{v(d)} v'\,dv = \int_{I_2} v'^2\,dx\,.$$

It follows that $Q(u) \le Q(v)$. ■

By definition, functions in Z_6 must change sign, and they can do so more than once. It would be nice if we could limit our search for the best K to those functions that change sign only once. Unfortunately such functions, under the constraint that they vanish with their first derivative at the endpoint 0, do not produce a quotient close to K. In the next lemma we prove that we can still recover K from functions that change sign only once, provided that we weaken the boundary condition at 0 to allow additional functions. At this point, we continue to admit functions (those in Z_7 defined below) that vanish with their first derivative at 0. Eventually, we shall see that these functions too can be thrown away.

Lemma 9. *Let*

$$Z_7 = \{z \in Z_6 : z \text{ changes sign exactly once at } \alpha \text{ and } z'(\alpha) \ne 0\}\,.$$

Furthermore, let $Z_8 \subset W([0,1])$ consist of functions satisfying the conditions:

(i) $z(0) < 0$, $z'(\alpha) \ne 0$.
(ii) z *changes sign exactly once at* α *and* $z'(\alpha) \ne 0$.
(iii $z(1) > 0$, $z'(1) > 0$.
(iv) $-\dfrac{z'^2(0)}{z(0)} = \dfrac{z'^2(1)}{z(1)}$.
(v) $|z(0)| < z(1)$.

Then $K = K(Z_7 \cup Z_8)$.

Proof. We first show that

$$K \le K(Z_7 \cup Z_8). \tag{3.6}$$

Let $z \in Z_6$ such that $Q(z) \geq K(Z_6) - \epsilon$. If z changes sign only once, then $z \in Z_7$ and we have $K(Z_7) \geq Q(z) \geq K - \epsilon$, which implies (3.6). Suppose that z changes sign more than once. Let α be the last zero of z.
Case 1: $z'(\alpha) = 0$. Then $z_{[\alpha,1]} \in Z_7$. It is easy to see that $Q(z_{[0,\alpha]}) \leq 1$. By Lemma 4, $Q(_{[\alpha,1]}) \geq Q(z) \geq K - \epsilon$, and (3.6) follows.
Case 2: $z'(\alpha) > 0$. Let τ be the last zero of z before α. We can dispose of the case when $z'(\tau) = 0$, as in Case 1 above, to obtain $z_{[\tau,1]} \in Z_7$.
Case 3: Thus we may assume that $z'(\tau) < 0$. Let $\sigma > \tau$ be the first zero of z' after τ. Then $z'(x) < 0$ in $[\tau, \sigma)$. Consider the function

$$r(x) = \left| \frac{z'^2(x)}{z(x)} \right| ,$$

which satisfies $\lim_{x \to \sigma} r(x) = 0$ and $\lim_{x \to \tau} r(x) = \infty$. By the intermediate value theorem, there is a $\gamma \in (\tau, \sigma)$ such that $r(\gamma) = r(1)$. (Note that this argument fails for functions in Cases 1 and 2, since $\lim_{x \to \tau} r(x)$ may be bounded.) The function $u = z_{[\gamma,1]}$ thus satisfies conditions (i)–(iv) in the definition of Z_8. We shall see later that condition (v) is also satisfied, so we can assume that $u \in Z_8$. If $Q(u) \geq K - \epsilon$, we have $K(Z_8) \geq Q(u) \geq K - \epsilon$ and (3.6) follows. If not, then by Lemma 4, $Q(z_{[0,\tau]} \geq K - \epsilon$. we can repeat the argument over the remaining oscillations of z in $[0, \tau]$. One of these must yield a section of z with $Q \geq K - \epsilon$, or else the infinite version of Lemma 4 will give $Q(z) < K - \epsilon$, a contradiction.

Let us show that condition (v) is satisfied by z. Suppose that $|z(0)| = z(1)$. Then

$$\int_0^1 z'^2 = \int_0^1 zz'' \Longrightarrow Q(z) \leq 1 ,$$

contradicting $Q(z) \geq K - \epsilon$. We extend z by piecing together to the right of z its own images scaled by compressing vertically with the ratio $|z(0)/z(1)|$ and vertically with the ratio $\sqrt{|z(0)/z(1)|}$. This gives a function u of compact support and $Q(z) = Q(u)$. The arguments used in Lemma 6, however, prove that $Q(u) \leq 1$.

The proof of the lemma will be complete if we can show that $K \geq K(Z_7 \cup Z_8)$. If $z \in Z_7$, then it is also in Z_2, and so $Q(z) \leq K(Z_2) = K$. On the other hand, if $z \in Z_8$, then we use the extension method given in the previous paragraph to piece together a chain of scaled images of z, but this time to the left, to obtain a function u of compact support, that is, in Z_2. Again $Q(z) = Q(u) \leq K(Z_2) = K$. ∎

In the following, we use the term *local maximum (minimum)* in a narrow sense, referring to one in the interior of the interval and *not* at the end points.

Lemma 10. *Let*

$$Z_9 = \{z \in Z_7 \cup Z_8 : z \text{ has no local maximum}\}.$$

Then $K(Z_9) = K$.

Proof. *Case 1:* z has a local maximum in $(\alpha, 1)$. In this case, there must also be a local minimum in $(\alpha, 1)$. Let μ be the last of these. First we extend z to $u \in Z_2$. Then $Q(u) = Q(z)$. But $Q(u_{[b,\mu]}) \le 1$ (Case 2 of Lemma 6) and $Q(u_{[\mu,1]}) \le K(Z_4) = 2$. Therefore, by Lemma 4, $Q(z) = Q(u) \le 2$ and z can be thrown away.

Case 2: z has a local maximum in $(0, \alpha)$. Let σ be a local maximum and $\mu < \sigma$ be the local minimum just to its left. Note that $Q(u_{[\mu,\alpha]}) \le 1$. By an argument similar to that used to prove Lemma 4, we see that

$$\frac{\int_0^\mu z'^2 + \int_\sigma^1 z'^2}{\left(\int_0^\mu |z| + \int_\sigma^1 |z|\right)} \ge Q(z).$$

Also note that $Q(z_{[0,\mu]}) \le 1$. Now scale $z_{[0,\mu]}$ to fit smoothly to the left of $z_{[\sigma,\alpha]}$ to form a new function $u \in Z_9$. The scaling is a compression since $|z(\mu)| > |z(\sigma)|$. Thus

$$Q(u) = \frac{\lambda \int_0^\mu z'^2 + \int_\sigma^1 z'^2}{\left(\lambda \int_0^\mu |z| + \int_\sigma^1 |z|\right) \max |z''|}$$

for some $\lambda < 1$. It is easy to see that the fraction in this expression is larger than that in (2.8). So $Q(u) \ge Q(z)$. ∎

By the above lemma, Z_9 can have only one local minimum, which we denote by β.

Lemma 11. *Let*

$$Z_{10} = \{z \in Z_9 : z''_{[\beta,1]} = C = \max_{[0,1]} z''\}.$$

Then $K(Z_{10}) = K$.

Proof. Let us show that any functions in $Z_9 \setminus Z_{10}$ can be skipped without affecting the best constant. By the definition of Z_9, z belongs to Z_7 or Z_8.

Case 1: Suppose $z \in Z_7$. If $z \notin Z_{10}$, then $z'' \ne C$ in $[\beta, 1]$. We can replace $z_{[\beta,1]}$ by a function u such that $u'' = C$, as in Lemma 8. Note that u will be defined in a shorter interval $[0, \gamma]$ with $\beta < \gamma < 1$. The inequality $Q(u) \ge Q(z)$ then follows from Lemma 8.

Case 2: If $z \in Z_8$, we first extend z to the left to a function $u \in Z_2$ as in the last part of the proof of Lemma 8. Suppose that the domain of u is now $[\delta, 1]$. Then $Q(u) = Q(z)$. We may assume without loss of generality that z has been chosen so that

$$Q(z) \geq Q(u_{(\delta,b]}) \quad \text{for all} \quad b \in (\delta, \beta). \tag{3.7}$$

Otherwise, we can replace z by a better choice constructed as follows. Let b be such that $Q(u_{(\delta,b)}) = \max_{\bar{b}} Q(u_{(\delta,\bar{b})}) > Q(z)$, and cut out a piece $\bar{z} = u_{[c,b]} \in Z_8$, as in Lemma 9, so that $Q(\bar{z}) > Q(z)$.

If $u'' \neq C$ in $[\beta, 1]$, we can modify u in that part to form a new function such that $v'' = C$ on $[\beta, \gamma]$, as in Case 1. Then $v = u$ in (δ, β), and

$$Q(v) \geq Q(u) = Q(z)\,. \tag{3.8}$$

Now we can cut out a section $v_{[b,\gamma]} \in Z_8$. Since $v_{(\delta,b]} = u_{(\delta,b]}$, by (3.7), (3.8), and Lemma 4, we have

$$Q(v_{[b,\gamma]}) > Q(z)\,.$$

■

Lemma 12. *Let*

$$\begin{aligned} Z_{11} = \{&z \in Z_{10} : \text{there exists } \gamma \in [0, \beta] \text{ such that} \\ &z'' = -C \ \text{ in } \ (0, \gamma) \ \text{ and } z'' = C \text{ in } (\gamma, 1)\} \end{aligned}$$

and

$$Z_{12} = \left\{z \in Z_{11} : Q(Z_{[0,\epsilon]}) < Q(z) \text{ for all } \epsilon \in (0, \gamma)\right\}\,.$$

Then $K = K(Z_{11}) = K(Z_{12})$.

Proof. Let z be in Z_{10}. We modify z in $[0, \beta]$ as follows, where β is the local minimum of z. We denote the new function by u. Starting at β, we let the graph bend upwards towards the left, with $u'' = C$. At a suitable point γ, to be determined below, we let u bend downwards, now with $u'' = -C$, until u reaches a height equal to $z(0)$, at some point $x = \zeta$.

The condition on choosing γ is such that $u'(\zeta) = z'(0)$. That γ exists follows from a continuity argument. If we let u change curvature at a point when the height of u is halfway between $z(\beta)$ and $z(0)$, then we end up with $u'(\zeta) = 0 \geq z'(0)$, while if we let u continue to bend upwards without changing its curvature, then $u'(\zeta) < z'(0)$. Therefore, somewhere between these two extreme cases, there must be a suitable choice of γ.

The new function $u \in Z_{11}$ and $u_{[\zeta,\beta]}$ compares favorably with $z_{[0,\beta]}$ in the sense of Lemma 8; hence $Q(u) > Q(z)$ and $K(Z_{11}) = K$ follows.

In view of Lemma 7, functions in $Z_{11} \cap Z_7$ can either be thrown away or are already in Z_{12}. So now suppose that $z \in Z_{11} \cap Z_8 \setminus Z_{12}$. Then there exists an $\epsilon \in (0, \gamma)$ such that $Q(z_{[0,\epsilon]}) \geq Q(z)$. Let η be the maximum of all such ϵ. It can happen that $\eta = \gamma$. In the contrary case, we observe that for every $\epsilon \in (\eta, \gamma)$,

$$Q(z_{[\eta,\epsilon]}) < Q(z). \tag{3.9}$$

Otherwise, if (3.9) is not true, then by Lemma 4,

$$Q(z_{[0,\epsilon]}) \geq \min\left\{Q(z_{[0,\eta]}), Q(z_{[\eta,\epsilon]})\right\} \geq Q(z)\,,$$

contradicting the definition of η.

Our next construction is to cut the part $z_{[0,\eta]}$ out from z, translate it to the right of z, stretch it appropriately, and then reattach it to z smoothly. It is easy to see now that the new function u will satisfy $u'' = C$ to the right of β and that $u \in Z_{12}$. The inequality $Q(u) \geq Q(z)$ follows from the facts that the part $z_{[0,\eta]}$ carries a larger (or same) quotient than the rest of z and that the weight carried by this quotient is magnified when $z_{[0,\eta]}$ has to be stretched before being reattached to the right hand side of z. ∎

Lemma 13. *Let*

$$Z_{13} = \{z \in Z_{12} : \gamma = 0\}\ ,$$

which consists of exactly the quadratic polynomials in Z_{10} such that $z'' = C$ on $[0,1]$. Then

$$K(Z_{13}) = K.$$

Proof. Let $z \in Z_{12}$. If $\gamma \neq 0$, we modify z to the left of γ to a new function u by bending the graph upward with $u'' = C$, until it reaches a point $x = \zeta$ at which $|u'^2(\zeta)/u(\zeta)| = r(1) = |z'^2(1)/z(1)|$. It is easy to see that this happens with $\zeta \in (0, \gamma)$ and that $u(\zeta) < z(0)$. Let $\epsilon \in (0, \zeta)$ be the point at which $z(\epsilon) = u(\zeta)$. By the definition of Z_{12}, $Q(z_{[0,\epsilon]}) < Q(z)$, so that $Q(z_{[\epsilon,1]}) > Q(z)$. Lemma 8 can now be invoked to show that $Q(u) > Q(z_{[\epsilon,1]})$, and the lemma is proved. ∎

Lemma 14. *An extremal exists in Z_{13} and is unique (modulo a constant multiple).*

Proof. Existence follows from the fact that the maximization problem is now reduced to one on a class of functions that depends only on one parameter, namely, $r(0)$ (and not on $C = \max|z''|$). We now show that if w is an extremal, then

$$-\frac{w'^2(0)}{w(0)C} = \frac{w'^2(1)}{w(1)C} = Q(w)\,.$$

To this end, we choose a particular quadratic function $y = x^2 - x$. It is easy to see that the extremal for K, after scaling and translation, must be of the form y_I, in other words, y restricted to some suitable interval $I = [a, b]$, such that $r(a) = r(b)$. Next extend $y_{[a,b]}$ to its left to a function u in Z_2. If we consider u as a function of its endpoint b, we see that when b gives the extremal, $dQ/db = 0$. But

$$\frac{dQ}{db} = \frac{u(b)}{C\int_{-\infty}^{b}|u|}(r(b) - CQ(b)).$$

Hence, $r(b) = CQ(b)$. This argument also implies the uniqueness of the extremal. The parameter $r(b)$ is a monotonically decreasing function of b so there can only be one choice of b that makes $r(b) = CK$. ■

Proof of Theorem 1. We apply a method suggested by Lemma 14 to compute K. Let $y = x^2 - x$ as in the proof of Lemma 14. We use $r \geq 2$ as our parameter and determine a and b so that $r(a) = r(b) = r$. The computations shown below were done using the symbolic manipulation software MAPLE V. The program will be given in the Appendix. We get

$$a = \frac{1 - \sqrt{1 - \frac{2}{r+2}}}{2} \tag{3.9}$$

and

$$b = \frac{1 + \sqrt{1 + \frac{2}{r-2}}}{2}. \tag{3.10}$$

Straightforward computation gives

$$q(r) = Q(y_I = \frac{[4b^3 - 6b^2 + 3b] - [4a^3 - 2a^2 + a]}{2 + 2a^3 - 3a^2 + 2b^3 - 3b^2}.$$

After substituting the expressions for a and b above and simplifying, we obtain

$$q(r) = \frac{r^{3/2}\left(\sqrt{r-2}r - 2\sqrt{r-2} + 2\sqrt{2+r} + \sqrt{2+r}r\right)}{D} \tag{3.11}$$

where the denominator D is given by

$$\begin{aligned} D = {} & \sqrt{2+r}\sqrt{r-2} + 2\sqrt{2+r}\sqrt{r-2}r^2 - 6\sqrt{r-2}\sqrt{r} + \sqrt{r-2}r^{5/2} \\ & + r^{3/2}\sqrt{r-2} + 6\sqrt{2+r}\sqrt{r} - \sqrt{2+r}r^{5/2} + r^{3/2}\sqrt{2+r}. \end{aligned} \tag{3.12}$$

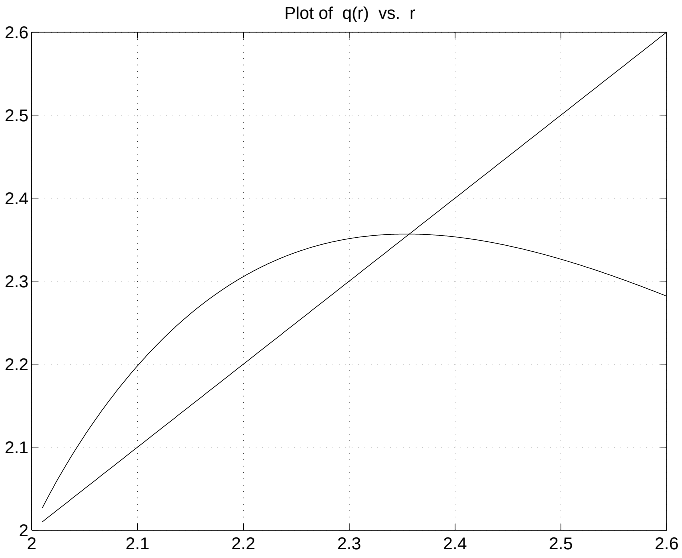

Figure 1. Plot generated with MATLAB

A plot of q versus r (constructed using MATLAB) is shown in Fig. 1. By Lemma 14 the maximum of the curve is also the intersection of the curve with the line $q = r$. The numerical solution of this equation is not difficult. The exact algebraic solution of the equation, however, seemed at first sight out of reach because of the formidable-looking expressions (3.11) and (3.12). Thanks to MAPLE, the simple command "`solve(r=q,r)`" gave us the answer as given by this lemma.

Once we know the answer, it is possible to work backwards to figure out how one could have obtained it by pen and paper. Denote by N the numerator of $q(r)$ in (3.11). Then the equation $q = r$, which is to be solved, is equivalent to $D - N/r = 0$, which can be factored as

$$(r+2)(r-2)\left(\sqrt{r(r-2)} - \sqrt{r(r+2)} + 2\sqrt{(r+2)(r-2)}\right) = 0\,.$$

Solving the equation obtained by the third factor is then straightforward. ∎

4. Further Inequalities

A more general inequality than that given in the title of the paper is

$$\int_0^\infty |y'|^{2p} \le K(p)\left(\int_0^\infty |y|^p\right) \max_{(0,\infty)} |y''|^p\,. \tag{4.1}$$

Notice that (4.1) is equivalent to Gabushin's inequality, with $q = 2p$, $r = \infty$, $m = 2$, and $k = 1$, and that $K(p) = K^p_{2p,p,\infty,2,1}$. This inequality can be studied by exactly the same method described in the early part of this paper. We merely state the final result.

Theorem 2. *The best constant $K(p)$ in inequality (4.1) is the maximum of the quotient*

$$q(r) = \frac{\int_a^b |y'|^{2p}}{2^p \left(\int_a^b |y|^p\right)},$$

where $y = x^2 - x$, $r > 2$, and a and b are given by (3.9) and (3.10).

Alternatively, $K(p)$ is the unique positive solution of the equation

$$r^p = q(r), \qquad r > 2. \tag{4.2}$$

The constant $K(2) = 36/5$ was determined exactly by using MAPLE, but the next one, $K(3)$, is the solution of an eighth-degree polynomial and is not a rational number. One can easily find the approximate values of $K(p)$ using an iterative fixed-point scheme based on (4.2). We summarize our results below. The numerical values were computed using the arbitrary-precision arithmetic in MAPLE to 50 significant places and then rounded off to 30 places after the decimal point.

Theorem 3.

$$\begin{aligned}
K(2) &= \frac{36}{5}\\
K(3) &\approx 25.01845\ 17898\ 28377\ 11460\ 58592\ 89421\\
K(4) &\approx 91.05280\ 88548\ 54808\ 83114\ 72060\ 68652\\
K(5) &\approx 339.42928\ 70993\ 18821\ 45933\ 06691\ 47941\,.
\end{aligned}$$

The algebraic equation obtained by MAPLE for $K(3)$ is

$$K(3) = r^3,$$

where r satisfies

$$\begin{aligned}
&\sqrt{r}(2\,r^2 + 10\,r + 15)(r-2)^{5/2} + 4\,(r-2)^{5/2}(2+r)^{5/2}\\
&\qquad - \sqrt{r}(2\,r^2 - 10\,r + 15)(2+r)^{5/2} = 0\,.
\end{aligned}$$

Appendix. The MAPLE Program

The following is the MAPLE program we used to set up the various variables.

```
 if not assigned (p) then p := 1; fi:
  q := 2* p;
  a := (1-(1-2/(2+r))^(1/2))/2:
  b := (1+(1+2/(r-2))^(1/2))/2:
 a1 := 1:
 a2 := p:
  Y := int( (1-2*x)^q, x=a..1/2)+int( (2*x-1)^q, x=1/2..b):
 Yp := int( (x-x^2)^p,x=a..1 ) + int((x^2-x)^p,x=1..b ):
  q := ndifferential^a1/2^a2:
  q := simplify(q):
 eq := denom(q)-numer(q)result^p:
 nq := proc(R) local RR:
    RR := convert(R,rational,exact);
       := evalf(subs(r=RR,Q),50);
end:
```

The first line sets up a default choice, namely 1, for p. In the second line, q is the exponent for y' in the inequality (4.1). The values a and b are then computed using (3.9) and (3.10). The program was originally written for an inequality even more general than (4.1), in which the exponent for z' is any given q not necessarily, $2p$. In such a case, the integrals on the righthand side of (4.1) will have to be raised to some suitable powers $a1$ and $a2$, respectively. The integrals Y and Yp of q are then computed and appropriate powers of them are used to give q. The next command calls a utility "`simplify`" in MAPLE to simplify the expression obtained for q and then store the result back to the variable q. This step helps to make it easier for MAPLE to try to solve the equation later. The equation "`eq`" obtained in the next line is equivalent to the equation $r^p = q(r)$. The last four lines defines a procedure (a function subroutine) "`nq`" to give the numerical value of $q(r)$ up to 50 significant decimal places, when r is given a numerical value `RR`.

Within a MAPLE session, one invoke the above program by issuing the commands

```
read FILE;
```

where `FILE` is the name of the file that contains the program. Depending on whether p has been previously assigned a value or not, the ensuing computing will be pertinent to K_p or the default K. To ask MAPLE to solve the equation $r^p = q(r)$ exactly, one issues the command

```
solve(eq);
```

and if MAPLE is able to find the solutions, they will be displayed. For larger values of p, MAPLE is not able to solve the equation exactly (in a reasonable time). Instead, one can use the iterative scheme

`K :=` *INITIAL GUESS* `for n from 1 to 50 do K := nq(K); od`

to obtain K as a fixed point.

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