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ASYMPTOTIC ANALYSIS

Working Note #3
BOUNDARY LAYERS

by

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Preface

For some time, we have been interested in the development and application of asymptotic methods for the numerical solution of boundary value problems with critical parameters—that is, parameters that determine the nature of the solution in some critical way. We are thinking, for example, of fluid flow (viscosity), combustion (Lewis number), and superconductivity (Ginzburg-Landau parameter) problems. Their solution may remain smooth over a wide range of parameter values, but as the parameters approach critical values, complicated patterns may emerge. Boundary layers may develop, or the region over which the solution extends may take on the appearance of a patchwork of subregions; on each subregion, the solution is smooth, but between subregions the solution undergoes dramatic changes over very short distances. Shock layers in fluid flow are a visible manifestation of this type of behavior.

Boundary value problems with critical parameters pose some of the most challenging problems in computational science, and much effort is being spent on developing new techniques for their numerical solution. Some of the most useful techniques, in particular on parallel computing architectures, are based on domain decomposition. In a domain decomposition method, one partitions the domain into subdomains, approximates the solution on each subdomain, and assembles these solutions to obtain an approximate solution on the entire domain. Many criteria, involving considerations from linear algebra to computer architecture, go into the design of a useful domain decomposition method. Our aim is to explore the use of asymptotic methods.

Asymptotic analysis, in particular singular perturbation theory, is the study of boundary value problems involving critical parameters. It provides a methodology to identify and characterize boundary layers, transition layers, and initial layers; hence, our idea to use asymptotic methods in the design of domain decomposition algorithms.

We have organized two workshops on the subject of asymptotic analysis and domain decomposition: a workshop at Argonne, jointly sponsored by the Department of Energy and the National Science Foundation (February 1990), and a NATO Advanced Research Workshop in Beaune, France (May 1992). Proceedings of these workshops have been published (*Asymptotic analysis and the numerical solution of partial differential equations*, edited by H. G. Kaper and M. Garbey, Lecture Notes in Pure and Applied Mathematics – Vol. 130, Marcel Dekker, Inc., New York, 1991; *Asymptotic and numerical methods for partial differential equations*, edited by H. G. Kaper and M. Garbey, NATO ASI Series C: Mathematical and Physical Sciences – Vol. 384, Kluwer Academic Publishers, Dordrecht, Neth., 1993).

We currently have plans to develop a full-length book on the subject. To formulate our thoughts before final publication, we intend to produce a series of *Working Notes* on various relevant topics. Some of the notes will contain new material; others may offer new presentations of existing material. We certainly expect the notes to evolve in time; the notes may or may not appear eventually as chapters of the book. The notes are intended for our own use, but we will be happy to supply copies to interested colleagues.

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Working Note #1:
Asymptotic Analysis—Basic Concepts and Definitions, ANL/MCS-TM-179 (July 1993)

Working Note #2:
Asymptotic Analysis—Approximation of Integrals, ANL/MCS-TM-180 (July 1993)

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ASYMPTOTIC ANALYSIS

Working Note #3

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Abstract

In this chapter we discuss the asymptotic approximation of functions that display boundary-layer behavior. Our purpose here is to introduce the basic concepts underlying the phenomenon, to illustrate its importance, and to describe some of the fundamental tools available for its analysis. To achieve our purpose in the clearest way possible, we shall work with functions that are assumed to be given explicitly—that is, functions $f:(0,\epsilon_0)\to X$ whose expressions are known, at least in principle. Only in the following chapter shall we begin the study of functions that are given implicitly as solutions of boundary value problems—the real stuff of which singular perturbation theory is made.

Boundary-layer behavior is associated with asymptotic expansions that are regular "almost everywhere"—that is, expansions that are regular on every compact subset of the domain of definition, but not near the boundary (Section 1). These regular asymptotic expansions can be continued in a certain sense all the way up to the boundary (Section 2), but a separate analysis is still necessary in the boundary layer. The boundary-layer analysis is purely local and aims at constructing local approximations in the neighborhood of each point of the singular part of the boundary (Section 3). The problem of finding an asymptotic approximation is thus reduced to matching the various local approximations to the existing regular expansion valid in the interior of the domain (Section 4).

1 Boundary-Layer Behavior

We now continue the discussion of the asymptotic approximation of functions begun in Chapter 2 and turn our attention to vector-valued functions that do not have regular asymptotic expansions on their entire domain of definition. We assume throughout this chapter that we are working with functions whose expressions are known, at least in principle, to better explain the phenomenon of boundary-layer behavior; the discussion of boundary-layer behavior for functions that are only known implicitly as solutions of singular boundary value problems will begin in the following chapter.

We first recall some relevant points of Chapter 2. Suppose that we are given a function f, which is defined on $(0, \epsilon_0) \times D$, where D is a fixed (ϵ-independent) domain in $\mathbf{R}^N$. We consider f as a vector-valued function, $f : (0, \epsilon_0) \to X$, where X is a normed space of functions defined on D, by making the identification $f(\epsilon)(x) = f(\epsilon, x)$ for all $x \in D$. The function f has a regular asymptotic expansion on D if there exist an asymptotic sequence $\{\delta_n : n = 0, 1, \ldots\}$ of order functions δ_n and a nontrivial sequence $\{f_n : n = 0, 1, \ldots\}$ of elements $f_n \in X$, which do not depend on ϵ, such that the function $Ef : (0, \epsilon_0) \to X$ defined by the expression $Ef(\epsilon) = \sum_n \delta_n(\epsilon) f_n$ is an asymptotic approximation of f on D. The (regular) asymptotic expansion may be defined up to a specified number of terms or up to a specified order of accuracy. The expansion coefficients f_n are uniquely determined,

$$f_n = \lim_{\epsilon \downarrow 0} f^{(n)}(\epsilon)/\delta_n(\epsilon), \quad n = 0, 1, \ldots,$$

where $f^{(0)}(\epsilon) = f(\epsilon)$ and $f^{(n)}(\epsilon) = f(\epsilon) - \sum_{p=0}^{n-1} \delta_p(\epsilon) f_p$ for $n = 1, 2, \ldots$. The limits are taken in X.

Observe that the asymptotic sequence $\{\delta_n : n = 0, 1, \ldots\}$ depends on f; indeed, different applications may require different asymptotic sequences.

It is sometimes advantageous to weaken the condition that the coefficients f_n be totally independent of ϵ and require instead that they are of the order $O^\sharp(1)$ as $\epsilon \downarrow 0$. This weaker condition leads to the theory of *generalized asymptotic expansions*. The coefficients f_n in such expansions are not unique. Much of what we will say about regular asymptotic expansions extends to generalized asymptotic expansions, but we prefer to avoid the technical complications and will not consider generalized expansions.

If the asymptotic expansion of f fails to be regular on the entire domain D, it is still possible that it is regular on a subdomain.

Definition 1 *The function $f : (0, \epsilon_0) \to X$ has a regular asymptotic expansion on a subdomain D_0 of D if there exist an asymptotic sequence $\{\delta_n : n = 0, 1, \ldots\}$ of order functions and a nontrivial sequence $\{f_n : n = 0, 1, \ldots\}$ of elements $f_n \in X$, which do not depend on ϵ, such that the function $E_0 f : (0, \epsilon_0) \to X$, defined by the expression $E_0 f(\epsilon) = \sum_n \delta_n(\epsilon) f_n$ is an asymptotic approximation of f on D_0.*

Again, if f has a regular expansion on some subdomain D_0 of D, its expansion coefficients f_n are unique and they are found by taking limits. The limits must be taken in the topology induced on D_0 by the norm of X, but only the restrictions of the functions $f^{(n)}(\epsilon)/\delta_n(\epsilon)$ to D_0 need to be considered.

Lemma 1 *Let f have regular asymptotic expansions $E_0 f$ and $E_1 f$ on the subdomains D_0 and D_1 of D, respectively. If $D_0 \supset D_1$, then $E_0 f$ extends $E_1 f$ to an asymptotic approximation of f on D_0.*

Proof. An immediate consequence of the uniqueness of the coefficients in a regular asymptotic expansion with respect to the asymptotic sequence. ∎

Functions that display "boundary-layer behavior" are functions that are approximated by a regular asymptotic expansion "almost everywhere" on D.

Definition 2 *The function* $f : (0, \epsilon_0) \to X$ *exhibits* boundary-layer behavior *on* D *if there exists a regular asymptotic expansion* Ef *approximating* f *on every* ϵ*-independent compact subset of* D*, but there is no regular asymptotic expansion approximating* f *on* D*.*

The terminology "boundary layer" has its origin in fluid dynamics. In the motion of a viscous fluid near a boundary, the viscosity causes rapid variations of the fluid velocity in a thin layer near the boundary, a phenomenon first studied by Prandtl in 1904 [1]. Later developments in continuum mechanics revealed that the phenomenon of boundary layer behavior was not confined to fluid dynamics, but that it is indeed a widespread phenomenon caused by "singular perturbations" in the mathematical model. The terminology has been generally adopted to loosely describe the breakdown of regularity near boundaries.

Boundary layers can occur anywhere along the boundary and don't have to happen everywhere along the boundary. Thus, the boundary is the union of two disjoint sets, the "regular part" and the "singular part."

Definition 3 *A point* $P \in \partial D$ *belongs to the* regular part of the boundary, $\partial_r D$*, if the regular asymptotic expansion* Ef *extends to an asymptotic approximation of* f *on every* ϵ*-independent compact subset of* $D \cup P$*. Every point* $P \in \partial D$ *that does not belong to the regular part of the boundary belongs to the* singular part of the boundary, $\partial_s D$*.*

The set of all points in D whose distance to a component of $\partial_s D$ is $o(1)$, is usually designated as a *boundary layer.* The terminology is vague and somewhat qualitative.

A simple example of a function that exhibits boundary-layer behavior is given by $f(\epsilon, x) = 1 - e^{-x/\epsilon}$ for $0 \le x < \infty$. If $X = (C[0, \infty), \| \cdot \|_\infty)$ and $f(\epsilon) \in X$ is defined by the identity $f(\epsilon)(x) = f(\epsilon, x)$, then f has a boundary layer near 0. The element $f_0 \in X$, given by $f_0(x) = 1$ for all $x \ge 0$, defines a regular approximation as $\epsilon \downarrow 0$ on every compact interval $[a, b]$ with $0 < a < b < \infty$, even on every interval $[a, \infty)$ with $a > 0$, but there is no regular approximation on the entire interval $[0, \infty)$.

Despite all physical evidence to the contrary, one could argue—and some mathematicians have argued—that boundary-layer behavior illustrates a shortcoming of the mathematical model, which could be easily "fixed." Boundary-layer behavior occurs because the topology in X is too strong to handle significant local variations that increase dramatically as the small parameter decreases to 0—for example, when X is a space of continuous functions and the topology is defined in terms of uniform convergence, as in the example. So, all one has to do is weaken the topology and admit generalized functions to the club of potential expansion coefficients. Of course, the argument is correct—from the theoretical point of view. But from the practical point of view it makes sense to consider functions that can be evaluated pointwise and are defined continuously everywhere. The notion of a classical solution is essential in the context of applications, and uniform error estimates do make sense in numerical computations. Furthermore, the fact that ϵ is a small parameter does not necessarily imply that it is infinitesimally small. For all these good reasons we will stick with the topology that causes the boundary-layer behavior and, if all goes well, we will show not only how one can live with it, but even how one take advantage of it.

A few more remarks about boundary layers.

If a boundary layer occurs along a manifold that is described in terms of a timelike variable t (usually $t = 0$), we may refer to it as an *initial layer*.

If D is the union of a finite number of disjoint subdomains, $D = \cup_i D_i$ with $D_i \cap D_j = \emptyset$ if $i \neq j$, we may find a boundary layer along the common boundary of two adjacent subdomains. In such cases, we may refer to the boundary layer as a *transition layer*. (Sometimes, the term *internal layer* is used.)

For an elementary example of a transition layer, consider the boundary value problem

$$\epsilon u_{xx} + 2uu_x = 0,\ x \in (-\infty, \infty); \quad \lim_{x\to\pm\infty} u(x) = \pm 1.$$

This nonlinear problem is a prototype of many other problems in the theory of nonlinear waves; it is a special case of Burgers equation [2] or [3, Chapter 4]. Its solution is known, $u(\epsilon, x) = \tanh(x/\epsilon)$ for $-\infty < x < \infty$. To put this problem in the functional framework, define $D_1 = (-\infty, 0)$ and $D_2 = (0, \infty)$ and put $D = D_1 \cup D_2$. Let X be the normed vector space $(C(D), \|\cdot\|_\infty)$, and consider u as a map from $(0, \epsilon_0)$ into X by making the usual identification, $u(\epsilon)(x) = u(\epsilon, x)$ for all $x \in D$. The function $u_0 \in X$ with values $u_0(x) = -1$ if $x \in D_1$ and $u_0(x) = 1$ if $x \in D_2$, defines an asymptotic approximation on every compact subset of D, even on every subset $\{x \in D : |x| \geq a\}$ with $a > 0$, but there is no ϵ-independent element in X that provides an asymptotic approximation of u on all of D. The origin makes up the singular part of the boundary, and a transition layer is located on either side of it.

Boundary layers may have very complicated structures. For example, boundary layers may be nested within each other when changes occur on several length scales simultaneously. Or transition layers may move and interfere with each other—a situation that arises frequently in the context of fluid dynamics, where shock waves present the most visible evidence of transition layers. Clearly, all this makes for a very complicated theory, whose development becomes cumbersome or even impossible unless one makes drastic simplifying assumptions. *Singular perturbation theory*, the study of boundary layers, initial layers, and transition layers, has made much progress, but remains the domain of a few. In particular, its application to the numerical solution of boundary value problems—the main objective of this book—is largely undeveloped.

Exercises

1. Consider the boundary value problem

$$u_t = \epsilon u_{xx}, \quad (t, x) \in (0, \infty) \times (-\infty, \infty),$$

$$\lim_{|x| \to \infty} u(t, x) = 0, \ t \in (0, \infty); \quad u(0, x) = \phi(x), \ x \in (-\infty, \infty),$$

where ϕ is a given function. This linear boundary value problem is a simple model for all sorts of diffusion phenomena; the coefficient ϵ measures the rate of diffusion.

(i) Verify that the solution of this boundary value problem is given by the integral

$$u(\epsilon, t, x) = \frac{1}{(4\pi\epsilon t)^{1/2}} \int_{-\infty}^{\infty} \phi(y) \exp\left[-\frac{(x-y)^2}{4\epsilon t}\right] \, dy.$$

(ii) Discuss the asymptotic behavior of u as $\epsilon \downarrow 0$.

2 Extension Theorems

Suppose that f has boundary-layer behavior in D. Then there exist ϵ-independent elements $f_0, f_1, \ldots$ in X such that the function $Ef = \sum_n \delta_n f_n$ is an asymptotic approximation of f on any compact subset K of D. If the boundary of D has a regular part $\partial_r D$, then Ef extends to an asymptotic approximation of f on every ϵ-independent compact subset of $D \cup \partial_r D$. However, Ef does not extend to an asymptotic approximation of f on D, because of the presence of the singular part of the boundary, $\partial_s D$. On the other hand, it is possible to extend Ef up to $\partial_s D$ in a well defined sense. The extension theorems that we establish in this section generalize a result of Kaplun [4, 5] and are a consequence of the following simple lemma.

Lemma 2 *Let the function g be positive on $(0, \epsilon_0) \times (0, d_0)$ for some $\epsilon_0 > 0$ and $d_0 > 0$, and let g satisfy the properties* (i) *$g(\epsilon, \cdot)$ is monotone nonincreasing for each $\epsilon \in (0, \epsilon_0)$, and* (ii) *$\lim_{\epsilon \downarrow 0} g(\epsilon, d) = 0$ for each $d \in (0, d_0)$. Then there exists an order function δ satisfying $\delta = o(1)$, such that $\lim_{\epsilon \downarrow 0} g(\epsilon, \delta(\epsilon)) = 0$.*

Proof. It follows from (ii) that, for any $p > 0$ and any fixed $d \in (0, d_0)$, we have $g(\epsilon, d) < p$ for all sufficiently small positive ϵ. Let $q(d, p) = \sup\{q : g(\epsilon, d) < p, \epsilon \in (0, q)\}$. Because of (i), $q(\cdot, p)$ is nondecreasing; furthermore, $\lim_{d \downarrow 0} q(d, p) = 0$. We now form a sequence of triples, $\{(d_n, p_n, q_n) : n = 1, 2, \ldots\}$, starting from two sequences $\{d_n : n = 1, 2, \ldots\}$ and $\{p_n : n = 1, 2, \ldots\}$, both monotonically decreasing toward 0, and using the definition

$q_n = \min\{q(p_k, d_k) : k = 1, 2, \ldots, n; 1/n\}$ to determine the third element of each triple. Then $g(\epsilon, d) < p_n$ for all $\epsilon \in (0, q_n)$ and $d \geq d_n$. We claim that any monotone continuous function δ that satisfies the relation $\delta(q_n) = d_{n-1}$ for $n = 2, 3, \ldots$ satisfies the conditions of the lemma.

Let p be any positive number. Then $p_m < p$ for some integer m. Any $\epsilon \in (0, q_m)$ satisfies $q_{n+1} \leq \epsilon < q_n$ for some $n \geq m$; hence, $d_n \leq \delta(\epsilon) < d_{n-1}$, and therefore $g(\epsilon, \delta(\epsilon)) \leq g(\epsilon, d_n) < p_n \leq p_m < p$. ∎

We will prove an extension theorem for the case that $\partial_s D$ is bounded, leaving the unbounded case to the exercises.

Theorem 1 *Suppose $\partial_s D$ is compact and the regular expansion $Ef = \sum_n \delta_n f_n$ is an asymptotic approximation of f on every ϵ-independent compact subset of $D \cup \partial_r D$. Then there exist an order function δ satisfying $\delta = o(1)$ and a nested family $\{K_{\delta(\epsilon)} : \epsilon \in (0, \epsilon_0)\}$ of compact subsets of $D \cup \partial_r D$ satisfying* $\operatorname{dist}(K_{\delta(\epsilon)}, \partial_s D) = \delta(\epsilon)$*, such that Ef extends to an asymptotic approximation of f on $K_{\delta(\epsilon)}$.*

Proof. If $\partial_s D$ is compact, we can associate with any compact subset K of $D \cup \partial_r D$ a finite number $\operatorname{dist}(K, \partial_s D) = \sup\{\inf\{\|x - y\| : y \in K\}, x \in \partial_s D\}$, where $\|\cdot\|$ denotes the Euclidean distance in $\mathbf{R}^N$. Furthermore, for any number $d \in (0, d_0)$ we can find a compact subset K_d of D for which $\operatorname{dist}(K_d, \partial_s D) = d$. Without loss of generality, we may assume that the family of subsets $\{K_d : d \in (0, d_0)\}$ is nested, in the sense that $K_d \subset K_{d'}$ if $d > d'$.

Let $\|\cdot\|_d$ denote the X-norm of the restriction of an element of X to K_d. We are given that there exist functions $f_n \in X$ such that $\lim_{\epsilon \downarrow 0} \|f^{(n)}(\epsilon)/\delta_n(\epsilon) - f_n\|_d = 0$, where $f^{(0)}(\epsilon) = f(\epsilon)$ and $f^{(n)}(\epsilon) = f(\epsilon) - \sum_{p=0}^{n-1} \delta_p(\epsilon) f_p$ for $n = 1, 2, \ldots$, for each $d \in (0, d_0)$. The assertion of the theorem follows from Lemma 2, where we take $g(\epsilon, d) = \|f^{(n)}(\epsilon)/\delta_n(\epsilon) - f_n\|_d$. ∎

We may summarize the statement of Theorem 1 by saying that Ef extends to an asymptotic approximation of f on $D \cup \partial_r D$ "from within D." We denote the extension by $E_I f$ and refer it as the *interior expansion* of f on D. The subscript I serves to distinguish $E_I f$ from a regular expansion Ef, which does not require an approximation of the underlying domain. Of course, if f has a regular asymptotic expansion on D, then $E_I f = Ef$.

To illustrate the effect of Theorem 1, we recall the example $f(\epsilon, x) = 1 - e^{-x/\epsilon}$ on $[0, \infty)$ from the previous section. As we have seen, the element $f_0 \in X$, given by $f_0(x) = 1$ for all $x \geq 0$, defines a regular asymptotic approximation on every interval $[a, \infty)$ with $a > 0$. According to Theorem 1, there exists an order function δ satisfying $\delta = o(1)$, such that f_0 extends to an asymptotic approximation on $[\delta(\epsilon), \infty)$. However, Theorem 1 does not give us any more information about δ, other than that it is $o(1)$ as $\epsilon \downarrow 0$. So, although we know that f has a regular asymptotic approximation on $(0, \infty)$ in the sense of the theorem, what actually happens near 0 remains undisclosed.

Another interesting application of Lemma 2 is given in the following extension theorem. It shows that one can sometimes sacrifice some of the accuracy of an asymptotic approximation to gain an extension of its domain of validity.

Theorem 2 *Suppose $\partial_s D$ is compact and Ef is an asymptotic approximation of the order of δ_{m-1} (i.e., Ef is a regular expansion to m terms) of f on any ϵ-independent compact subset of $D \cup \partial_r D$. Then there exist, for each $p \in \{0, \ldots, m-1\}$, an order function δ'_p satisfying $\delta'_p = o(1)$ and a nested family $\{K_{\delta'_p(\epsilon)} : \epsilon \in (0, \epsilon_0)\}$ of compact subsets of $D \cup \partial_r D$ satisfying $\mathrm{dist}(K_{\delta'_p(\epsilon)}, \partial_s D) = \delta'_p(\epsilon)$, such that Ef extends to an asymptotic approximation of the order of δ_{m-1-p} of f on $K_{\delta'_p(\epsilon)}$. The order functions δ'_p satisfy the relation $\delta'_p = O(\delta'_{p-1})$ for $p = 1, \ldots, m-1$.*

Proof. If $f - Ef = o(\delta_{m-1})$ on any compact subset of $D \cup \partial_r D$, then certainly $f - Ef = o(\delta_{m-1-p})$ for $p = 0, \ldots, m-1$ on the same subset. The proof of the theorem is similar to the proof of Theorem 1, where in the final step we take $g(\epsilon, d) = \|(1/\delta_{m-1-p})(f - Ef)\|$. It suffices to take $\delta'_p = \delta'_{p-1}$ to prove the last assertion of the theorem. ∎

Although the statement of the theorem is impressively complicated, its proof is disappointingly simple; in fact, the bottom line is that by taking $\delta'_p = \delta'_{p-1}$ we do not achieve any extension at all. The point is, however, that it may just be possible to sharpen the estimate $\delta'_p = O(\delta'_{p-1})$ to $\delta'_p = o(\delta'_{p-1})$ and thus obtain a real extension of Ef, albeit at the price of a reduction in the order of accuracy.

Here is a simple example, where all the details can be worked out. Consider the function $f(\epsilon, x) = (\epsilon + x)^{-1}$ on $[0, \infty)$ as a mapping from $(0, \epsilon_0)$ into $X = (C[0, \infty), \| \cdot \|_\infty)$. The m-term expansion $Ef(\epsilon, x) = \sum_{n=0}^{m-1} (-1)^n \epsilon^n x^{-n-1}$ defines an asymptotic approximation of the order of ϵ^{m-1} of f, $f - Ef = o(\epsilon^{m-1})$, on any interval $[a, \infty)$ with $a > 0$. The remainder can be calculated explicitly, $(f - Ef)(\epsilon, x) = (-1)^m \epsilon^m x^{-m} (\epsilon + x)^{-1}$, and one readily verifies that, for each $p \in \{0, \ldots, m-1\}$, $f - Ef = o(\epsilon^{m-1-p})$ on any interval of the form $[a\epsilon^{p/(m+1)}, \infty)$. The left endpoint of this interval actually gets closer to 0 as p increases, while ϵ is being kept fixed. Thus, a real extension of Ef is obtained. In the notation of Theorem 2, we have $\delta'_p(\epsilon) = \epsilon^{p/(m+1)}$ and, indeed, $\delta'_p = o(\delta'_{p-1})$ for $p = 1, \ldots, m-1$.

Exercises

1. Prove the following counterpart of Lemma 2.

   Lemma 3 *Let the function g be positive on $(0, \epsilon_0) \times (d_0, \infty)$ for some $\epsilon_0 > 0$ and some real number d_0, and let g satisfy the properties* (i) *$g(\epsilon, \cdot)$ is monotone nondecreasing for each $\epsilon \in (0, \epsilon_0)$, and* (ii)

$\lim_{\epsilon \downarrow 0} g(\epsilon, d) = 0$ *for each* $d \in (d_0, \infty)$. *Then there exists an order function* δ *satisfying* $\delta = o(1)$, *such that* $\lim_{\epsilon\downarrow 0} g(\epsilon, 1/\delta(\epsilon)) = 0$.

2. Prove the following counterpart of Theorem 1.

   Theorem 3 *Let* B_R *denote the (open) ball of radius* R *centered at the origin in* $\mathbf{R}^N$. *Suppose* D *is unbounded and* f *has a regular asymptotic expansion* $Ef = \sum_n \delta_n f_n$ *on every* ϵ*-independent compact subset* $K_R = D \cap \overline{B}_R$ *of* D. *Then there exist an order function* δ *satisfying* $\delta = o(1)$ *and a nested family* $\{B_{1/\delta(\epsilon)} : \epsilon \in (0, \epsilon_0)\}$ *of balls of radius* $1/\delta(\epsilon)$ *centered at the origin in* $\mathbf{R}^N$, *such that* Ef *extends to an asymptotic approximation of* f *on* $K_{1/\delta(\epsilon)} = D \cap \overline{B}_{1/\delta(\epsilon)}$.

3. Prove the following counterpart of Theorem 2.

   Theorem 4 *Let* B_R *denote the (open) ball of radius* R *centered at the origin in* $\mathbf{R}^N$. *Suppose* D *is unbounded and* Ef *is an asymptotic approximation of the order of* δ_{m-1} *(i.e.,* Ef *is a regular expansion to* m *terms) of* f *on every* ϵ*-independent compact subset of* $K_R = D \cap \overline{B}_R$ *of* D. *Then there exist, for each* $p \in \{0, \ldots, m-1\}$, *an order function* δ'_p *satisfying* $\delta'_p = o(1)$ *and a nested family* $\{B_{1/\delta'_p(\epsilon)} : \epsilon \in (0, \epsilon_0)\}$ *of balls of radius* $1/\delta'_p(\epsilon)$ *centered at the origin in* $\mathbf{R}^N$, *such that* Ef *extends to an asymptotic approximation of the order of* δ_{m-1-p} *of* f *on* $K_{1/\delta'_p(\epsilon)} = D \cap \overline{B}_{1/\delta'_p(\epsilon)}$. *The order functions* δ'_p *satisfy the relation* $\delta'_p = O(\delta'_{p-1})$ *for* $p = 1, \ldots, m-1$.

4. Prove the following extension of Lemma 3.

   Lemma 4 *Let* ρ *be a real-valued function on* $(0, \epsilon_0)$, *which grows beyond bounds as* $\epsilon \downarrow 0$. *Let the function* g *be positive on the set* $\{(\epsilon, d) : \epsilon \in (0, \epsilon_0), d \in (d_0, \rho(\epsilon))\}$ *for some* $\epsilon_0 > 0$ *and some real number* d_0, *and let* g *satisfy the properties* (i) $g(\epsilon, \cdot)$ *is monotone nondecreasing for each* $\epsilon \in (0, \epsilon_0)$, *and* (ii) $\lim_{\epsilon\downarrow 0} g(\epsilon, d) = 0$ *for each* $d \in (d_0, \infty)$. *Then there exists an order function* δ *satisfying* $\delta = o(1)$, *such that* $\lim_{\epsilon\downarrow 0} g(\epsilon, 1/\delta(\epsilon)) = 0$.

5. Prove the following extension of Theorem 3.

   Theorem 5 *Let* ρ *be a real-valued function on* $(0, \epsilon_0)$ *which grows beyond bounds as* $\epsilon \downarrow 0$, *and let* B_ρ *denote the (open) ball of radius* $\rho(\epsilon)$ *centered at the origin in* $\mathbf{R}^N$. *Suppose* D *is unbounded and* f *has a regular asymptotic expansion* Ef *on every compact subset* $D_\rho = D \cap \overline{B}_\rho$ *of* D. *Then there exist an order function* δ *satisfying* $\delta = o(1)$ *and a nested family* $\{B_{1/\delta(\epsilon)} : \epsilon \in (0, \epsilon_0)\}$ *of balls of radius* $1/\delta(\epsilon)$ *centered at the origin in* $\mathbf{R}^N$, *such that* Ef *extends to an asymptotic approximation of* f *on* $K_{1/\delta(\epsilon)} = D \cap \overline{B}_{1/\delta(\epsilon)}$.

6. Let $X = C([0,\infty), \|\cdot\|_\infty)$, and let $\phi \in X$ have a convergent Taylor series expansion $\phi(x) = \sum_{i=0}^{\infty} a_i x^i$ for all $x \in [0, \infty)$. Consider the function $f(\epsilon, x) = \phi(\epsilon x)$ as a map from $(0, \epsilon_0)$ into X. Find the m-term asymptotic expansion Ef of f and verify that it can be extended to an asymptotic approxiation of the order of ϵ^{m-1-p} $(p = 0, \ldots, m-1)$ on any interval $[0, b\epsilon^{-p/m}]$, where $b > 0$. (Cf. Theorem 4.)

3 Regularization

The extension theorems are important for the development of singular perturbation theory, but of limited practical use. First, they are typical *existence* theorems; they do not say anything about the order function δ, except that it is $o(1)$. Second, they do not give any insight as to what actually happens inside the boundary layer. This information can only come from a local analysis.

The idea behind the local analysis that we are about to present is fairly simple. Boundary layers occur because the function under discussion undergoes rapid variations near the singular part of the boundary, and these variations happen on ever smaller scales as $\epsilon \downarrow 0$. So, if we want to find out what actually happens inside the boundary layer, we must look on a different scale—that is, introduce a new independent variable—, where things are stretched out in the transverse direction across the boundary layer. In fact, the amount of stretching must increase as ϵ decreases, to counteract the "thinning" of the boundary layer.

The choice of the new independent variable depends of course on the function under consideration and may vary from one component of the singular part of the boundary to another. In general, the new variable, y, will be a function of ϵ and the old variable x, $y = \tau(\epsilon, x)$ say. If $P \in \partial_s D$ is a representative point on the particular component of the singular part of the boundary near which we wish to perform a local boundary-layer analysis, then τ must be sufficiently regular (for example, a homeomorphism), at least on some subset $D_P = D \cap B_R(P)$ of D with $R > 0$. ($B_P(R)$ is the open ball of radius R centered at P in $\mathbf{R}^N$.) Usually, one defines τ in such a way that P is sent to the origin, so $\tau(\epsilon, x^P) = 0$, and τ is a linear function of x. For example, $y = (x - x^P)/\delta(\epsilon)$ with $\delta = o(1)$ maps a one-dimensional boundary layer near $x = x^P$ onto an interval whose length grows beyond bounds as $\epsilon \downarrow 0$. If $\tau(D_P)$ is the image of the set D_P under the change of variables, $\tau(D_P) = \{\tau(\epsilon, x) : x \in D_P, \epsilon \in (0, \epsilon_0)\}$, then $\tau(D_P)$ looks like a semi-infinite cylinder based at the origin.

Now, consider the effect of such a transformation on f, which is a function of ϵ and x. The change of variables $y = \tau(\epsilon, x)$ transforms f into another function, Tf say, of ϵ and y through the formula $Tf(\epsilon, y) = Tf(\epsilon, \tau(\epsilon, x)) = f(\epsilon, x)$. This function can be interpreted as a mapping $Tf : (0, \epsilon_0) \to Y$, where Y is an appropriately defined normed vector space of functions on $\tau(D_P)$, whose asymptotic behavior as $\epsilon \downarrow 0$ can be analyzed in the framework of Y. Since the dependence of Tf upon ϵ differs from the dependence of f upon ϵ, there is a chance that Tf has a regular asymptotic expansion on $\tau(D_P)$ and that *the origin belongs to the regular part of the boundary.* (Recall that the origin is the image of P under the change of variables; it is a boundary point of $\tau(D_P)$.) If this is the case, we have achieved a *regularization* of f at P. We can construct the regular expansion in Y and then transform everything back to D_P by means of the inverse transformation τ^{-1}. The result is an expansion of f, which is a *local asymptotic approximation* of f near P.

Of course, there is no guarantee that a regularization exists, and if it exists it may not be unique, but this all depends on the function f.

Before moving on to the details, we discuss a simple example, where everything can be worked out. The example is again provided by the function $f(\epsilon, x) = 1 - e^{-x/\epsilon}$ on $[0, \infty)$, which, considered as a map from $(0, \epsilon_0)$ into $X = (C[0, \infty), \|\cdot\|_\infty)$, has a boundary layer near 0. The transformation $y = \tau(\epsilon, x) = x/\epsilon$ maps the interval $[0, \infty)$ onto itself, and we can take $Y = X$. The transformations τ induce a regularization of f near 0, $Tf(\epsilon, y) = 1 - e^{-y}$ for $y \in [0, \infty)$. The function Tf is independent of ϵ and is therefore identical with its (regular) asymptotic expansion on its entire domain of definition. The inverse transformation $x = \tau^{-1}(\epsilon, y) = \epsilon y$ yields the expansion $E_0 f(\epsilon, x) = 1 - e^{-x/\epsilon}$, which is a local asymptotic approximation of f on every compact subinterval of $[0, \infty)$.

We now turn to the details of a local boundary-layer analysis.

Suppose $f : (0, \epsilon_0) \to X$ has boundary-layer behavior on D. The singular part $\partial_s D$ of the boundary of D may consist of several components, each of which needs to be analyzed separately, but throughout the following discussion we focus on one single component. To avoid extraneous technical complications, we assume that this component is a smooth $(N-1)$-dimensional manifold, where the normal direction is well defined at each point. Let P be a representative point on this manifold. The position vector of P is $x^P = (x_1^P, \ldots, x_N^P)$, the tangent space at the point P is spanned by the unit vectors $t_1^P, \ldots, t_{N-1}^P$, and the unit normal vector at P is n^P. For definiteness, we assume that n^P points into D; if P is identified with a transition layer, we treat each side of the boundary separately.

Let $Ef(\epsilon) = \sum_n \delta_n(\epsilon) f_n$ be the regular asymptotic expansion of f on every ϵ-independent compact subset of $D \cup \partial_r D$. As we have seen in the previous section, Ef extends to an asymptotic approximation of f on ϵ-dependent compact subsets, possibly with some loss of the order of accuracy. We assume that this extension has been done and that Ef is an asymptotic approximation of f on the compact subset $K_{\delta_p'}$ of $D \cup \partial_r D$ for some δ_p' satisfying $\delta_p' = o(1)$. The distance from P to $K_{\delta_p'}$ is $O^\sharp(\delta_p')$.

As a first step, we change variables to magnify the boundary layer in the "thin" (normal) direction. The change is accomplished by an ϵ-dependent transformation,

$$\tau(\epsilon) : x \mapsto y = \tau(\epsilon, x).$$

In most cases, $\tau(\epsilon)$ can be an affine transformation—that is, a shift to put P at the origin, followed by an ϵ-dependent linear transformation; in exceptional cases, it may be necessary to generalize to a general homeomorphism. In either case, one must assume that $\tau(\epsilon)$ is defined on some subset $D_P = D \cap B_R(P)$ of D and continuously extended to P. ($B_R(P)$ is the open ball of radius R centered at P in $\mathbf{R}^N$.) To keep things simple, we will take

$$\tau(\epsilon, x) = \tau(\epsilon)(x - x^P), \quad x \in D_P,$$

where $\tau : (0, \epsilon_0) \mapsto \mathbf{R}_+^{N \times N}$ is a nonsingular matrix-valued function, which is asymptotically equal to the direct sum of the identity matrix on the tangent space and a scalar multiplication by $1/\delta$ in the direction of the normal vector at P, for some $\delta \in \mathcal{E}$,

$$\|\tau(\epsilon) t_i^P\| = O^\sharp(1), \; i = 1, \ldots N-1; \quad \|\tau(\epsilon) n^P\| = O^\sharp(1/\delta(\epsilon)).$$

Here, $\| \cdot \|$ is the Euclidian length in $\mathbf{R}^N$. We require that δ satisfy the order relation $\delta = o(\delta_p')$, to provide $\tau(\epsilon)$ with sufficient magnification in the boundary layer as $\epsilon \downarrow 0$.

The transformations $\tau(\epsilon)$ form a family,

$$\tau = \{\tau(\epsilon) : \epsilon \in (0, \epsilon_0)\},$$

which we assume to be ordered, $\tau(\epsilon_1)(D_P) \subset \tau(\epsilon_2)(D_P)$ whenever $\epsilon_1 \geq \epsilon_2$. We put $\tau(D_P) = \cup_\epsilon \tau(\epsilon)(D_P)$, say that τ is of the order of $1/\delta$, and write $\tau = O^\sharp(1/\delta)$.

A first observation is that $\tau(D_P)$ is an unbounded subset of $\mathbf{R}^N$. In fact, $\tau(\epsilon)(D_P)$ looks like an open cylinder, whose cross section remains asymptotically the same, but whose dimension in the axial direction is stretched more and more as $\epsilon \downarrow 0$.

Second, the origin belongs to the boundary of $\tau(\epsilon)(D_P)$ for every $\epsilon \in (0, \epsilon_0)$ and so, by induction, to the boundary $\partial\tau(D_P)$ of $\tau(D_P)$. In the image of the previous paragraph, the origin is at the center of the base of every cylinder.

Third, because we have given τ sufficient magnification, all points beyond the boundary layer (that is, all points belonging to the domain of validity of the interior approximation Ef of f) are sent to infinity. Consequently, the pre-image of any ϵ-independent compact subset of $\tau(D_P \cup P) = \tau(D_P) \cup 0$ is a subset of the boundary layer near P.

Now, we consider the effect of the transformation τ on elements of X, or rather, their restrictions to D_P.

Let X_P denote the normed vector space of functions defined on D_P with the topology induced by the norm of X. Elements of X_P are functions defined on the domain D_P, which is ϵ-independent. But by applying the coordinate transformation $\tau(\epsilon)$, we transform them into functions defined on $\tau(\epsilon)(D_P)$, which is ϵ-dependent. The purpose of the following arguments is to construct a normed vector space Y_P of functions defined on the ϵ-independent domain $\tau(D_P)$. The tool we use is that of the inductive limit; a discussion of inductive limits and their properties can be found, for example, in [6, Section XI.5].

Let $Y_P(\epsilon)$ be the set of all functions ψ defined on $\tau(\epsilon)(D_P)$ by the identity

$$\psi(y) = \psi(\tau(\epsilon, x)) = \phi(x), \quad y \in \tau(\epsilon)(D_P),$$

for some $\phi \in X_P$. We make $Y_P(\epsilon)$ into a normed vector space by introducing a norm $\|\cdot\|_Y$ that is commensurate with the norm in X,

$$\|\psi\|_Y = \|\phi\|_X, \quad \phi \in X_P.$$

The spaces $\{Y_P(\epsilon) : \epsilon \in (0, \epsilon_0)\}$ form a nested family of normed vector spaces, because of the ordering within the family τ. Their inductive limit defines the space Y_P,

$$Y_P = \lim\ \mathrm{ind}_{\epsilon\downarrow 0} Y_P(\epsilon).$$

Again, the precise details of this construction are less important; what is important is the intuitive idea that we end up with a normed vector space Y_P of functions defined on an

ϵ-independent domain and that we have a framework to relate elements of X_P (functions defined on D_P) to elements of Y_P (functions defined on $\tau(D_P)$). The relation above between the functions $\phi \in X_P$ and $\psi \in Y_P$ defines a transformation $T_\epsilon : X_P \to Y_P$,

$$(T_\epsilon \phi)(y) = \phi(x), \quad y = \tau(\epsilon, x),\ x \in D_P; \quad \phi \in X_P.$$

Let us see what this transformation does to the restriction of $f(\epsilon)$ to D_P. For each $\epsilon \in (0, \epsilon_0)$, we obtain the element $T_\epsilon f(\epsilon) \in Y_P$, so if we do this for every $\epsilon \in (0, \epsilon_0)$, we obtain a new vector-valued function, $T_P f : (0, \epsilon_0) \to Y_P$,

$$(T_P f)(\epsilon) = T_\epsilon f(\epsilon), \quad \epsilon \in (0, \epsilon_0).$$

The pointwise expression is

$$(T_P f)(\epsilon, y) = f(\epsilon, x), \quad y = \tau(\epsilon, x),\ x \in D_P; \quad \epsilon \in (0, \epsilon_0).$$

The function $T_P f$ gives us indirect access to f, and our goal is to use $T_P f$ to investigate the asymptotic behavior of f near P. (Recall that P is mapped into the origin, so we will pay particular attention to the behavior of $T_P f$ near the origin.)

Definition 4 *The transformation τ is a* regularizing transformation *for f at P if the origin belongs to the regular part $\partial_r \tau(D_P)$ of the boundary of $\tau(D_P)$ for the function $T_P f : (0, \epsilon_0) \to Y_P$. The function $T_P f$ defined by a regularizing transformation τ is called a* regularization *of f at P.*

Thus, if $T_P f$ is a regularization of f at P, then $T_P f$ has a regular asymptotic expansion $\tilde{E} T_P f(\epsilon) = \sum_n \delta_n(\epsilon)(T_P f)_n$ on every ϵ-independent compact subset of $\tau(D_P) \cup 0$, where $(T_P f)_n \in Y_P$ for $n = 0, 1, \ldots$ The tilde $\tilde{}$ indicates that the expansion defines an asymptotic approximation in Y_P.

A regularization may or may not exist. For example, there is no transformation that regularizes the function $(1 - e^{-x/\epsilon}) \sin(1/x)$ at 0. Also, if a regularization exists, it is not necessarily unique. Again, we illustrate with an example,

$$f(\epsilon, x) = -(x/\epsilon) + \sqrt{(x/\epsilon)^2 + 2/\epsilon + 1},$$

on $[0, \infty)$. This function, considered as a mapping from $(0, \epsilon_0)$ into $X = C([0, \infty), \| \cdot \|_\infty)$, has a boundary layer at 0 ($P = 0$). Each transformation $\tau_\nu(\epsilon)(x) = x/\epsilon^\nu$ with $\nu \geq 0$ maps the interval $[0, \infty)$ onto itself and $Y_P = X_P = X$. We have

$$T_P^\nu f(\epsilon, y) = -\epsilon^{\nu - 1} y + \sqrt{\epsilon^{2\nu - 2} y^2 + 2\epsilon^{-1} + 1}.$$

If $0 \leq \nu < \frac{1}{2}$, then $T_P^\nu f(\epsilon) \sim \epsilon^{-\nu} f_{0,\nu}$ with $f_{0,\nu}(y) = 1/y$ on any interval $[a, \infty)$ with $a > 0$; if $\nu \geq \frac{1}{2}$, then $T_P^\nu f(\epsilon) \sim \epsilon^{-1/2} f_{0,\nu}$ with $f_{0,\nu}(y) = -y + \sqrt{2 + y^2}$ if $\nu = \frac{1}{2}$ and $f_{0,\nu}(y) = \sqrt{2}$ if

$\nu > \frac{1}{2}$ on any compact interval $[0, b]$ with $b > 0$. Thus, every transformation τ_ν with $\nu \geq \frac{1}{2}$ defines a regularization $T_P^\nu f$ of f at 0.

As can be seen from the example, some regularizing transformations look more promising than others: one has the feeling that the transformation τ_ν with $\nu = \frac{1}{2}$ is somehow more significant, because it gives more information at less magnifying power than the same transformation with $\nu > \frac{1}{2}$.

We can compare the magnification of different transformations and thus their ability to induce a regularization.

Lemma 5 *Suppose τ induces a regularization of f at P, and $\tau = O^\sharp(1/\delta)$. Then every transformation τ' which is $O^\sharp(1/\delta')$ with $\delta' = o(\delta)$ induces another regularization of f at P.*

Proof. The assertion of the lemma is an immediate consequence of the ordering within the families τ and τ' and the order relation $\delta' = o(\delta)$. ∎

Definition 5 *A* significant regularization *of f is induced by any transformation τ with minimal order—that is, if $\tau = O^\sharp(1/\delta)$ and τ' is another regularizing transformation for f at the same point, which satisfies the order relation $\tau' = O^\sharp(1/\delta')$, then $\delta' = o(\delta)$. A variable $\{\tau(\epsilon, x) : x \in D\}$ that defines a significant regularization is called a* boundary-layer variable.

In the example above, the significant regularization is induced by the transformation τ_ν with $\nu = \frac{1}{2}$; less magnification does not yield a regularization, more magnification does not yield enough detail. The boundary layer variable is $y = x\epsilon^{-1/2}$.

Significant regularizations are obviously more significant than ordinary regularizations and will usually be the ones of interest. We shall therefore ignore the latter in favor of the former and simply refer to the former as "regularizations." We repeat, however, that the existence of a regularization is in no way guaranteed; in fact, the characterization of the class of functions f for which a regularization exists is an open and interesting problem of asymptotic analysis.

The (significant) regularization will generally be the same at points on the same component of $\partial_s D$, but vary from one component to another if $\partial_s D$ consists of more than one component. It then becomes a matter of piecing the various regularizations together to arrive at a global regularization.

Now suppose that we have found a regularization $T_P f$ of f at P through a regularizing transformation τ of the order $O^\sharp(\delta_r)$, and let $\tilde{E}T_P f(\epsilon) = \sum_n \delta_n(\epsilon)(T_P f)_n$ be the regular asymptotic expansion of $T_P f$ on every ϵ-independent compact subset of $\tau(D_P) \cup 0$. (We recall that δ_r satisfies the order relation $\delta_r = o(\delta'_p)$, where δ'_p is a measure of the thickness of the boundary layer.) As we have seen in the previous section, the asymptotic approximation defined by $\tilde{E}T_P f$ extends to ϵ-dependent compact subsets, possibly with some loss of the order of accuracy if one goes for the largest possible domain. (In the present case, the extension process must be based on Theorem 3, or its generalization Theorem 4, because the set $\tau(D_P)$ is unbounded.)

Let us formulate the result of the extension in precise terms. Suppose $\tilde{E}T_P f$ is an asymptotic approximation of the order of δ_{n-1} of $T_P f$ (i.e., $\tilde{E}T_P f$ is an n-term asymptotic expansion). Then there exist, for each $q \in \{0, \ldots, n-1\}$, an order function δ''_q satisfying $\delta''_q = o(1)$ and a nested family $\{B_{1/\delta''_q(\epsilon)} : \epsilon \in (0, \epsilon_0)\}$ of balls of radius $1/\delta''_q(\epsilon)$ centered at the origin in $\mathbf{R}^N$, such that $\tilde{E}T_P f$ extends to an asymptotic approximation of the order of δ_{n-1-q} of $T_P f$ on $\tilde{K}_{1/\delta''_q} = (\tau(D_P) \cup 0) \cap \overline{B}_{1/\delta''_q(\epsilon)}$. The order functions δ''_q satisfy the relation $\delta''_q = O(\delta''_{q-1})$ for $q = 1, \ldots, n-1$; possibly, this order relation can be sharpened to a o-relation.

If we transfer these results to the space X_P by means of the transformation T_P^{-1}, we obtain the following theorem.

Theorem 6 *Suppose $T_P f$ is a regularization of f at P and $\tilde{E}T_P f$ is its asymptotic approximation of the order of δ_{n-1} (i.e., $\tilde{E}T_P f$ is an n-term asymptotic expansion) on any ϵ-independent compact subset of $\tau(D_P) \cup 0$. Then there exist, for each $q \in \{0, \ldots, n-1\}$, an order function δ''_q satisfying $\delta''_q = o(1)$ and a nested family $\{B_{(\delta_r/\delta''_q)(\epsilon)}(P) : \epsilon \in (0, \epsilon_0)\}$ of balls of radius $\delta_r(\epsilon)/\delta''_q(\epsilon)$ centered at P in $\mathbf{R}^N$, such that $T_P^{-1}\tilde{E}T_P f$ is an asymptotic approximation of f on $K_{(\delta_r/\delta''_q)(\epsilon)} = (D_P \cup P) \cap \overline{B}_{(\delta_r/\delta''_q)(\epsilon)}(P)$. The expansions $\tilde{E}T_P f$ and $T_P^{-1}\tilde{E}T_P f$ have the same number of terms and define asymptotic approximations to the same order of accuracy.*

Proof. If we apply T_P^{-1} to the regular asymptotic expansion $\tilde{E}T_P f$, we obtain the function $T_P^{-1}\tilde{E}T_P f$ in X_P, which is an expansion of f, although not necessarily a regular asymptotic expansion. If $\tilde{E}T_P f$ is an asymptotic approximation of $T_P f$ of a certain order of accuracy or of a certain number of terms on some compact subset of $\tau(D_P) \cup 0$, then $T_P^{-1}\tilde{E}T_P f$ is an asymptotic approximation of f of the same order of accuracy or the same number of terms on the pre-image of the set under the transformation τ. The fact that the order of accuracy does not change is an immediate consequence of the fact that the norms in X_P and Y_P are commensurate. The pre-image of a compact subset of $\tau(D_P) \cup 0$ is a compact subset of $D_P \cup P$, and the pre-image of a ball of radius $1/\delta_q(\epsilon)$ centered at the origin in $\mathbf{R}^N$ is a "flattened ball" centered at P in $\mathbf{R}^N$, whose dimensions are asymptotically of the order $O^\sharp(1/\delta''_q)$ in the tangential directions and $O^\sharp(\delta_r/\delta''_q)$ in the normal direction. ∎

By imposing the condition $\delta_r = o(\delta''_q)$, we achieve that the subsets $K_{(\delta_r/\delta''_q)(\epsilon)}$ are located entirely within the boundary layer near P; they fill the boundary layer in the normal direction as $\epsilon \downarrow 0$.

We may summarize the statement of the theorem by saying that $T_P^{-1}\tilde{E}T_P f$ extends to an asymptotic approximation of f in the boundary layer near P. We denote the extension by $E_P f$ and refer to the function $E_P f$ thus defined as a *local expansion* of f near P. The expansion can be to a specified number of terms or to a specified order of accuracy. If τ is the regularizing transformation that defines $T_P f$, then we have the pointwise expression

$$E_P f(\epsilon)(x) = E_P f(\epsilon, x) = \sum_n \delta_n(\epsilon)(T_P f)_n(y), \quad y = \tau(\epsilon, x), \quad x \in D_P.$$

The functions $(T_P f)_n$ are uniquely determined; they are obtained by taking limits in Y,

$$(T_P f)_n = \lim_{\epsilon \downarrow 0} \frac{(T_P f)^{(n)}(\epsilon)}{\delta_n(\epsilon)}, \quad n = 0, 1, \ldots,$$

where $(T_P f)^{(0)}(\epsilon) = (T_P f)(\epsilon)$ and $(T_P f)^{(n)}(\epsilon) = (T_P f)(\epsilon) - \sum_{p=0}^{n-1} \delta_p(\epsilon)(T_P f)_p$ for $n = 1, 2, \ldots$ Pointwise,

$$(T_P f)(\epsilon)(y) = (T_P f)(\epsilon, y) = f(\epsilon, x), \quad y = \tau(\epsilon, x), \quad x \in D_P \cup P.$$

We conclude with a simple example to illustrate Theorem 6.

Consider the function $f(\epsilon, x) = e^{-x/\epsilon} + \phi(x)$ on $[0, \infty)$, where ϕ has a convergent Taylor series expansion for all x, $\phi(x) = \sum_{i=0}^{\infty} a_i x^i$. As usual, we consider this function as a mapping from $(0, \epsilon_0)$ into $X = (C[0, \infty), \|\cdot\|_\infty)$. The transformation $y = x/\epsilon$ regularizes f at the origin, $T_P f(\epsilon, y) = e^{-y} + \phi(\epsilon y)$, and the asymptotic approximation to the accuracy of ϵ^{n-1} is $\tilde{E}T_P f(\epsilon, y) = e^{-y} + \sum_{i=0}^{n-1} a_i \epsilon^i y^i$. The remainder satisfies the estimate $T_P f - \tilde{E}T_P f = O^\sharp(\epsilon^n) = o(\epsilon^{n-1})$ on any interval $[0, b]$ with $b > 0$. But, in agreement with Theorem 6, we also have, for each $q \in \{0, \ldots, n-1\}$, $T_P f - \tilde{E}T_P f = o(\epsilon^{n-1-q})$ on any interval of the form $[0, b\epsilon^{-q/n}]$. When we translate these results back to the original variable, we obtain the local expansion $E_P f(\epsilon, x) = e^{-x/\epsilon} + \sum_{i=0}^{n-1} a_i x^i$ for f, which satisfies the asymptotic estimate $f - E_P f = o(\epsilon^{n-1-q})$ on any interval $[0, b\epsilon^{1-q/n}]$. Notice that the length of this interval always goes to zero as $\epsilon \downarrow 0$, but the rate decreases as q increases. In the notation of Theorem 6, we have $\delta''_q(\epsilon) = \epsilon^{q/n}$.

Exercises

1. Consider the function $f(\epsilon, x) = \epsilon^2(x + \epsilon^2)^{-1}e^{-x/\epsilon}$ for $0 \leq x < \infty$ as a map into $X = (C[0, \infty), \|\cdot\|)$. Verify that f has boundary-layer behavior near 0, find a significant regularization of f at 0, and discuss the local expansion of f near 0.

4 Matching Asymptotic Approximations

We now come to the important problem of matching local expansions valid in the boundary layer to the interior expansion valid in D.

Suppose f has boundary-layer behavior on D and P is a point on $\partial_s D$, the singular part of the boundary of D. Let $E_I f$ be the asymptotic approximation of f obtained by approximating D from within, as described in Theorem 1 or its generalization, Theorem 2, and let $E_P f$ be the local asymptotic approximation obtained through a regularization of f near P, as described by Theorem 6. Then two things are needed to match $E_P f$ to $E_I f$: (i) $E_I f$ and $E_P f$ need to share a common domain of validity, and (ii) $E_I f$ and $E_P f$ need to have commensurate orders of approximation. Let us consider the situation in more detail.

First, we look at $E_I f$. Arguing from within D and starting from an m-term regular asymptotic expansion Ef of f, we have proven the existence of order functions $\delta'_0, \ldots, \delta'_{m-1}$, all $o(1)$ as $\epsilon \downarrow 0$, and of nested sets of compact subsets $K_{\delta'_0(\epsilon)}, \ldots, K_{\delta'_{m-1}(\epsilon)}$ satisfying $\mathrm{dist}(K_{\delta'_p}, \partial_s D) = O^\sharp(\delta'_p)$, such that $f - Ef = o(\delta_{m-1-p})$ on $K_{\delta'_p(\epsilon)}$ for $p = 0, \ldots, m-1$. The order functions satisfy the relation $\delta'_p = O(\delta'_{p-1})$ for $p = 1, \ldots, m-1$, but it is not ruled out that the symbol O may be strengthened to o, in which case the compact subsets actually expand as p increases while ϵ is being kept fixed.

Next, consider $E_P f$. The fact that we talk about $E_P f$ presupposes that there exists a regularization $T_P f$ of f near P. Let us assume that it has been obtained through a regularizing transformation (i.e., a change of coordinates) of the order of $1/\delta_r$. By imposing the condition $\delta_r = o(\delta'_p)$ for $p = 0, \ldots, m-1$, we have achieved that the entire domain of validity of the inner expansion is sent to infinity by the regularizing transformation. Then, starting from an n-term regular asymptotic expansion $\tilde{E} T_P f$ of $T_P f$, we have proven the existence of order functions $\delta''_0, \ldots, \delta''_{n-1}$, all $o(1)$ as $\epsilon \downarrow 0$, and of nested sets of compact subsets $K_{(\delta_r/\delta''_0)(\epsilon)}, \ldots, K_{(\delta_r/\delta_{n-1})''(\epsilon)}$, each containing P and all filling the boundary layer near P in the normal direction as $\epsilon \downarrow 0$, such that $f - T_P^{-1} \tilde{E} T_P f = o(\delta_{n-1-q})$ on $K_{(\delta_r/\delta''_q)(\epsilon)}$ for $q = 0, \ldots, n-1$. The dimensions of the set $K_{(\delta_r/\delta''_q)(\epsilon)}$ are $O^\sharp(1/\delta''_q)$ in the tangential directions and $O^\sharp(\delta_r/\delta''_q)$ in the normal direction. We assume that $\delta_r = o(\delta''_q)$ for $q = 0, \ldots, n-1$. The order functions satisfy the order relation $\delta''_q = O(\delta''_{q-1})$ for $q = 1, \ldots, n-1$, but, again, it is not ruled out that the symbol O may be strengthened to o, in which case the compact subsets actually expand as q increases while ϵ is being kept fixed.

Given these details, let us first consider the most straight-forward case, where $m = n$ and $p = q = 0$.

Definition 6 *The function f and its regularization $T_P f$ at P satisfy the* strong overlap condition *if the intersection of the extended domain of validity of the m-term interior expansion $E_I f$ and the extended domain of validity of the m-term local expansion $E_P f$ has a* *nonempty interior for every $m = 1, 2, \ldots$*

Theorem 7 *The function f and its regularization $T_P f$ at P satisfy the strong overlap condition if, for every $m = 1, 2, \ldots$, there exist order functions δ_0' and δ_0'' such that $\delta_0' = o(\delta_r/\delta_0'')$.*

Proof. According to Theorem 2, the m-term expansion $E_I f$ extends to an asymptotic approximation of f on $K_{\delta_0'(\epsilon)}$. The distance from this set to $\partial_s D$ is $O^\sharp(\delta_0'(\epsilon))$. According to Theorem 6, the m-term expansion $E_P f$ extends to an asymptotic approximation of f on $K_{(\delta_r/\delta_0'')(\epsilon)}$. The latter set extends over a distance $O^\sharp(\delta_r/\delta_0'')(\epsilon))$ from P into D in the normal direction. If $\delta_0' = o(\delta_r/\delta_0'')$, the two sets certainly overlap as $\epsilon \downarrow 0$. ∎

In the case of strong overlap, there is no need to extend the domains of validity of $E_I f$ and $E_P f$ at the expense of accuracy. Obviously, this is the best of all possible worlds. If there is no strong overlap, it becomes a matter of balancing the accuracy of each approximation (i.e., the number of terms in the expansion) against the domain of validity, and we may have to increase the number of terms in either or both of the expansions to achieve the desired order of accuracy in the region of overlap.

Definition 7 *The function f and its regularization $T_P f$ at P satisfy the* overlap condition *if, for every $k = 1, 2, \ldots$, there exist integers $m, n \in \{k, k+1, \ldots\}$ such that the intersection of the extended domain of validity of the m-term interior expansion $E_I f$ and the extended domain of validity of the n-term local expansion $E_P f$ has a nonempty interior, where $f - E_I f = o(\delta_{k-1})$ and $f - E_P f = o(\delta_{k-1})$.*

Theorem 8 *The function f and its regularization $T_P f$ at P satisfy the overlap condition if, for every $k = 1, 2, \ldots$, there exist integers $m, n \in \{k, k+1, \ldots\}$ such that the order functions δ_{m-k}' and δ_{n-k}'' satisfy the relation $\delta_{m-k}' = o(\delta_r/\delta_{n-k}'')$.*

Proof. According to Theorem 2, the m-term expansion $E_I f$ extends to an asymptotic approximation of f of the order of δ_{k-1} on $K_{\delta_{m-k}'(\epsilon)}$. The distance from this set to $\partial_s D$ is $O^\sharp(\delta_{m-k}'(\epsilon))$. According to Theorem 6, the n-term expansion $E_P f$ extends to an asymptotic approximation of f of the order of δ_{k-1} on $K_{(\delta_r/\delta_{n-k}'')(\epsilon)}$. This set extends over a distance $O^\sharp(\delta_r/\delta_{n-k}'')(\epsilon))$ from P into D in the normal direction. If $\delta_{m-k}' = o(\delta_r/\delta_{n-k}'')$, the two sets certainly overlap as $\epsilon \downarrow 0$. ∎

If there is no strong overlap, one will generally try to keep m and n as close to k as possible, so as to minimize the number of extra terms one has to carry to achieve the desired order of accuracy on the region of overlap. The following examples illustrate the matching mechanism.

First, an example where the strong overlap condition is met.

Consider again the function $f(\epsilon, x) = e^{-x/\epsilon} + \phi(x)$ on $[0, \infty)$ from the previous section. The m-term interior expansion is $E_I f(\epsilon, x) = \phi(x)$ for any positive integer m, and the n-term local expansion is $E_P f(\epsilon, x) = e^{-x/\epsilon} + \sum_{i=0}^{n-1} a_i x^i$. The former satisfies the estimate $f - E_I f = o(\epsilon^{m-1})$ on any interval $[a\epsilon^\mu, \infty)$ with $\mu > 0$ and $a > 0$, the latter the estimate $f - E_P f = O^\sharp(\epsilon^{n\nu})$ on any interval $[0, b\epsilon^\nu]$ with $\nu > 0$ and $b > 0$. Because of the conditions $\delta_r = o(\delta_0')$ and $\delta_r = o(\delta_0'')$, μ and ν are further restricted to the interval $(0, 1)$. We claim that we can take $m = n$ and choose μ and ν so Theorem 7 applies.

We have $\delta_0'(\epsilon) = \epsilon^\mu$ and $\delta_r/\delta_0''(\epsilon) = \epsilon^\nu$. Overlap is achieved whenever $\mu > \nu$. For matching, it suffices to choose ν such that $\epsilon^{m\nu} = o(\epsilon^{m-1})$, or $\nu > 1 - 1/m$. For example, by choosing $\nu = 1 - 1/(2m)$ and $\mu = 1 - 1/(4m)$, we achieve that the interior and local approximations overlap on the interval $[\epsilon^{1-1/(2m)}, \epsilon^{1-1/(4m)}]$, and that they match there to the order of δ_{m-1}, for $m = 1, 2, \ldots$

Next, an example where the strong overlap condition is not met, but where we still have overlap.

Consider the function $f(\epsilon, x) = \epsilon/(\epsilon + x) + \phi(x)$ on $[0, \infty)$, with ϕ as before. The m-term interior expansion is $E_I f(\epsilon, x) = \phi(x) + \sum_{i=0}^{m-2} (-1)^i \epsilon^{i+1} x^{-(i+1)}$ and the n-term local expansion is $E_P f(\epsilon, x) = \epsilon/(\epsilon + x) + \sum_{i=0}^{n-1} a_i x^i$. The former satisfies the estimate $f - E_I f = O^\sharp(\epsilon^{m(1-\mu)})$ on any interval $[a\epsilon^\mu, \infty)$ with $\mu > 0$ and $a > 0$, the latter the estimate $f - E_P f = O^\sharp(\epsilon^{n\nu})$ on any interval $[0, b\epsilon^\nu]$ with $\nu > 0$ and $b > 0$. Because the regularizing transformation is of the order of $1/\epsilon$, we restrict μ and ν further to the interval $(0, 1)$. We have $\delta_{m-k}'(\epsilon) = \epsilon^\mu$ and $(\delta_r/\delta_{n-k}'')(\epsilon) = \epsilon^\nu$.

For a given k, we must determine m, n, μ, and ν so the domains of validity overlap and the matching condition is satisfied. The two approximations $E_I f$ and $E_P f$ match to the order of δ_{k-1} (i.e., to k terms) if $(k-1)/n < \nu < \mu < 1 - (k-1)/m$. The choice $m = n = k$ works only for $k = 1$; any μ and ν with $0 < \nu < \mu < 1$ will do in this case. For larger values of k, we can take $m = k + 1$, $n = k$, and $1 - 1/k < \nu < \mu < 1 - (k-1)/(k+1)$.

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