MATHEMATICAL ANALYSIS FOR THE RATIONAL LARGE EDDY SIMULATION MODEL

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In this paper we consider the Rational Large Eddy Simulation model recently introduced by Galdi and Layton. We briefly present this model, which (in principle) is similar to others commonly used, and we prove the existence and uniqueness of a class of strong solutions. Contrary to the gradient model, the main feature of this model is that it allows a better control of the kinetic energy. Consequently, to prove existence of strong solutions, we do not need subgrid-scale regularization operators, as proposed by Smagorinsky. We also introduce some breakdown criteria that are related to the Euler and Navier-Stokes equations.

Keywords: Large eddy simulation, strong solutions, existence, uniqueness, blow up.

1. Introduction

We consider the well-known Navier-Stokes equations for a fluid filling a smooth, bounded open set $\Omega \subset \mathbf{R}^3$,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{1}{Re}\Delta u + (u \cdot \nabla)\, u + \nabla p = f & \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $Re > 0$ is the Reynolds number. The phenomena of instability of fluid motion at high Reynolds number lead to the study of *turbulent flows*. The main idea underlying the study of turbulent motion can be traced back to Leonardo da Vinci[34] (at the beginning of the 16^{th} century), who was the first to observe that the motion of vortices trailing a blunt body can be understood as a mean motion plus some *turbulent* fluctuations (this term being introduced by Lord Kelvin[19] though; see also Chapter 11 in Lamb[23]). The first mathematical model using this idea was introduced by Reynolds[29]. In fact, Reynolds proposed to consider the velocity as decomposed in

$$u = \overline{u} + u', \tag{1.2}$$

where the *mean velocity* $\overline{u}$ is the time average of the real velocity, while u' represents the turbulent fluctuations. It is clear that the averaging operator commutes with linear differential operators, but

$$\overline{u \otimes u} \neq \overline{u} \otimes \overline{u}.$$

Substituting the decomposition (1.2) into the Navier-Stokes equations (1.1), we do not have a closed set of equation, and some extra assumptions are needed. In particular, we need to model the *Reynolds stress tensor*

$$\tau = -\overline{u \otimes u}.$$

Using the assumption that a turbulent flow is "dissipative in mean", Boussinesq[7] proposed the tensor

$$\tau = \nu_t(\nabla\overline{u} + \nabla\overline{u}^{\mathrm{T}}),$$

where ν_t is function of the turbulent flow. Later Smagorinsky[31] (see also the work of Ladyžhenskaya[20] in the context of regularity results) proposed the following constitutive relation for the turbulent stress tensor:

$$\tau = \left(c_1 + c_2|\nabla\overline{u} + \nabla\overline{u}^{\mathrm{T}}|^{2\mu}\right)\left(\nabla\overline{u} + \nabla\overline{u}^{\mathrm{T}}\right) \qquad 0 < c_1, c_2 \in \mathbf{R}. \tag{1.3}$$

An approach different from that of Reynolds is so-called Large Eddy Simulation (LES), which uses space averaging instead of time averaging. The main objective of LES is to derive equations for a "mean velocity" that does not have high frequencies in its spectrum; equivalently, the LES equations resolve only scales bigger than a given positive averaging radius. The methods of LES were introduced by Deardorff[12], and they are essentially based on averaging operators acting as *low-pass filters* (see Section 2). In this paper we analyze the "Rational" LES (RLES) model, recently introduced by Galdi and Layton[16]:

$$\begin{cases} \dfrac{\partial w}{\partial t} + \nabla q + (w\cdot\nabla)\,w + \nabla\cdot\left(\mathrm{I} - \dfrac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\dfrac{\delta^2}{2\gamma}\nabla w \nabla w\right] - \dfrac{1}{Re}\Delta w = \overline{f}, \\ \nabla\cdot w = 0, \\ w(x,0) = w_0(x). \end{cases} \tag{1.4}$$

This model is based (see Section 2.1) on a rational approximation of the Fourier transform of the Gaussian filter. The positive *averaging radius* is denoted by δ, and γ is a positive constant (generally $\gamma = 6$).

The variables (w, q) are approximations of the averaged flow variables $(\overline{u}, \overline{p})$. In fact, the system (1.4) is obtained by disregarding, from the exact filtered equations, terms that are formally $O(\delta^4)$; consequently, the above system models the motion in which the solution does not contain scales of size smaller than $O(\delta)$. Here, to avoid the delicate problems related with the boundary conditions, we consider the case of space-periodic boundary conditions. Other boundary conditions of Navier type were introduced by Galdi and Layton[16]. For a recent study that also introduces nonlinear boundary conditions, see Sahin[30].

The main result we prove is an existence theorem for a class of solutions that have the same regularity as the *strong solutions* to the Navier-Stokes equations. Our results differ from other LES models proposed in literature, in which the existence for weak solutions is obtained by adding an extra dissipative term of Smagorinsky type.

In Section 2 we briefly introduce the model we will study, with its physical justification. In Section 3 we prove a result of existence and uniqueness of strong solution. In Section 4 we consider the problem of the global existence of strong solutions, and we mention some numerical results.

2. Large Eddy Simulation Models

We present some basic facts related to LES; we refer the readers to Aldama[2] for other details. Complete derivation of the model we will consider can be found in the recent study by Galdi and Layton[16].

Given a function (as well as a vector field) $f(x,t)$, one can define its corresponding filtered variable $\overline{f}(x,t)$ by means of a convolution integral

$$\overline{f}(x,t) = [H * f](x,t) = \int_{\mathbf{R}^3} H(x-\xi)\, f(\xi,t)\, d\xi, \tag{2.5}$$

where H is a suitably defined smooth *filter function.* An ideal *low-pass filter* is one such that $\widehat{H} = 0$ for $|k| > k_c$, where the hat denotes the Fourier transform of a function. For our purposes (as well as for the practical purposes of numerical simulations) a filter that is rapidly decreasing is enough. In particular, we consider in (2.5) convolution with a *Gaussian filter*

$$g_\delta(x) = \left(\frac{\gamma}{\pi}\right)^{3/2} \frac{1}{\delta^3}\, \mathrm{e}^{-\frac{\gamma |x|^2}{\delta^2}} \qquad \text{and} \qquad \widehat{g}_\delta(k) = \mathrm{e}^{-\frac{\delta^2}{4\gamma}|k|^2}.$$

Extending all the variables by zero outside Ω, we can study the system (1.1) by means of the Fourier transform and $\overline{u} = g_\delta * u$. Recall that convolution in the physical space becomes multiplication in the frequency space. Then, the Fourier transform of (1.2) gives

$$\widehat{\overline{u}}(k) = \widehat{g}_\delta(k)\widehat{\overline{u}}(k) + \widehat{g}_\delta(k)\widehat{u}'(k).$$

Thus, $\widehat{u}'$ is given exactly, in terms of $\widehat{\overline{u}}$, by

$$\widehat{u}' = \left(\frac{1}{\widehat{g}_\delta(k)} - 1\right)\widehat{\overline{u}}.$$

One possible approach to model the terms arising from the filtering of $\overline{uu}$ is to use a Taylor series expansion with respect to δ. The first model obtained by using this approach was proposed by Leonard[24]. Further studies by Clark et al.[9], Bedford and Yeo[4], and Cantekin et al.[8] led to the following expansions:

$$\widehat{g}_\delta(k) = 1 - \frac{\delta^2}{4\gamma}|k|^2 + O(\delta^4) \qquad \text{and} \qquad \frac{1}{\widehat{g}_\delta(\kappa)} - 1 = \frac{\delta^2}{4\gamma}|k|^2 + O(\delta^4). \tag{2.6}$$

Disregarding terms that are $O(\delta^4)$, we have a poor filtering operator. In fact, the Fourier transform of the averaging kernel satisfies $\widehat{g}_\delta(k) \to 0$ as $|k| \to \infty$; on the other hand $\left(1 - \frac{\delta^2}{4\gamma}|k|^2\right) \to \infty$ as $|k| \to \infty$. Consequently, for high wave numbers the Taylor approximation may act as an *antismoothing* operator: the velocity u belongs to L^2, while the $\overline{u}$ may not belong to the same space.

When one disregards terms that are $O(\delta^4)$ and applies the inverse Fourier transform, the Taylor approximation (2.6) leads to the following nonlinear system of partial differential equations, known as the gradient model[9], for $U \simeq g_\delta * u$,

$$\frac{\partial U}{\partial t} + (U \cdot \nabla)\, U - \frac{1}{Re}\Delta U + \nabla \cdot \left[\frac{\delta^2}{2\gamma}\nabla U \nabla U\right] = \overline{f}, \tag{2.7}$$

together with the incompressibility condition $\nabla \cdot U = 0$. In the notation that we will use in the sequel, we have, for a given vector field ϕ,

$$[\nabla\phi\nabla\phi]_{ij} \stackrel{\text{def}}{=} \sum_{l=1}^{3} \frac{\partial \phi_i}{\partial x_l}\frac{\partial \phi_j}{\partial x_l}.$$

Unfortunately, for the reason explained above, the model (2.7) *might have unbounded kinetic energy.* Hence the existence theorems for the mean velocity U need, for instance, an additional Smagorinsky dissipative term; see Coletti[10]. Moreover, the numerical experiments performed by Coletti[11] and Iliescu et al.[18] show that the kinetic energy of the solution to (2.7) blows up in finite time if there is not a (very accurately tuned) additional dissipative term.

2.1. *The Rational LES Model*

In this section we introduce the RLES model (1.4). This model is based on the following **(0,1) subdiagonal Padé** approximation of the exponential function:

$$\widehat{g}_\delta(k) = \frac{1}{1 + \dfrac{\delta^2}{4\gamma}|k|^2} + O(\delta^4). \tag{2.8}$$

The term $1/\widehat{g}_\delta(\kappa)-1$ is approximated as in (2.6). When we disregard terms that are $O(\delta^4)$ in the $(0,1)$ Padé approximant, the resulting expression in (2.8) vanishes as $|k|$ goes to infinity. This seems a more promising method of approximate filtering. In fact, the weak solutions to the Navier-Stokes equations satisfy $u \in L^2(\mathbf{R}^3)$, and the Plancherel theorem implies that $\widehat{u} \in L^2(\mathbf{R}^3)$. Consequently, we have also that $\frac{\widehat{u}}{(1+\frac{\delta^2}{4\gamma}|k|^2)}$ belongs to $L^2(\mathbf{R}^3)$. By using this approximation and by applying the inverse-Fourier transform in the same way used to derive (2.7), one obtains the system RLES (1.4). This is not simply a differential system.

In fact, the system (1.4) is identical to (2.7) except for the presence of the *nonlocal* regularizing term

$$\left(\mathrm{I} - \frac{\delta^2}{4\gamma}\Delta\right)^{-1}. \tag{2.9}$$

This term, which at first glance makes the equations more complicated, is in effect a smoothing term and is responsible for better existence results from the analytical point of view. From the numerical point of view, the presence of this term (which is not difficult to handle with a Fast Poisson Solver) requires one to solve an additional linear problem. The particular form of the term (2.9) and the way (1.4) has been derived make the use of spectral methods very promising. This is the topic of work in progress.

Regarding the known analytical results, if a Smagorinsky term is added to system (1.4), Galdi, Iliescu, and Layton[17] proved the following result of existence of weak solutions.

Theorem 1. *Let μ be greater than or equal to* 0.1. *Let* $\overline{w}_0 \in L^2$, $w_t(0) \in L^2$, $\nabla \overline{w}_0 \in L^{2+2\mu}$, $\overline{f} \in L^2(0,T;L^2)$, *and* $\overline{f}_t \in L^2(0,T;L^2)$. *Moreover, assume that* $\|\overline{w}_0\|_{L^2}$ *and* $\|\overline{f}\|_{L^2(0,T;L^2)}$ *are small enough. Then, there exists a unique weak solution to* (1.4) *(together with the additional term* (1.3)*) in* $L^\infty(0,T;L^2) \cap L^{2+2\mu}(0,T;W_0^{1,2+2\mu})$.

Remark 1. The hypotheses assumed in Theorem 1 are weaker than that required by Coletti[10,11] for the system (2.7). In particular, $\mu \geq 0.1$ is needed for the model we consider, while for the gradient model[9] (2.7) a Smagorinsky dissipative term with $\mu \geq 0.5$ is required. For our RLES model (1.4) we conjecture the existence of weak solutions also for $\mu = 0$, that is, without extra dissipative terms.

Indeed, the main purpose of this paper is to prove the existence of strong solutions to (1.4) without additional dissipative terms.

Remark 2. For the sake of completeness, we mention the Approximate Deconvolution Models (ADM) recently introduced and studied numerically by Stolz and Adams[33] and Adams et al.[1], and references therein. These models are based on the approximate inversion of the filtering operation through repeated filtering. For the particular case of Gaussian filtering, Adams, Stolz, and Kleiser[1] noticed that the ADM coincides up to $O(\delta^4)$ with the model (2.7) (Appendix B, pp. 1013–1014). An approach similar (at least in principle) to the Fourier transform is that of the wavelets transform. This method is expected to be able to capture different patterns and not only to cutoff small eddies or high frequencies, but its use is far different from the purposes of this paper. For a recent review regarding wavelets methods in turbulence, see Farge et al.[13].

3. Existence and Uniqueness of Strong Solutions

In this section we prove the existence of strong solutions for the RLES system (1.4). The main result is that we prove the existence of such solutions without extra dissipative terms, as are required in other LES models previously proposed[10,11,17].

3.1. *Functional Setting*

Since we will consider the problem in the space-periodic setting, we recall the basic function spaces needed to deal with this functions. We denote by $H^m_{per}(Q)$, $m \in \mathbf{N}$, the space of functions that are in $(H^m_{loc}(\mathbf{R}^3))^3$ (i.e., $u_{|\mathcal{O}} \in H^m(\mathcal{O})$ for every bounded set $\mathcal{O}$) and that are periodic with period $\mathcal{L} > 0$:

$$u(x + \mathcal{L}\, e_i) = u(x), \qquad i = 1, 2, 3,$$

where $< e_1, e_2, e_3 >$ represents the canonical basis of $\mathbf{R}^3$, and $Q =]0, \mathcal{L}[^3$ is a cube of side length $\mathcal{L}$.

For $m = 0$, $H^0_{per}(Q)$ coincides simply with the Lebesgue space $L^2(Q)$. For an arbitrary $m \in \mathbf{N}$, $H^m_{per}(Q)$ is a Hilbert space. The functions in $H^m_{per}(Q)$ are easily characterized by the Fourier expansion

$$H^m_{per}(Q) = \left\{ u = \sum_{k \in \mathbf{Z}^3} c_k e^{\frac{2i\pi k\cdot x}{\mathcal{L}}}, \quad \overline{c}_k = c_{-k}, \quad \sum_{k \in \mathbf{Z}^3} (1 + |k|)^{2m} |c_k|^2 < \infty \right\}. \tag{3.10}$$

The definition (3.10) allows also us to consider $m \in \mathbf{R}$. We set

$$H^m = \left\{ u \in H^m_{per}(Q) \text{ of type (3.10), such that } c_0 = 0 \right\}.$$

For $m \in \mathbf{R}$, H^m is a Hilbert space for the norm $\{\sum_{k \in \mathbf{Z}^3} |k|^{2m} |c_k|^2\}^{1/2}$; furthermore, H^m and H^{-m} are in duality.

The norm (of functions, vectors, and tensors) in the Lebesgue space $L^2 := L^2(Q)$ is denoted by $\|\,.\,\|$, while the scalar product is written simply $(\,.\,,\,.\,)$. The norm of L^p, $p \neq 2$, is denoted by $\|\,.\,\|_{L^p}$. We also use the customary Sobolev spaces $W^{k,p}$, $k \in \mathbf{N}$, defined as the closure of smooth, periodic functions with respect to the norm

$$\|f\|_{W^{k,p}} = \left[\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p} \right]^{1/p};$$

the space $W^{-k,q}$, for $q = p/(p-1)$, denotes the topological dual of $W^{k,p}$.

Two spaces frequently used in the theory of Navier-Stokes equations are

$$V = \left\{ u \in H^1, \ \nabla \cdot u = 0 \right\} \qquad \text{and} \qquad H = \left\{ u \in H^0, \ \nabla \cdot u = 0 \right\}.$$

If $\Gamma_i = \partial Q \cap \{x_i = 0\}$, while $\Gamma_{i+3} = \partial Q \cap \{x_i = \mathcal{L}\}$, we have that if $u \in V$, then $u_{|\Gamma_{j+3}} = u_{|\Gamma_j}$. Let G be the orthogonal complement of H in H^0. We have

$$G = \left\{ u \in L^2 : \ u = \nabla q, \ q \in H^1_{per}(Q) \right\}.$$

The **Stokes operator** associated with the space-periodic functions is the following one. Given $f \in H^{-1}$, we solve

$$-\Delta u + \nabla p = f \text{ in } Q, \quad \nabla \cdot u = 0 \text{ in } Q. \tag{3.11}$$

We observe that if f belongs to H (in particular $\sum_{k \in \mathbf{Z}^3} k \cdot f_k = 0$, where f_k are the Fourier coefficients of f), then the Fourier coefficients $\{u_k, p_k\}$ of the solution of (3.11) are given by

$$u_k = -\frac{f_k\, \mathcal{L}^2}{4\pi^2 |k|^2} \quad \text{and} \quad p_k = 0, \qquad k \in \mathbf{Z}^3.$$

We define a one-to-one mapping $f \to u$ from H onto

$$\mathcal{D}(A) = \{u \in H, \ \Delta u \in H\} = H^2 \cap H.$$

Its inverse from $\mathcal{D}(A)$ onto H is denoted by A and, in fact,

$$Au = -\Delta u, \quad \forall\, u \in \mathcal{D}(A).$$

Remark 3. In absence of boundaries (in this case, the space-periodic setting) the Stokes and the Laplace operator coincide, apart from the domain of definition.

If $\mathcal{D}(A)$ is endowed with the norm induced by L^2, then A becomes an isomorphism from $\mathcal{D}(A)$ onto H. It follows that the norm $\|Au\|$ on $\mathcal{D}(A)$ is equivalent to the norm induced by H^2. It is well known that A is an unbounded, positive, linear, and self-adjoint operator on H. We can define the powers A^α and, if we set $V_\alpha = \mathcal{D}(A^{\alpha/2})$,

$$V_\alpha = \{v \in H^\alpha, \ \nabla \cdot v = 0\}.$$

Furthermore, the operator A^{-1} is linear continuous and compact. Hence A^{-1} possesses a sequence of eigenfunctions $\{\mathcal{W}_j\}_{j \in \mathbf{N}}$ that form an orthonormal basis of H,

$$\begin{cases} A\,\mathcal{W}_j = \lambda_j\, \mathcal{W}_j, \quad \mathcal{W}_j \in \mathcal{D}(A), \\ 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \ldots, \quad \text{and} \quad \lambda_j \to \infty \text{ for } j \to \infty. \end{cases} \tag{3.12}$$

In the sequel we denote by c several (possibly different also in the same line) positive constants not depending on w, but at most on $Re, \overline{f}$, and $\mathcal{L}$. All the norms that appear in the paper are evaluated on $Q =]0, \mathcal{L}[^3$.

3.2. *Proof of the Existence and Uniqueness Theorems*

In this section we prove the existence and uniqueness of a particular class of solutions for system (1.4).

Definition 1. *We say that w is a strong solution to system (1.4) if*

$$w \in L^{\infty}(0,T;V) \cap L^{2}(0,T;\mathcal{D}(A)) \qquad \textit{and} \qquad \frac{\partial w}{\partial t} \in L^{2}(0,T;H) \tag{3.13}$$

and if w satisfies, for each $\phi \in V$,

$$\begin{aligned} \frac{d}{dt}(w,\phi)+&\frac{1}{Re}(\nabla w,\nabla\phi)+((w\cdot\nabla)\,w,\phi)\\ &-\left(\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w\nabla w\right],\nabla\phi\right)=(\overline{f},\phi). \end{aligned} \tag{3.14}$$

Since w satisfies (3.13), we have that $w \in C([0,T];V)$ and the condition $w(x,0) = w_0(x)$ makes sense.

The main result we prove is the following.

Theorem 2. *Let be given $w_0 \in V$ and $\overline{f} \in L^2(0,T;H)$. Then there exists a strictly positive $T^* = T^*(w_0, Re, \overline{f})$ such that there exists a strong solution to (1.4) in $[0,T^*)$. A lower bound for T^* depending on $\|\nabla w_0\|, Re, \|\overline{f}\|_{L^2(0,T;L^2)}$ is obtained in (3.23).*

Proof. We consider the Faedo-Galerkin approximation of problem (1.4), that is, we look for approximate functions

$$w_m(x,t) = \sum_{k=1}^{m} g_m^i(t)\mathcal{W}_i(x),$$

satisfying for $k = 1,\dots,m$

$$\begin{cases} \dfrac{d}{dt}(w_m,\mathcal{W}_k)+\dfrac{1}{Re}(\nabla w_m,\nabla\mathcal{W}_k)+((w_m\cdot\nabla)\,w_m,\mathcal{W}_k)\\[2ex] \qquad -\left(\left(\mathrm{I}-\dfrac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\dfrac{\delta^2}{2\gamma}\nabla w_m\nabla w_m\right],\nabla\mathcal{W}_k\right)=(\overline{f},\mathcal{W}_k),\\[2ex] w_m(x,0)=P_m(w_0(x)), \end{cases} \tag{3.15}$$

with the $g_m^i(t)$ functions of class C^1, while $\{\mathcal{W}_i\}_{i\in\mathbf{N}}$ is the basis of eigenfunctions in (3.12). The operator P_m denotes the orthogonal projection

$$P_m : H \to \mathrm{Span} < \mathcal{W}_1,\dots,\mathcal{W}_m > .$$

To obtain *a priori* estimates, we multiply (3.15) by $\mathcal{A}\,w_m$, defined by

$$\mathcal{A}\,w_m := w_m + \frac{\delta^2}{4\gamma}A\,w_m,$$

and use suitable integration by parts to get

$$\frac{1}{2}\frac{d}{dt}\left(\|w_m\|^2+\frac{\delta^2}{4\gamma}\|\nabla w_m\|^2\right)+\frac{1}{Re}\left(\|\nabla w_m\|^2+\frac{\delta^2}{4\gamma}\|A w_m\|^2\right)=(\overline{f},\mathcal{A}\,w_m)$$

$$-\left((w_m\cdot\nabla)\,\nabla w_m,\mathcal{A}\,w_m\right)+\left(\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w_m\nabla w_m\right],\nabla\mathcal{A}\,w_m\right).$$

The first term on the right-hand side can be estimated simply by the Schwartz inequality

$$|(\overline{f},\mathcal{A}w_m)|\leq|(\overline{f},w_m)|+\frac{\delta^2}{4\gamma}|(\overline{f},Aw_m)|\leq\frac{1}{6\,Re}\left(\|\nabla w_m\|+\frac{\delta^2}{4\gamma}\|Aw_m\|^2\right)+c\|\overline{f}\|^2. \tag{3.16}$$

We also use the fact that $A\mathcal{W}_m=\lambda_m\mathcal{W}_m$ to increase the L^2 norm of w_m with that in V. The second term can be estimated by observing that

$$((w_m\cdot\nabla)\,w_m,\,w_m)=0$$

and by using the following classical inequality (see, for instance, Prodi[28]):

$$|((u\cdot\nabla)\,v,w)|\leq c\|\nabla u\|\|\nabla v\|^{1/2}\|Av\|^{1/2}\|w\|,\quad\forall\,u\in V,\ \forall\,v\in\mathcal{D}(A),\ \forall\,w\in H. \tag{3.17}$$

We obtain

$$\begin{aligned}|((w_m\cdot\nabla)\,\nabla w_m,\mathcal{A}w_m)| &\leq\frac{c\,\delta^2}{4\gamma}\|\nabla w_m\|^{3/2}\|Aw_m\|^{3/2}\\ &\leq\frac{1}{Re}\frac{\delta^2}{24\gamma}\|Aw_m\|^2+\frac{c\,\delta^2 Re^3}{4\gamma}\|\nabla w_m\|^6.\end{aligned} \tag{3.18}$$

Concerning the last term, we use the following identity. Given a linear, self-adjoint, and unbounded operator B acting from $\mathcal{D}(B)\subseteq X$ into the Hilbert space X, then we have

$$(Bx,y)=(x,By)\qquad\forall\,x,y\in\mathcal{D}(B). \tag{3.19}$$

In particular, if $B=\mathcal{A}^{-1}$, we have

$$(\mathcal{A}^{-1}x,\mathcal{A}y)=(x,y).$$

We observe that, since we are working in the space periodic setting, if $\mathcal{W}_k$ is in the domain of A, its partial derivatives also belong to the same subspace of H. We have then, by using (3.19),

$$\left|\left(\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w_m \nabla w_m\right], \nabla \mathcal{A}\, w_m\right)\right|$$

$$=\left|\left(\mathcal{A}^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w_m \nabla w_m\right], \mathcal{A}\,\nabla w_m\right)\right| = \frac{\delta^2}{2\gamma}|(\nabla w_m \nabla w_m, \nabla w_m)|$$

$$\leq \frac{\delta^2}{2\gamma}||\nabla w_m \nabla w_m||\,||\nabla w_m|| \leq \frac{\delta^2}{2\gamma}||\nabla w_m||^2_{L^4}||\nabla w_m||.$$

Now, by using the classical interpolation inequality,

$$||u||_{L^4} \leq c||u||^{1/4}||\nabla u||^{3/4} \qquad \forall\, u \in V, \tag{3.20}$$

(see, for instance, Ladyžhenskaya[21]), we obtain

$$\begin{aligned}\left|\left(\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w_m \nabla w_m\right], \nabla \mathcal{A} w_m\right)\right| &\leq \frac{c\,\delta^2}{2\gamma}||\nabla w_m||^{3/2}||Aw_m||^{3/2} \\ &\leq \frac{1}{12\,Re}\frac{\delta^2}{2\gamma}||Aw_m||^2 + \frac{c\,\delta^2 Re^3}{2\gamma}||\nabla w_m||^6.\end{aligned} \tag{3.21}$$

By collecting estimates (3.16), (3.18), and (3.21), we get

$$\begin{aligned}&\frac{d}{dt}\left(||w_m||^2 + \frac{\delta^2}{4\gamma}||\nabla w_m||^2\right) + \frac{1}{Re}||\nabla w_m||^2 \\ &\quad + \frac{1}{Re}\frac{\delta^2}{4\gamma}||Aw_m||^2 \leq c||\overline{f}||^2 + c\,\delta^2 Re^3||\nabla w_m||^6.\end{aligned} \tag{3.22}$$

The Gronwall lemma (provided $\overline{f}$ belongs to $L^2(0,T;H)$) and the results of existence for systems of ordinary differential equations imply that there exists $T^* > 0$ such that there exists a solution w_m to (3.15) in $[0,T^*)$ and

$$\{w_m\} \text{ is bounded uniformly with respect to } m \text{ in } \; L^\infty(0,T^*;V)\cap L^2(0,T^*;\mathcal{D}(A)).$$

A lower bound on the time T^* can be deduced as follows. Let us set $y(t) = ||w_m|| + \delta^2/4\gamma\,||w_m||^2$. Then we study (recall (3.22)) the differential inequality

$$\frac{dy}{dt} \leq c_1||\overline{f}||^2 + \frac{c_2\,Re^3}{\delta^4}y^3.$$

Dividing both sides by $(1+y)^3 \geq 1$, we obtain

$$\frac{dy}{dt}\frac{1}{(1+y)^3} \leq c_1||\overline{f}||^2 + \frac{c_2\,Re^3}{\delta^4}.$$

This equation can be explicitly integrated to get

$$1 + y(t) \le \frac{1 + y(0)}{\sqrt{1 - (1 + y(0))^2 \left[c_1 \int_0^t \|\overline{f}(\tau)\|^2 \, d\tau + \frac{c_2 Re^3}{\delta^4} t\right]}}.$$

Consequently, a condition that bounds T^* from below is the following:

$$c_1 \int_0^{T^*} \|\overline{f}(\tau)\|^2 \, d\tau + \frac{c_2 Re^3}{\delta^4} T^* \le \frac{1}{(1 + \|\nabla w_0\|^2)^2}. \tag{3.23}$$

Remark 4. The same result can be written also as follows: There exists $\epsilon = \epsilon(T, \overline{f}, Re) > 0$ such that if $\|\nabla w_0\| < \epsilon$ and $\|\overline{f}\|_{L^2(0,T;L^2)} < \epsilon$, then $\{w_m\}$ is uniformly bounded in

$$L^\infty(0, T; V) \cap L^2(0, T; \mathcal{D}(A)). \tag{3.24}$$

Remark 5. The result of existence is given for a fixed averaging radius δ. The basic theory of differential inequalities implies that, if all the other quantities (w_0, Re, and $\overline{f}$) are fixed, then the *life-span* of w_m is $O(\delta^4)$.

Let us turn to an estimate of the time derivative of w_m. By comparison, we have to estimate

$$\left| \left(\frac{\partial w_m}{\partial t}, \mathcal{W}_j \right) \right| \le |((w_m \cdot \nabla)\, w_m, \mathcal{W}_j)| + \frac{1}{Re} |(A w_m, \mathcal{W}_j)|$$

$$+ \left| \left(\nabla \cdot \left(\mathrm{I} - \frac{\delta^2}{4\gamma} \Delta \right)^{-1} \left[\frac{\delta^2}{2\gamma} \nabla w_m \nabla w_m \right], \mathcal{W}_j \right) \right| + \left| (\overline{f}, \mathcal{W}_j) \right|.$$

Some care is needed to estimate the highly nonlinear term, while the others are treated in a standard way (see, for instance, Galdi[15]). This one can be estimated as follows:

$$\left| \left(\nabla \cdot \left(\mathrm{I} - \frac{\delta^2}{4\gamma} \Delta \right)^{-1} \left[\frac{\delta^2}{2\gamma} \nabla w_m \nabla w_m \right], \mathcal{W}_j \right) \right|$$

$$\le \left\| \nabla \cdot \left(\mathrm{I} - \frac{\delta^2}{4\gamma} \Delta \right)^{-1} \left[\frac{\delta^2}{2\gamma} \nabla w_m \nabla w_m \right] \right\| \|\mathcal{W}_j\| \tag{3.25}$$

$$\le \left\| \left(\mathrm{I} - \frac{\delta^2}{4\gamma} \Delta \right)^{-1} \left[\frac{\delta^2}{2\gamma} \nabla w_m \nabla w_m \right] \right\|_{W^{1,2}} \|\mathcal{W}_j\|.$$

Recalling the Sobolev embedding $W^{2,3/2} \hookrightarrow W^{1,2}$, and classical results of elliptic regularity, the expression in (3.25) is bounded by

$$c \left\| \left(\mathrm{I} - \frac{\delta^2}{4\gamma}\Delta \right)^{-1} \left[\frac{\delta^2}{2\gamma} \nabla w_m \nabla w_m \right] \right\|_{W^{2,3/2}} \|\mathcal{W}_j\| \le c \|\nabla w_m \nabla w_m\|_{L^{3/2}} \|\mathcal{W}_j\|.$$

Next, we use the convex-interpolation inequality that holds for $f \in L^p(\Omega) \cap L^q(\Omega)$: if $p \le r \le q$, then

$$\|f\|_{L^r} \le \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta}, \quad \text{for} \quad \theta = \frac{pq - pr}{r(q-p)}. \tag{3.26}$$

The latter, together with the Sobolev embedding $H^1(Q) \subset L^6(Q)$, implies that the term in (3.25) is bounded by

$$c\|\nabla w_m\|_{L^3}^2 \|\mathcal{W}_j\| \le c\|\nabla w_m\| \, \|\nabla w_m\|_{L^6} \|\mathcal{W}_j\| \le c\|\nabla w_m\| \, \|A w_m\| \|\mathcal{W}_j\|.$$

Multiplying (3.15) by $dg_m^i(t)$, summing over $i = 1, \dots, m$, and using the last inequality (together with well-known estimates for the other terms), we obtain

$$\frac{1}{2Re} \frac{d}{dt} \|\nabla w_m\|^2 + \left\| \frac{\partial w_m}{\partial t} \right\|^2 \le c(1 + \|\nabla w_m\|^2) \|A w_m\|^2 + c\|\nabla w_m\|^6 + c\|\overline{f}\|^2.$$

The last differential inequality, together with (3.24), shows that $\partial w_m / \partial t$ is uniformly bounded in $L^2(0, T^*; L^2)$.

We now recall the following compactness result (see, for instance, Lions[25], Chap. 1).

Lemma 1. *Let, for some $p > 1$, the set Y be bounded in*

$$\mathcal{X} := \left\{ u \in L^p(0, T^*; X_1) : \quad \frac{du}{dt} \in L^p(0, T^*; X_3) \right\}.$$

If $X_1 \subset X_2 \subset X_3$ are reflexive Banach spaces and the first inclusion is compact, while the second is continuous, then Y is compactly included in $L^p(0, T^*; X_2)$.

By using the above lemma with

$$p = 2, \quad X_1 = \mathcal{D}(A), \quad X_2 = V, \quad \text{and} \quad X_3 = H,$$

we see that it is possible to extract from $\{w_m\}_{m \in \mathbf{N}}$ a subsequence (relabeled for notational convenience again as $\{w_m\}$) such that

$$\begin{cases} w_m \overset{*}{\rightharpoonup} w & \text{in } L^\infty(0, T^*; V) \\ w_m \rightharpoonup w & \text{in } L^2(0, T^*; \mathcal{D}(A)) \\ w_m \to w & \text{in } L^2(0, T^*; V) \quad \text{and a.e. in } (0, T) \times Q. \end{cases} \tag{3.27}$$

With this convergence it is easy to pass to the limit in (3.15) and to prove that w satisfies (1.4); hence, it is a solution to (1.4). Without loss of generality, along the same subsequence, we have also

$$\frac{\partial w_m}{\partial t} \rightharpoonup \frac{\partial w}{\partial t} \quad \text{in } L^2(0,T^*;L^2).$$

By using a classical interpolation argument (see Lions and Magenes[26]), the function w belongs also to

$$C(0,T^*;V).$$

For the reader's convenience, we show how the proof concludes; namely, that w satisfies (3.14), and hence it is a strong solution. The passage to the limit is done in a standard way (the same as for the Navier-Stokes equations; see, for instance, Galdi[15]) for all terms appearing in (3.15), except for

$$\left(\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w_m \nabla w_m\right], \nabla \mathcal{W}_k\right).$$

To pass to the limit in the above expression, we recall (see, for instance, Lemma 6.7, Chap. 1, in Lions[25]) that

$$\nabla f \in L^\infty(0,T^*;L^2)\cap L^2(0,T^*;H^1)$$

implies, by the Hölder inequality,

$$\nabla f \in L^4(0,T^*;L^3).$$

Thus, $\nabla w_m \nabla w_m$ is bounded in $L^2(0,T^*;L^{3/2})$, and $(3.27)_3$ implies that

$$\nabla w_m \nabla w_m \rightharpoonup \nabla w \nabla w \quad \text{in } L^2(0,T^*;L^{3/2}).$$

This implies that $\forall\, \phi \in C^\infty_{per}(Q)$

$$\left(\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w_m \nabla w_m\right], \nabla \phi\right) =$$

$$\left(\frac{\delta^2}{2\gamma}\nabla w_m \nabla w_m, \left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\nabla\phi\right) \to \left(\frac{\delta^2}{2\gamma}\nabla w \nabla w, \left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\nabla\phi\right)$$

in $L^2(0,T^*)$. The proof concludes with a density argument. □

Theorem 3. *Under the same hypotheses as in Theorem 2, there exists at most one strong solution to* (1.4).

Proof. Let us suppose that we have two solutions w_1, w_2 relative to the same external force $\overline{f}$ and the same initial datum w_0. Furthermore, let us suppose that both the solutions exist in some interval $[0,T]$. We subtract the equation satisfied by w_2 from that one satisfied by w_1, and we multiply the equations by Aw, where $w := w_1 - w_2$. The most dangerous term is that corresponding to

$$\left(\nabla\cdot\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\frac{\delta^2}{2\gamma}\left[\nabla w_1\nabla w_1-\nabla w_2\nabla w_2\right], A(w_1-w_2)\right).$$

By adding and subtracting $\nabla\cdot\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w_1\nabla w_2\right]$ on the left term, we get

$$\left(\nabla\cdot\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\frac{\delta^2}{2\gamma}\left[\nabla w_1\nabla w-\nabla w\nabla w_2\right], A(w_1-w_2)\right). \tag{3.28}$$

The first term in (3.28) can be estimated as follows:

$$\begin{aligned}
I_1 &= \left|\left(\nabla\cdot\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w_1\nabla w\right], A(w_1-w_2)\right)\right| \\
&\le \left\|\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w_1\nabla w\right]\right\|_{W^{2,2}}\|\nabla A(w_1-w_2)\|_{W^{-2,2}} \\
&\le c\,\|\nabla w_1\nabla w\|\,\|\nabla w\| \le c\|\nabla w_1\|_{L^4}\|\nabla w\|_{L^4}\|\nabla w\|.
\end{aligned}$$

By using again the interpolation inequality (3.20), we obtain

$$I_1 \le c\|\nabla w_1\|_{L^4}\|\nabla w\|^{5/4}\|Aw\|^{3/4} \le \frac{1}{8Re}\|Aw\|^2 + c\|\nabla w_1\|_{L^4}^{8/5}\|\nabla w\|^2.$$

The other term leads, *mutatis mutandis*, to

$$I_2 = \left|\left(\nabla\cdot\left(\mathrm{I}-\frac{\delta^2}{4\gamma}\Delta\right)^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w\nabla w_2\right], Aw\right)\right| \le \frac{1}{8Re}\|Aw\|^2 + c\|\nabla w_2\|_{L^4}^{8/5}\|\nabla w\|^2.$$

For the sake of completeness let us see how to estimate the other nonlinear terms

$$I_3 = \left|\left(\left(w_1\cdot\nabla\right)w_1 - \left(w_2\cdot\nabla\right)w_2, Aw\right)\right|.$$

Again, by adding and subtracting the term $(w_2\cdot\nabla)\,w_1$, we obtain

$$I_3 = \left|\left(\left(w\cdot\nabla\right)w_1 - \left(w_2\cdot\nabla\right)w, Aw\right)\right|.$$

Using again estimate (3.17), we obtain

$$\left|\left(\left(w_2\cdot\nabla\right)w, Aw\right)\right| \le \|\nabla w_2\|\,\|\nabla w\|^{1/2}\|Aw\|^{3/2} \le \frac{1}{8Re}\|Aw\|^2 + c\|\nabla w_2\|^4\|\nabla w\|^2.$$

The other term is easier to handle, since

$$I_4 = |((w \cdot \nabla)\, w_1, Aw)| \le \|w\|_{L^4}\|w_1\|_{L^4}\|Aw\|$$

$$\le c\|w_1\|_{L^4}\|Aw\|\,\|\nabla w\| \quad \text{by a Sobolev embedding theorem}$$

$$\le \frac{1}{8Re}\|Aw\|^2 + c\|\nabla w_1\|^2_{L^4}\|\nabla w\|^2 \quad \text{by the Young inequality.}$$

Collecting all the above estimates, we obtain

$$\frac{d}{dt}\|\nabla w\|^2 + \frac{1}{Re}\|Aw\|^2 \le c\left(\|\nabla w_1\|^{8/5}_{L^4} + \|\nabla w_2\|^{8/5}_{L^4} + \|\nabla w_2\|^4 + \|\nabla w_1\|_{L^4}\right)\|\nabla w\|^2.$$

We recall that $w(x,0) = w_1(x,0) - w_2(x,0) = 0$; by (3.24) we get

$$\left(\|\nabla w_1\|^{8/5}_{L^4} + \|\nabla w_2\|^{8/5}_{L^4} + \|\nabla w_2\|^4 + \|\nabla w_1\|_{L^4}\right) \in L^1(0,T),$$

and since $\|u\| \le 1/\lambda_1\,\|\nabla u\|$, for each $u \in V$, the Gronwall lemma directly implies that $w \equiv 0$ in V. $\square$

4. Analytical and Numerical Results concerning the Breakdown of Strong Solutions

In this section we introduce some criteria for the breakdown (and also for the continuation) of strong solutions, and we report some numerical results recently obtained. We compare these criteria with others in the literature, and we use them in interpreting the numerical simulations.

4.1. *On the Breakdown of Strong Solutions*

In this section, for simplicity (but it is easy to include a smooth external force), we set $\overline{f} = 0$. We start with the following theorem.

Theorem 4. *Let w be a strong solution in the time interval $[0,\overline{T})$. If it cannot be continued in* (3.13) *to $t = \overline{T}$, then we have*

$$\lim_{t\to\overline{T}^-} \|\nabla w(t)\| = +\infty. \tag{4.29}$$

Furthermore, we have the following blow-up estimate

$$\|\nabla w(t)\| \ge \frac{C\delta}{Re^{3/4}}\frac{1}{(\overline{T}-t)^{1/4}}, \qquad t < \overline{T}. \tag{4.30}$$

Proof. We observe that if $\overline{f} = 0$, the estimate (3.23) of the life span of the strong solution such that $w(x,0) = w_0$ can be replaced by the more explicit

$$T^* \ge \frac{C\delta^4}{Re^3\|\nabla w_0\|^4},$$

as can be easily seen by using the same technique of Section 3.2. We now prove (4.29) by contradiction. Let us assume that (4.29) does not hold. Then, there would exist a sequence $\{t_k\}_{k\in\mathbf{N}}$ (such that $t_k \uparrow \overline{T}$) and a positive number M such that

$$\|\nabla w(t_k)\| \leq M.$$

Since $w(t_k) \in H^1$, by using Theorem 2 we may construct a solution $\overline{w}$ with initial datum $w(t_k)$ in a time interval $[t_k, t_k + T^*)$, where

$$T^* \geq \frac{C}{\|\nabla w(t_k)\|^4} \geq \frac{C}{M^4} := T^0.$$

By using the uniqueness Theorem 3, we have $w \equiv \overline{w}$ in $[t_k, t_k + T^0)$. We may now select $k_0 \in \mathbf{N}$ such that $t_{k_0} + T^0 > \overline{T}$ to contradict the assumption on the boundedness of $\|\nabla w(t)\|$. This proves (4.29).

To obtain the estimate on the growth of $\|\nabla w(t)\|$, we argue as in the proof of Theorem 2. We multiply (1.4) by Aw, and we get that $Y(t) := \|\nabla w(t)\|^2$ satisfies, in the time interval $[0, \overline{T})$,

$$\frac{dY(t)}{dt} \leq \frac{c\,Re^3}{\delta^4}[Y(t)]^3.$$

Integrating the above equation, we find

$$\frac{1}{\|\nabla w(t)\|^4} - \frac{1}{\|\nabla w(\tau)\|^4} \leq \frac{c\,Re^3}{\delta^4}(\tau - t) \qquad 0 < t < \tau < \overline{T}.$$

Letting $\tau \to \overline{T}$, and recalling (4.29), we obtain (4.30). □

By using the above theorem, we can prove the following blow-up criteria, involving other norms of the gradient of w.

Theorem 5. *Let w be a strong solution to (1.4), and suppose that there exists a time $\overline{T}$ such that the solution cannot be continued in the class (3.13) to $T = \overline{T}$. Assume that $\overline{T}$ is the first such time. Then*

$$\int_0^{\overline{T}} \|\nabla w(\tau)\|_{L^\beta}^\alpha \, d\tau = \infty, \quad \textit{for} \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2, \quad 1 \leq \alpha < \infty, \quad 3/2 < \beta \leq \infty. \quad (4.31)$$

Observe that the condition (4.31) is the same as that involved in the study of the breakdown (or the global regularity) for the 3D Navier-Stokes equations; see Beirão da Veiga[5] for the Cauchy problem (also in $\mathbf{R}^n$) and Berselli[6] for the initial boundary value problem. In the limit case $\beta = \infty$, condition (4.31) is related to the Beale-Kato-Majda[3] criterion for the 3D Euler equations.

Proof. The proof is done by contradiction. We assume that

$$\int_0^{\overline{T}} \|\nabla w(\tau)\|_{L^\beta}^\alpha \, d\tau \leq C < \infty \qquad (4.32)$$

and we use estimates similar to that ones derived in the existence theorem. Let us suppose that $[0, \overline{T})$ is the maximal interval of existence of the unique strong solution starting from w_0 at time $t = 0$. We multiply (1.4) by (recall Remark 3)

$$\mathcal{A}\, w = w + \frac{\delta^2}{4\gamma} A\, w = w - \frac{\delta^2}{4\gamma}\Delta w,$$

and we obtain, with suitable integrations by parts,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|w_m\|^2 + \frac{\delta^2}{4\gamma}\|\nabla w\|^2\right) + \frac{1}{Re}\left(\|\nabla w\|^2 + \frac{\delta^2}{4\gamma}\|Aw\|^2\right)& \\ \leq \frac{\delta^2}{4\gamma}\left|((w\cdot\nabla)\,\nabla w, \Delta\, w)\right| + \left|\left\langle \mathcal{A}^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w \nabla w\right], \nabla \mathcal{A}\, w\right\rangle_{V,V'}\right|&,\end{aligned} \tag{4.33}$$

where $< \,.\,,\,.\, >_{V,V'}$ denotes the pairing between V and its topological dual V'. The first term on the right-hand side can be estimated with an integration by parts. We have, in fact,

$$\int_Q (w\cdot\nabla)\, w\,\Delta w\, dx = -\sum_{i,j,k=1}^{3}\left(\int_Q \frac{\partial w_j}{\partial x_k}\frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_k}\, dx - \int_Q w_j \frac{\partial^2 w_i}{\partial x_j \partial x_k}\frac{\partial w_i}{\partial x_k}\, dx\right). \tag{4.34}$$

The term

$$\sum_{i,j,k=1}^{3}\int_Q w_j \frac{\partial^2 w_i}{\partial x_j\partial x_k}\frac{\partial w_i}{\partial x_k}\, dx = \sum_{i,j,k=1}^{3}\frac{1}{2}\int_Q w_j\frac{\partial}{\partial x_j}\left(\frac{\partial w_i}{\partial x_k}\right)^2 dx$$

is identically zero, as can be seen with another integration by parts, since $\nabla\cdot w = 0$.

The other term on the right-hand side of (4.34) can be estimated in the following manner, for $3/2 < \beta \leq \infty$:

$$\left|\sum_{i,j,k=1}^{3}\int_Q \frac{\partial w_j}{\partial x_k}\frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_k}\, dx\right| \leq c\|\nabla w\|^2_{L^{2\beta'}}\|\nabla w\|_{L^\beta} \quad \text{for} \quad \frac{1}{\beta}+\frac{1}{\beta'} = 1.$$

Then, we use the interpolation inequality (3.26) (observe that $1 \leq \beta' < 3$, and if $\beta' = 1$ there is nothing to do), together with the Sobolev embedding $H^1(Q) \subset L^6(Q)$, to obtain

$$\left|\sum_{i,j,k=1}^{3}\int_Q \frac{\partial w_j}{\partial x_k}\frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_k}\, dx\right| \leq c\|\nabla w\|^{\frac{2\beta-3}{\beta}}\|\Delta w\|^{\frac{3}{\beta}}\|\nabla w\|_{L^\beta}.$$

We use Young's inequality with exponents $x = 2\beta/3$, $x' = 2\beta/(2\beta-3)$, and we obtain

$$\left|\sum_{i,j,k=1}^{3}\int_Q \frac{\partial w_j}{\partial x_k}\frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_k}\, dx\right| \leq \frac{1}{4\,Re}\|\Delta w\|^2 + c\|\nabla w\|^{\frac{2\beta}{2\beta-3}}_{L^\beta}\|\nabla w\|^2. \tag{4.35}$$

The other term in (4.33) can be estimated as follows:

$$\left\langle \mathcal{A}^{-1}\left[\frac{\delta^2}{2\gamma}\nabla w \nabla w\right], \mathcal{A}\nabla w\right\rangle_{V,V'} = \frac{\delta^2}{2\gamma}(\nabla w\nabla w, \nabla w),$$

and the latter can be treated as in (4.35).

The above estimates lead to

$$\frac{d}{dt}\left(\|w_m\|^2 + \frac{\delta^2}{4\gamma}\|\nabla w\|^2\right) \leq c\|\nabla w\|_{L^\beta}^\alpha \|\nabla w\|^2, \quad \text{where} \quad \alpha = \frac{2\beta}{2\beta - 3},$$

and hence α, β are as in (4.31). The Gronwall lemma, together with (4.32), implies that

$$\nabla w \in L^\infty(0, \overline{T}; L^2).$$

The last condition implies (from Theorem 4) that the solution w can be uniquely continued beyond $\overline{T}$, and this contradicts the maximality of the existence interval $[0, \overline{T})$. □

Remark 6. The same techniques may be used to prove that there exists $\eta > 0$ such that, if

$$\sup_{0<t<\overline{T}} \|\nabla w(t)\|_{L^{3/2}} < \eta,$$

then the strong solution exists up to $\overline{T}$. The constant η does not depend on w but only on Re, δ, γ, and $\mathcal{L}$. The proof easily follows by observing that

$$\left|\sum_{i,j,k=1}^{3}\int_Q \frac{\partial w_j}{\partial x_k}\frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_k}\,dx\right| \leq c\|\nabla w\|_{L^6}^2 \|\nabla w\|_{L^{3/2}} \leq c\|Aw\|^2\|\nabla w\|_{L^{3/2}}.$$

Consequently, in

$$\frac{1}{2}\frac{d}{dt}\left(\|w_m\|^2 + \frac{\delta^2}{4\gamma}\|\nabla w\|^2\right) + \frac{1}{Re}\left(\|\nabla w\|^2 + \frac{\delta^2}{4\gamma}\|Aw\|^2\right) \leq c\|Aw\|^2\|\nabla w\|_{L^{3/2}}$$

we can apply the Gronwall lemma to deduce a bound for $\|\nabla w\|_{L^\infty(0,\overline{T};L^2)}$, provided

$$\eta < \frac{\delta^2}{c\,4\gamma Re}.$$

Remark 7. From the Sobolev embedding theorem, we have

$$W^{1,p} \subset L^{p^*}, \quad \text{with} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{3},$$

for $1 \leq p < 3$. Consequently, if ∇w belongs to $L^\alpha(0, T; L^\beta)$ (with α, and β as in Theorem 5, $\beta < 3$), then

$$w \in L^r(0, T; L^s) \quad \text{for} \quad \frac{2}{r} + \frac{3}{s} = 1, \quad 2 < r < \infty, \quad 3 < s < \infty. \tag{4.36}$$

The above class (4.36) is a classical uniqueness and regularity class for weak solutions to the 3D Navier-Stokes equations; see, for instance, Galdi[15]. Furthermore, in the context of LES *scale similarity models*, if the approximate mean velocity w belongs to (4.36), then

$$w_\delta \to w \qquad \text{as} \quad \delta \to 0,$$

where w_δ is the solution corresponding to the averaging radius δ; see Layton[22]. Then, in such a class, w is a "good" approximation of u.

By using some classical results on elliptic systems and on singular integrals, we can also introduce breakdown criteria involving the vorticity $\omega = \operatorname{curl} w$. We start by observing that, for a divergence-free function w, we have $-\Delta w = \operatorname{curl}(\operatorname{curl} w)$, and the following estimate holds:

$$\|\nabla w\|_{L^p} \leq c_p \|\operatorname{curl} w\|_{L^p} \qquad \text{for} \quad 1 < p < \infty, \tag{4.37}$$

with c_p a positive constant depending only on p. The above estimate follows by observing that the Biot-Savart law implies

$$w(x) = \int G(x - x')\ \operatorname{curl} w(x')\, dx', \tag{4.38}$$

where $G(y)$ is given explicitly by

$$G(y) = \nabla \left[\frac{1}{4\pi} \lim_{N \to \infty} \sum_{k \in \mathbf{Z}^3,\, |k| \leq N} \left(\frac{1}{|y + L\,k|} + \frac{1}{|k\,L|} \right) \right].$$

By taking the gradient of (4.38) (with respect to the variable x), we obtain that

$$\nabla w = P(\operatorname{curl} w),$$

with P a (linear) singular operator of Calderón-Zygmund type. The estimate (4.37) follows by using the properties of such operators; see Stein[32].

Using estimate (4.37) and Theorem 5, one can easily prove the following result.

Corollary 1. *Let w be a strong solution to (1.4) in the time interval $[0, \overline{T})$. If it cannot be continued in (3.13) to $t = \overline{T}$, then*

$$\int_0^{\overline{T}} \|\operatorname{curl} w(\tau)\|_{L^\beta}^\alpha \, d\tau = \infty \quad \textit{for} \quad \frac{2}{\alpha} + \frac{3}{\beta} = 2,\ 1 < \alpha < \infty,\ 3/2 < \beta < \infty.$$

This breakdown criterion is slightly more interesting (from the physical point of view) than that involving the gradient of the velocity. In fact, if $\beta = 2$, and consequently $\alpha = 4$, we have the blow-up criterion

$$\int_0^{\overline{T}} \|\operatorname{curl} w(\tau)\|^4 \, d\tau = \infty$$

involving the so-called enstrophy, that is, the L^2-norm of the vorticity field.

4.2. *Remarks on Some Numerical Experiments*

The Rational LES model (1.4) was studied numerically for the 2D and 3D driven cavity[18] and for the 3D channel flow[14].

The driven cavity is an accepted and common test problem with many benchmarks available. Its solution, however, lies outside the known mathematical theory; since the boundary values are discontinuous, no solution can have an L^2 gradient.

On coarse meshes the finite element approximate solution to (1.4) was found to give an accurate description of moderate Re laminar flow. At higher Re, the approximate solution to (1.4) gave excellent qualitative agreement with the large eddies observed in simulations using other, accepted models when those models appeared sensible. Further, a quantitative comparison of the various models' kinetic energy yielded interesting results. The kinetic energy of the solution to the model (2.7) blew up in finite time unless very large amounts of additional dissipation were added to the model. At much higher Re and longer times, the kinetic energy in the RLES model (1.4) was also seen to blow up; a small amount of additional dissipation, modeling turbulent fluctuations, was seen to suffice to control its breakdown.

The RLES model (1.4) was also applied to the 3D channel flow at a Reynolds number based on the wall-shear velocity $Re_\tau = 180$ (or, equivalently, at a Reynolds number based on the mid-channel velocity $Re_c \approx 3300$).

Fischer and Iliescu[14] used a spectral element code to compare the RLES model (1.4) with the Smagorinsky model with Van Driest damping. The numerical results included plots of the following space-time averaged (denoted by $\ll \cdot \gg$) quantities: the mean streamwise velocity $\ll \overline{u} \gg$, the xy-component of the Reynolds stress $\ll u'v' \gg$, and the root mean square values of the streamwise $\ll u'u' \gg$, wall-normal $\ll v'v' \gg$, and spanwise $\ll w'w' \gg$ velocity fluctuations.

The RLES model (1.4) yielded improved results, showing good agreement with the fine Direct Numerical Simulations (DNS) calculation of Moser et al.[27].

It should be noted that, for this low Re simulation of the 3D channel flow, the RLES model (1.4) was successfully used without any additional dissipative term.

Further testing for higher Re flows will probably shed new light on the need for and character of extra dissipation in the use of the RLES model (1.4) in turbulent flow simulations.

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